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It is well-known that the class of right-ordered groups and the class of regular ordered permutation groups coincide. In this paper, we exploit this connection to investigate the component parts of an arbitrary regular o-permutation group. We show that there exist regular o-permutation groups with nonregular o-primitive components. We show how to construct a regular o-permutation group which has any given o-primitive o-permutation group as its largest component. We investigate consequences of this construction when o-primitive l-permutation groups are used. We also derive some of the necessary relationships which must exist between the o-primitive components of a regular o-permutation group, and we derive a collection of necessary and sufficient conditions for a regular o-permutation group, which has a finite number of o-primitive components, to have all its components regular.

In [2], Paul Conrad studied the structure of an arbitrary right-ordered group, and showed there is a natural correspondence between the class of right-ordered groups and the class of regular ordered permutation groups. He also investigated those right-ordered groups which have the property that for every pair of positive elements a, b, there exists a positive integer n such that $(ab)^n > ba$. These groups are called Conrad right-ordered groups. Those regular ordered permutation groups which are matched to the class of Conrad right-ordered groups under the natural correspondence mentioned above are distinguished by the property that each of their o-primitive components is order-isomorphic to a right regular representation of a subgroup of the reals. In this note, we show that there are regular ordered permutation groups with non-Archimedean, regular o-primitive components, and we also show that there exist regular ordered permutation groups with nonregular o-primitive components. In fact, each transitive, o-primitive ordered permutation group may occur as the o-primitive component of a regular ordered permutation group. Consequences of this theorem are explored. Next, we derive some of the necessary relationships which must exist between the components of a regular ordered permutation group. Finally, we briefly study a more general class of right-ordered groups than the class of Conrad right-ordered groups. This class has the property that each of the o-primitive components of the corresponding

ordered permutation group is regular, although they need not be Archimedean.

A group G together with a total order $\leqq$ of the set G is a *right-ordered group* (ro-group) if $a \leqq b$ implies $ac \leqq bc$ for all a, b, c in G. Throughout this note, the identity element of any group G will be denoted by e_G. For a totally ordered set S, $\mathscr{A}(S)$, the collection of all order preserving permutations of S, is a lattice-ordered group (*l*-group) under the induced order, and the pair $(\mathscr{A}(S), S)$ is a *lattice-ordered permutation group* (*l*-permutation group). If G is a subgroup of $\mathscr{A}(S)$, G is a partially ordered group (po-group), and (G, S) is an *ordered permutation group* (o-permutation group). If G is a sublattice of $\mathscr{A}(S)$ as well as a subgroup, (G, S) is also an *l*-permutation group. Let (G, S) be a regular o-permutation group and s_0 be a fixed reference point in the set S. Then, the triple (G, S, s_0) is *a regular o-permutation group with a distinguished element*. Suppose (G, S, s_0) and (H, T, t_0) are two such permutation groups. Then (G, S, s_0) and (H, T, t_0) are *order-isomorphic* if and only if (i) There exists an order-isomorphism $\psi \colon G \to H$, (ii) There exists a set order-isomorphism $\phi \colon S \to T$, (iii) $s_0\phi = t_0$, and (iv) For each s in S and for all g in G, $(sg)\phi = s\phi g\psi$. For further information on ordered permutation groups, the reader is referred to Glass [3].

A form of the following correspondence theorem was proved by Conrad in [1].

THEOREM 1. *There is a one-to-one correspondence between isomorphism classes of right-ordered groups and isomorphism classes of regular o-permutation groups with distinguished elements.*

Proof. Let $(G, \leqq)$ be an ro-group. G is a totally ordered set, and (G, G) denotes the right regular representation of G. Then (G, G, e_G) is a regular o-permutation group with a distinguished element. The isomorphism class of $(G, \leqq)$ will be matched to the isomorphism class of (G, G, e_G) by a map χ. Now let (G, S, s_0) be any regular o-permutation group with a distinguished element. To redefine the order relation on the set G, for any g in G, say g is positive if $s_0 g \geqq s_0$ in S. Let P denote the collection of positive elements of G. P has the properties of a positive cone for a right total order of G. Let $(G, \leqq_{s_0})$ denote this ro-group, and match (G, S, s_0) *with* $(G, \leqq_{s_0})$ by a map n. It is straightforward to check that χ is a one-to-one correspondence and $\chi^{-1} = n$.

We will refer to the correspondence χ set up in the above theorem as the *Conrad correspondence* between the isomorphism classes of right-ordered groups and the isomorphism classes of regular o-

permutation groups with distinguished elements.

Let (G, S) be an ordered permutation group. A *G-congruence on S* is an equivalence relation $\mathscr{C}$ on S such that if $s\mathscr{C}t$, then $sg\mathscr{C}tg$ for all s, t in S and g in G. $\mathscr{C}$ is a *convex G-congruence* if the classes of $\mathscr{C}$ are convex. A convex subset T of S is an *o-block* if $T \neq \varnothing$ and whenever $g \in G$ either $Tg = T$ or $Tg \cap T = \varnothing$. If G is transitive on S, every o-block T is a class of a unique convex G-congruence and, conversely, the classes of each convex G-congruence are o-blocks. Moreover, the collection of convex G-congruences forms a tower under inclusion [3]. Convex congruences can be used to form the *o-primitive components* of an ordered permutation group, and *ordered wreath products* are then used to sew the o-primitive components together. The knowledge of the possible o-primitive components which may occur in a transitive l-permutation group has proved to be an invaluable tool in understanding their overall structure. This is one of the primary reasons for examining o-primitive components when studying the more general o-permutation groups. A knowledge of these subjects is assumed in this note. The notation and terminology can be found in Glass [3].

Next, let $(G, \leqq)$ be an ro-group. If H is a subgroup of G, H is *convex* if for every g in G and h in H, $e_G \leqq g \leqq h$ in G implies $g \in H$. The collection of convex subgroups of $(G, \leqq)$ forms a tower under containment [2].

Suppose the isomorphism class of $(G, \leqq)$ is matched to the isomorphism class of (H, T, t_0) under the Conrad correspondence. The following theorem relates the convex subgroups of $(G, \leqq)$ to the o-blocks (and hence the convex H-congruences) of (H, T, t_0).

THEOREM 2. *There is a one-to-one correspondence between the collection of convex subgroups of G and the collection of o-blocks of the set T containing the point t_0. This correspondence preserves containment.*

Proof. It suffices to examine $(G, \leqq)$ and (G, G, e_G). If K is a convex subgroup of G, then K is a convex subset of the chain G which contains e_G. It is straightforward to check that K is an o-block. The second part of the theorem will then be clear.

The class of Conrad right-ordered groups and the corresponding class of regular o-permutation groups are described in the following theorem.

THEOREM 3. *Let $(G, \leqq)$ be a right-ordered group with positive cone P and let (G, G, e_G) be the corresponding regular o-permutation*

group. Then the following conditions are equivalent:

(i) *For all a, b in P there exists a positive integer n such that* $(ab)^n > ba$.

(ii) *For all covering pairs* (C_γ, C^γ) *of convex subgroups of G, C_γ is normal in C^γ and C^γ/C_γ is o-isomorphic to a subgroup of* **R**.

(iii) *Every o-primitive component of* (G, G, e_G) *is o-isomorphic to a right regular representation of a subgroup of* **R**.

Proof. (i) $\Rightarrow$ (ii) is proved in [2]. The equivalence of (ii) and (iii) is immediate.

Every regular *l*-permutation group has every *o*-primitive component *o*-isomorphic to a right regular representation of a subgroup of **R** [3]. In view of the above theorem, the natural questions to ask are whether the *o*-primitive components of every regular *o*-permutation group have to be regular, and whether the regular components of such a group have to be right regular representations of subgroups of **R**. The answer to both of these questions is no as we shall show by example.

We now consider the second question. The following example of an ro-group is due to Smirnov, although it originally appeared in another form. Let $G = \{(x, y) \in \boldsymbol{Q} \times \boldsymbol{Q} \mid y > 0\}$. Define an operation on G by: for $(x_1, y_1), (x_2, y_2)$ in G, $(x_1, y_1)(x_2, y_2) = (x_1 + x_2 y_1, y_1 y_2)$. Fix a positive irrational β and define a positive cone P for a right total order of G by $P = \{(x, y)\varepsilon G \mid x\beta + y \geq 1\}$. Then G is a nonabelian ro-group with no proper convex subgroups, and it is not Archimedean [6]. Since G has no proper convex subgroups, by Theorem 2, G has no proper *o*-blocks and thus (G, G, e_G) is a regular, *o*-primitive *o*-permutation group which is not isomorphic to a right regular representation of a subgroup of **R**. As remarked before, this is in sharp contrast to the situation for *o*-primitive regular *l*-permutation groups.

The following example gives a negative answer to the first question asked.

EXAMPLE 4. Let $(\mathscr{A}(\boldsymbol{R}), \boldsymbol{R})$ denote the *l*-permutation group of order preserving permutations of the real line. Well order the set $\boldsymbol{R}$ by $r_0 \prec r_1 \prec \cdots$ and right order $\mathscr{A}(\boldsymbol{R})$ by the following rule: for $f \in \mathscr{A}(\boldsymbol{R})$, call f strictly positive if the first point r in the well order of $\boldsymbol{R}$ that f moves is moved strictly up in the original total order of $\boldsymbol{R}$. (This method was introduced by Conrad in [2].) The collection of all strictly positive elements forms a strict positive cone for a right total order $\leqq$ of $\mathscr{A}(\boldsymbol{R})$. Let $(\mathscr{A}(\boldsymbol{R}), \mathscr{A}(\boldsymbol{R}), e)$ denote the regular *o*-permutation group which corresponds to $(\mathscr{A})\boldsymbol{R}), \leqq)$ under the Conrad correspondence. We show $(\mathscr{A}(\boldsymbol{R}), \mathscr{A}(\boldsymbol{R}))$ has

nonregular o-primitive components. In fact, each o-primitive component is nonregular, there are infinitely many o-primitive components, and they are all order-isomorphic to the o-2 transitive l-permutation group $(\mathscr{A}(R), R)$.

For each $r \in R$, let G_r denote the stabilizer subgroup of r in $\mathscr{A}(R)$. The tower of convex subgroups of $(\mathscr{A}(R), \leqq)$ is $\mathscr{A}(R) \supseteqq G_{r_0} \supseteqq G_{r_0} \cap G_{r_1} \supseteqq G_{r_0} \cap G_{r_1} \cap G_{r_2} \supseteqq \cdots$, and, at each stage in the tower, equality holds or the lower group is a maximal convex subgroup of the group immediately preceding it. (Equality will hold once a dense subset of R has been fixed.)

To show this is true, we argue that G_{r_0} is a maximal convex subgroup of $(\mathscr{A}(R), \leqq)$. The other needed arguments are similar. Clearly G_{r_0} is convex. Let $g \in \mathscr{A}(R) \backslash G_{r_0} \cdot \langle G_{r_0}, g \rangle$ denotes the convex subgroup of $\mathscr{A}(R)$ generated by G_{r_0} and g. We show $\langle G_{r_0}, g \rangle = \mathscr{A}(R)$. Without loss of generality, $g > e$. Choose any $f > e \in \mathscr{A}(R)$. $g \in \mathscr{A}(R) \backslash G_{r_0}$ so $r_0 < r_0 g$. If $r_0 \leqq r_0 f < r_0 g$, then $e < f < g$ and $f \in \langle G_{r_0}, g \rangle$. Assume $r_0 < r_0 g \leqq r_0 f$. Choose $h \in G_{r_0}$ so that $r_0 f < r_0 g h$. $g h \in \langle G_{r_0}, g \rangle$ and the first point in the well order of R which $g h$ moves is r_0. Therefore, by definition of the right order on $\mathscr{A}(R)$, $e < f < g h$ and so $f \in \langle G_{r_0}, g \rangle$. Thus $\langle G_{r_0}, g \rangle = \mathscr{A}(R)$.

By using the above information, we see that the largest o-primitive component of $(\mathscr{A}(R), \mathscr{A}(R))$ is $(\mathscr{A}(R), \mathscr{A}(R)/G_{r_0})$. This is o-isomorphic to $(\mathscr{A}(R), R)$ by letting ψ in the definition of o-isomorphism be the identity map and by letting ϕ be defined by: for $f G_{r_0} \in \mathscr{A}(R)/G_{r_0}$, $(f G_{r_0}) \phi = r_0 f$. Thus $(\mathscr{A}(R), \mathscr{A}(R))$ has a nonregular o-primitive component which is o-isomorphic to an o-2 transitive l-permutation group.

In fact, when descending the tower of convex subgroups, we see that each o-primitive component of $(\mathscr{A}(R), \mathscr{A}(R))$ is l-isomorphic to $(\mathscr{A}(R), R)$, and there will be infinitely many such components in the wreath product of o-primitive factors of $(\mathscr{A}(R), \mathscr{A}(R))$.

To check this statement, we describe what happens at the second and third stages. The component immediately below $(\mathscr{A}(R), \mathscr{A}(R)/G_{r_0})$ in the tower is $(G_{r_0}, G_{r_0}/G_{r_0} \cap G_{r_1})$. The l-group G_{r_0} has three orbits in the set R, namely all points which are to the left of r_0 in the natural order on R, $\{r_0\}$, and all points which lie to the right of r_0. The two nonsingleton orbits are o-isomorphic to R. Without loss of generality, suppose r_1 lies to the left of r_0. Call the left orbit $\mathscr{O}_L$. We may define an o-isomorphism ϕ from $G_{r_0}/G_{r_0} \cap G_{r_1}$ to $\mathscr{O}_L$ by $(f G_{r_0} \cap G_{r_1}) \phi = r_1 f$. G_{r_0} acts like $\mathscr{A}(R)$ when restricted to $\mathscr{O}_L$. Thus $(G_{r_0}, G_{r_0}/G_{r_0} \cap G_{r_1})$ is o-isomorphic to $(G_{r_0}, \mathscr{O}_L)$ which is in turn o-isomorphic to $(\mathscr{A}(R), R)$. At the third stage in the tower the o-primitive component is $(G_{r_0} \cap G_{r_1}, G_{r_0} \cap G_{r_1}/G_{r_0} \cap G_{r_1} \cap G_{r_2})$. Again, if r_1 lies to the left of r_0, $G_{r_0} \cap G_{r_1}$ has three nonsingleton orbits in R: all points to the

left of r_1, all points between r_1 and r_0, and all points to the right of r_0. We may then locate the orbit of r_2 and show that the resulting o-permutation group is o-isomorphic to $(\mathscr{A}(R),\, R)$.

To show that the above example is not pathological, we now describe which o-primitive o-permutation groups may occur as the o-primitive component of a regular o-permutation group.

THEOREM 5. *Let (G, S) be a transitive, o-primitive o-permutation group. Then there exists a regular o-permutation group which has (G, S) as an o-primitive component.*

Proof. We first prove the following lemma.

LEMMA. *Suppose (G, S) is a transitive, o-primitive, o-permutation group. Well order S by $s_0 \prec s_1 \prec s_2 \prec \cdots$, and right order G according to this well order (as was done in Example 4). Then G_{s_0} is a maximal convex subgroup of the ro-group $(G, \leqq)$.*

Proof. First, G_{s_0} is convex. For suppoe $g \in G_{s_0}$, $f \in G$, and $e_G < f < g$. There exists s_α in S such that s_α is the first element in the well order of S which g moves and $s_\alpha < s_\alpha g$. Similarly, there exists an element s_β in S for f with the same properties. $e_G < f < g$ implies, by definition of the right order on G, that $s_\alpha \leqslant s_\beta$. Since $g \in G_{s_0}$, $s_0 \prec s_\alpha$. Thus $s_0 \prec s_\beta$ and $f \in G_{s_0}$.

Next, G_{s_0} is maximal. Suppose there exists a convex subgroup C of G such that $G_{s_0} \subsetneqq C$. Let $s_0 C$ denote the orbit of s_0 under the group C. We check $s_0 C$ is an o-block. It is easy to see that $s_0 C$ is a G-block. To check the convexity, suppose $s_0 c_1$ and $s_0 c_2$ are in $s_0 C$ and there is a t in S such that $s_0 c_1 \leqq t \leqq s_0 c_2$ in the original order on S. G is transitive so there is f in G such that $s_0 f = t$. Thus $s_0 c_1 \leqq s_0 f \leqq s_0 c_2$. Since C is convex, $f \in C$. Thus $t \in s_0 C$. It follows $s_0 C$ is an o-block of S. $C \supsetneqq G_{s_0}$ so there exists $g \in C$ such that $s_0 g \neq s_0$. Therefore $s_0 C \supseteq \{s_0, s_0 g\}$. $(G, S\}$ is o-primitive so it must be that $s_0 C = G$.

We now show $C = G$. Choose $e_G < f \in G$. If $s_0 f = s_0$ then $f \in C$. Therefore assume $s_0 < s_0 f$. Since $s_0 C = S$, there exists c in C such that $s_0 < s_0 f < s_0 c$. By definition of the right order on G and the convexity of C, $f \in C$. Thus G_{s_0} is maximal in G.

Now right order G according to the well order in the lemma and let (G, G, e_G) be the corresponding regular o-permutation group of G so $(G, G/G_{s_0})$ is the largest o-primitive component of (G, G, e_G). Using the methods of Example 4, we can show $(G, G/G_{s_0})$ is o-isomorphic

to (G, S). Thus (G, G, e_G) has (G, S) as an *o*-primitive component.

When considering Example 4 and Theorem 5, several questions immediately come to mind. Suppose (G, S) is any *o*-primitive *o*-permutation group, $s_0 \prec s_1 \prec s_2 \cdots$ is a well order of the set S, and G is right-ordered according to this well order. We may form a tower of convex subgroups of G by $G_{s_0} \supseteq G_{s_0} \cap G_{s_1} \supseteq G_{s_0} \cap G_{s_1} \cap G_{s_2} \supseteq \cdots$. Some questions are: (i) at each stage of the tower is the lower group always maximal in or equal to the upper group?; (ii) does one have to go through infinitely many stages in the tower before any equality relations hold?; and (iii) are all the *o*-primitive components of a regular *o*-permutation group constructed by using the methods of Theorem 5 always order-isomorphic?

We examine these questions in the context of *o*-primitive *l*-permutation groups. S. H. McCleary has shown that all *o*-primitive *l*-permutation groups are either regular, *o*-2 transitive, or periodic [4]. If (G, S) is a a regular *o*-primitive *l*-permutation group, then by going through the process of Theorem 5, we simply recover (G, S), and there are no proper convex subgroups to consider.

We now consider the second case.

THEOREM 6. *Suppose (G, S) is an o-2 transitive l-permutation group, and $s_0 \prec s_1 \prec s_2 \prec \cdots$ is a well ordering of the set S. Right order G according to this well order. Then:*

(i) $G_{s_0} \supseteq G_{s_0} \cap G_{s_1} \supseteq G_{s_0} \cap G_{s_1} \cap G_{s_2} \supseteq \cdots$ *is a tower of convex subgroups of G.*

(ii) *For every positive integer $n (\geq 1)$, $\bigcap_{i=0}^{n} G_{s_i}$ is proper maximal convex subgroup of $\bigcap_{i=0}^{n-1} G_{s_i}$.*

(iii) *The o-primitive components which correspond to the countable collection of convex subgroups mentioned in (ii) are all o-2 transitive.*

Proof. (i) has been done. To prove the maximality part of (ii), use the identical argument given in Example 4. The fact that these relations will hold for at least countably many steps follows from the fact that *o*-2 transitivity implies *o-m* transitivity for every finite integer m. (iii) also follows from the same statement.

We may not, however, abstract the properties of Example 4 to show that all the *o*-primitive components of a regular *o*-permutation group *constructed from an o-2 transitive l*-permutation group are order-isomorphic. We demonstrate this in the following examples.

EXAMPLE 7. Let L denote the long line. Any initial segment of L is order-isomorphic to R, any final segment of L is order-

isomorphic to L, and $(\mathscr{A}(L), L)$ is o-2 transitive [3].

Fix any $r_0 \in L$. Next, choose any r_1 to the left of r_0 and choose an r_2 to the right of r_0. Well order L by $r_0 \prec r_1 \prec r_2$ and then complete the well ordering in any way obtaining $r_0 \prec r_1 \prec r_2 \prec r_3 \prec \cdots$.

Right order $\mathscr{A}(L)$ according to this well order. By Theorem 7, the tower of convex subgroups of $\mathscr{A}(L)$ begins with $G_{r_0} \supseteq G_{r_0} \cap G_{r_1} \supseteq G_{s_0} \cap G_{r_1} \cap G_{r_2}$. By checking the method of Example 4, the largest o-primitive component of $\mathscr{A}(L)$ is order-isomorphic to $(\mathscr{A}(L), L)$. The second largest o-primitive component is $(G_{r_0}, G_{r_0}/G_{r_0} \cap G_{r_2})$. Since r_1 is located in the left orbit of G_{r_0}, $G_{r_0}/G_{r_0} \cap G_{r_1}$ is order-isomorphic to R. It follows that the second largest component is order-isomorphic to $(\mathscr{A}(R), R)$. The point r_2 is located in a final segement of L, so $(G_{r_0} \cap G_{r_1}, G_{r_0} \cap G_{r_1}/G_{r_0} \cap G_{r_1} \cap G_{r_2})$, the third component, is order-isomorphic to $(\mathscr{A}(L), L)$.

Before giving the next example, we slightly generalize the method of right-ordering an ordered permutation group which was given in Example 4. For a chain S, let $\bar{S}$ denote the Dedekind completion of S. For any $f \in \mathscr{A}(S)$, f can be uniquely extended to an order preserving permutation $\bar{f}$ of $\bar{S}$ [3]. Identify each permutation f with its extension $\bar{f}$.

PROPOSITION 8. *Let* (G, S) *be an ordered permutation group. Well order the set* $\bar{S}$ *by* $\bar{s}_0 \prec \bar{s}_1 \prec \bar{s}_2 \prec \cdots$. *Order the set* G *by the rule: for any* f *in* $\mathscr{A}(S)$, f *is strictly positive if the first point* f *moves in the well order of* $\bar{S}$ *is moved strictly up in the original order on* $\bar{S}$. *Then, the collection of all strictly positive element forms a strict positive cone for a right total order of* G.

EXAMPLE 9. A totally ordered set S is an η_1-set if whenever $A, B \subseteq S$, $A < B$, and $|A|, |B| < \aleph_1$, there exists s in S such that $A < s < B$. Any η_1-set of cardinality $\aleph_1$ is a 1-*set*. Let S be a 1-set. Then $(\mathscr{A}(S), S)$ is o-2 transitive. Both the initial and final characters of S are ω_1. Let $\bar{S}$ denote the Dedekind completion of S. Then each point of S has character c_{11}, whereas for each $\bar{s} \in \bar{S} \backslash S$, $\bar{s}$ may have character c_{01}, c_{10}, or c_{11}. For verification of the above facts, the reader is referred to Glass [3].

We will right-order $\mathscr{A}(S)$ using Proposition 8. We will specify a well ordering of $\bar{S}$. Let $\bar{s}_0 \in \bar{S}$ be any point of character c_{01}, and $\bar{s}_1$ be any point of $\bar{S}$ which lies to the left of $\bar{s}_1$. Complete the well ordering in any way obtaining $\bar{s}_0 \prec \bar{s}_1 \prec \bar{s}_2 \prec \cdots$. Right order $\mathscr{A}(S)$ according to Proposition 8. As before, the tower of convex subgroups of $\mathscr{A}(S)$ starts with $G_{\bar{s}_0} \supseteq G_{\bar{s}_0} \cap G_{\bar{s}_1}$.

The largest o-primitive component of $(\mathscr{A}(S), \mathscr{A}(S))$ is $(\mathscr{A}(S), \mathscr{A}(S)/G_{\bar{s}_0})$. $\mathscr{A}(S)/G_{\bar{s}_0}$ is order-isomorphic to $\bar{s}_0.\mathscr{A}(S)$, the orbit of $\bar{s}_0$ under the functions in $\mathscr{A}(S)$. This consists of all points of character c_{01}, and so $\bar{s}_0.\mathscr{A}(S)$ is a dense subset of $\bar{S}$ disjoint from S. The largest o-primitive component is therefore something which is closely related to the original group $(\mathscr{A}(S), S)$.

The second o-primitive component is $(G_{\bar{s}_0}, G_{\bar{s}_0}/G_{\bar{s}_0} \cap G_{\bar{s}_1})$. $G_{\bar{s}_0}/G_{\bar{s}_0} \cap G_{\bar{s}_1}$ is a dense subset of the segment of $\bar{S}$ which lies to the left of $\bar{s}_0$ and has final character ω_0. Because of the vast difference between the final characters of this chain and the original chain S, we have no hope of marking an identification between this component and $(\mathscr{A}(S), S)$.

We now consider the final case where (G, S) is a periodically o-primitive l-permutation group.

THEOREM 10. *Suppose (G, S) is a periodically o-primitive l-permutation group, $s_0 \prec s_1 \prec s_2 \prec \cdots$ is a well ordering of the set S, G is right ordered according to this well ordering, and $G_{s_0} \supseteq G_{s_0} \cap G_{s_1} \supseteq G_{s_0} \cap G_{s_1} \cap G_{s_2} \supseteq \cdots$ is the tower of convex subgroups of G corresponding to this well ordering. Then,*

(i) *For every positive integer $n \geq 1$, either $\bigcap_{i=0}^{n} G_{s_i} = \bigcap_{i=0}^{n-1} G_{s_i}$ or $\bigcap_{i=0}^{n} G_{s_i}$ is a maximal convex subgroup of $\bigcap_{i=0}^{n-1} G_{s_i}$.*

(ii) *The largest o-primitive component of (G, G) is periodically o-primitive.*

(iii) *For every positive integer $n \geq 1$, either*

$$\left(\bigcap_{i=0}^{n-1} G_{s_i}, \bigcap_{i=0}^{n-1} G_{s_i} \Big/ \bigcap_{i=0}^{n} G_{s_i} \right)$$

is the trivial permutation group or it is an o-2 transitive l-permutation group.

Proof. (ii) has been shown. We check (i) at the first stage. G_{s_0} has a countable number of fixed points in the $\bar{S}$. If s_1 is one of these fixed points, then $G_{s_0} = G_{s_0} \cap G_{s_1}$ and we are done. Therefore assume $G_{s_0} \cap G_{s_1} \subsetneqq G_{s_0}$. s_1 is contained in one of the proper periodic intervals of G_{s_0}, and G_{s_0}, when restricted to this interval, is on o-2 transitive l-permutation group. Since G_{s_0} is o-2 transitive on its proper periodic intervals, we may slightly modify the argument in Example 4 to show $G_{s_0} \cap G_{s_1}$ is maximal in G_{s_0}.

The rest of (i) and (iii) follows from the facts that the stabilizer subgroups are o-2 transitive on their periodic intervals, that o-2 transitive implies o-m transitive for every finite integer m, and by the construction of the o-primitive components.

Note that this theorem shows that any regular o-permutation group constructed from a periodically o-primitive l-permutation group has the same periodic group as its largest o-primitive component, and then, minimally, countably many o-2 transitive l-permutation groups as o-primitive components. We now give examples of the types of o-2 transitive l-permutation groups which will be encountered in this construction.

EXAMPLE 11. Let $+1$ denote that element of $\mathscr{A}(R)$ which is translation by the real number 1, and let $G = \{f \in \mathscr{A}(R) \mid f(+1) = (+1)f\}$. Then G is a periodically o-primitive l-permutation group [4]. Well order the set R by $r_0 \prec r_1 \prec r_2 \prec \cdots$ and right order G according to this well order. Then, the largest o-primitive component of (G, G) will be order-isomorphic to (G, S) and the remaining o-primitive components will be order-isomorphic to $(\mathscr{A}(R), R)$.

To construct an example where all the o-2 transitive components are not order-isomorphic, we modify Example 9. Use is made of a construction given by McCleary in [5]. Let S be a 1-set and let S_{11} be all points in $\bar{S} \backslash S$ which have character c_{11}. Then, there is a periodically o-primitive l-permutation group G of Config (1) having the proper periodic orbits of the stabilizers isomorphic to S_{11}. Call the totally ordered set G acts on T. Then, by using the techniques of Example 9, (G, T) will give the desired example.

In view of the examples just presented, it is clear that only knowing the possible o-primitive components of a regular o-permutation group is not sufficient for the understanding of the overall structure of such a group. Further knowledge can be derived by understanding how the o-primitive components fit together. The following theorem is a step in this direction.

THEOREM 12. *Suppose (G, S) and (H, T) are transitive o-permutation groups, and let $(W, R) = (G, S)Wr(H, T)$. If there exists a regular o-permutation group $(K, R) \subseteq (W, R)$ which induces (G, S) and (H, T) as components (these are not necessarily o-primitive), then*
   (i) *(G, S) is a regular o-permutation group.*
   (ii) *For each $t_0 \in T$, there is an epimorphism $*_{t_0} \colon G \to H_{t_0}$.*

Proof. The regularity of (G, S) follows easily from the regularity of (K, R).

For (ii), fix $t_0 \in T$. H_{t_0} is the stabilizer of t_0 in H. Define $K_{t_0} = \{(\phi; h) \in K \mid h \in H_{t_0}\}$. We claim for each g in G there exists a unique $h \in H_{t_0}$ such that there exists a $(\phi; h)$ in K_{t_0} with $\phi(t_0) = g$. The existence of such an h follows from the fact that (K, R) has com-

ponents (G, S) and (H, T).

For uniqueness, suppose there exists $h_1, h_2 \in H_{t_0}$ such that $(\phi_1; h_1)$, $(\phi_2; h_2) \in K_{t_0}$ where $\phi_1(t_0) = g = \phi_2(t_0)$. Then, for all points $(s, t_0) \in R$, $(s, t_0)(\phi_1; h_1) = (sg, t_0) = (s, t)(\phi_2, h_2)$. By the regularity of K, $(\phi_1; h_1) = (\phi_2; h_2)$ and so $h_1 = h_2$.

Define $*_{t_0}: G \to H_{t_0}$ by: for g in G, $g*_{t_0} = h$, where h is the unique element of H_{t_0} defined in the above claim. Then, by using the regularity of K, it is straightforward to check $*_{t_0}$ is an epimorphism.

In the examples which preceded this theorem, we examined the largest o-primitive components of serveral regular o-primitive o-permutation groups. From this theorem we see that if a regular o-permutation group is locally o-primitive, it is locally regular.

From (ii) it is now easy to see that there is no regular o-permutation group which is globally an o-2 transitive l-permutation group and locally order-isomorphic to a regular representation of an abelian ordered group (o-group).

If the regular o-permutation group (K, R) in the theorem has local component order-isomorphic to a regular representation of an abelian o-group, we can conclude the following about the o-permutation group (H, T).

COROLLARY 13. *Suppose* $(K, R) \subseteq (G, S) Wr (H, T)$ *and* (K, R) *satisfies the hypothesis of the previous paragraph and Theorem 12. Then, for each* $t \in T$, H_t *is regular on each of its orbits in* T.

Proof. By the theorem, for each $t_0 \in T$, H_{t_0} is a homomorphic image of G and is therefore abelian. Let tH_{t_0} denote an arbitrary orbit of H_{t_0} in T. Then (H_{t_0}, tH_{t_0}) is a transitive, abelian ordered permutation group and hence is regular.

We now briefly study a class of ordered permutation groups which is somewhat more general than the class of regular o-permutation groups which corresponds to the class of Conrad right-ordered groups.

We consider regular o-permutation groups which have only a finite number of o-primitive components. For ease of notation, assume that the o-permutation groups we consider have two o-primitive components, namely (G, S) aad (H, T).

Let $(W, R) = (G, S) Wr (H, T)$: (K, R) is a *large subaction of* (W, R) if (K, R) has global action (H, T), and for each fixed $t_0 \in T$ and $g \in G$, there exists $(\phi; e_H)$ in K such that $\phi(t_0) = g$. This is stronger, in general, than saying (K, R) has local action (G, S) and global action (H, T). Note that the small wreath product and the diagonal

wreath product are always large subactions of the usual wreath product.

THEOREM 14. *Suppose* $(G, \leqq)$ *is an ro-group and* $\{e_G\} \subsetneqq H \subsetneqq G$ *is the collection of convex subgroups of* G. *Then, then following are equivalent*:

(i) G *is an extension of the ro-group* H *by the ro-group* G/H.

(ii) H *is a normal, convex subgroup of* G.

(iii) (H, H), *the local component of* (G, G), *and* $(G/\bigcap_{g \in G} H^g, G/H)$, *the global component of* (G, G), *are both regular o-permutation groups*.

(iv) *If* $(W, G) = (H, H)Wr(G/\bigcap_{g \in G} H^g, G/H)$, *then* (G, G) *is a large subaction of* (W, G).

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear.

Next, (ii) $\Rightarrow$ (iv). Since H is normal in G, $\bigcap_{g \in G} H^g = H$, and so the global action of (G, G) is $(G/H, G/H)$, a regular o-permutation group. $(G/H, G/H)$ is regular so the point stabilizers of G/H in G/H are trivial. It is now clear that (G, G) has local action (H, H) and global action $(G/H, G/H)$ imply that (G, G) is a large subaction of (W, G).

The only implication left is (iv) $\Rightarrow$ (iii). Clearly (H, H) is regular. For ease of notation, let $(K, T) = (G/\bigcap_{g \in G} H^g, G/H)$, so (G, G) is a large subaction of $(H, H)Wr(K, T)$. Suppose there exist k in K and $t_0 \in T$ such that $t_0 k = t_0$. We show $k = e_K$. (G, G) has global action (K, T) so there exists $(\phi_1; k) \in G$. Suppose $\phi_1(t_0) = h_0$. (G, G) is large in $(H, H)Wr(K, T)$ so there exists $(\phi_2; e_K) \in G$ such that $\phi_2(t_0) = h_0$. Then, for all $(h, t_0) \in H \overset{\rightarrow}{\times} T$, $(h, t_0)(\phi_1; k) = (h; t_0)(\phi_2; e_K)$. By the regularity of G, $(\phi_1; k) = (\phi_2; e_K)$ and so $k = e_K$. Thus $(K, T) = (G/\bigcap_{g \in G} H^g, G/H)$ is a regular o-permutation group.

The reader may generalize the definition of a large subaction for an o-permutation group with a finite number of o-primitive components. In the case where (G, G) is a regular o-permutation group, the following conditions are equivalent: (i) $(G, \leqq)$ has a finite number of convex subgroups, each of which is normal in its cover; (ii) (G, G) has a finite number of o-primitive components, all of which are regular; (iii) (G, G) is a large subaction of the wreath product of its o-primitive components. Even in the case where (G, G) has an infinite number of o-primitive components, obviously (i) all convex subgroups of $(G, \leqq)$ are normal in their covers, is equivalent to (ii) each o-primitive component of (G, G) is regular.

We offer the following example to demonstrate that the conditions studied in Theorem 14 are not always satisfied by regular o-permu-

tation groups with a finite number of *o*-primitive components.

EXAMPLE 15. Let G be the subgroup of $\mathscr{A}(\mathbf{R})$ consisting of all lines of positive slope. $(G, \mathbf{R})$ is a sharply *o*-2 transitive ordered permutation group, i.e., if we fix two points of $\mathbf{R}$ by an element of G, we are forced to fix every point of $\mathbf{R}$.

Let r_0 be the real number 0, r_1 be the real number 1, and complete a well ordering of $\mathbf{R}$ in any way obtaining $0 \prec 1 \prec r_2 \prec r_3 \prec \cdots$. Right order G according to this well order, and let (G, G) denote the regular *o*-permutation group corresponding to $(G, \leqq)$.

Since G is sharply *o*-2 transitive, the entire tower of convex subgroups of $(G, \leqq)$ is $G \supseteqq G_0 \supseteqq \{e_G\}$. By the methods of Theorem 5, $(G, G/G_0)$, the global component of (G, G) is order-isomorphic to $(G, \mathbf{R})$, a nonregular *o*-permutation group. The local component of (G, G) is (G_0, G_0). We describe this group more explicitly. The function f in G are of the form $xf = ax + b$ where a is any fixed strictly positive real number and b is any fixed real number. If $f \in G_0$, then $b = 0$ in the above formula. Thus G_0 is the multiplicative group of positive real numbers, and (G_0, G_0) is a regular, abelian ordered permutation group. (G, G) is therefore a regular ordered permutation group which has only two *o*-primitive components, one of which is regular and one of which is not.

The ideas of Example 15 may be generalized to construct further examples of right-ordered groups which have not yet been studied, and which may prove useful in constructing counterexamples to various conjectures. We briefly outline this method.

Let G be any ro-group (written additively) and let $\Phi(G)$ denote the collection of all order-automorphisms of the ro-group G. $\mathscr{A}(G)$ denotes the collection of order preserving permutations of the ordered set G, and G will also denote the subgroup of $\mathscr{A}(G)$ which is translation by elements of G. Let $\langle \Phi(G), G \rangle$ denote the group generated by $\Phi(G)$ and G. $\langle \Phi(G), G \rangle$ is a subgroup of $\mathscr{A}(G)$. For any x in G and for any $f \in \langle \Phi(G), G \rangle$, $xf = x\psi + g$, where ψ is a fixed element of $\Phi(G)$ and g is a fixed element of G. (In Example 15, $\Phi(G)$ was the multiplicative group of positive reals, G was the additive group of reals, and $\langle \Phi(G), G \rangle$ was the group of lines of positive slope.) We may then well order G and right order $\langle \Phi(G), G \rangle$ according to this well order.

There are a number of questions related to this study which would be interesting to answer. We list some of them.

1. Classify all *o*-primitive, regular *o*-permutation groups. (The known examples are the right regular representations of subgroups of $\mathbf{R}$ and Smirnov's group.)

2. In Smirnov's example of an ro-group of G, the collection of elements whose powers are unbounded forms a subsemigroup of G. Is this a common property of all non-Archimedean ro-groups with no proper convex subgroups?

3. In Theorem 5, we constructed ro-groups from arbitrary o-primitive o-permutation groups. Is the tower of intersections of stabilizer subgroups always the complete tower of convex subgroups of the ro-group so constructed?

4. Investigate the structure of the ro-groups $\langle \Phi(G), G \rangle$ constructed after Example 15 for various right-ordered groups G.

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