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**ON THE REDUCTION OF CERTAIN DEGENERATE PRINCIPAL  
SERIES REPRESENTATIONS OF  $SP(n, C)$**

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# ON THE REDUCTION OF CERTAIN DEGENERATE PRINCIPAL SERIES REPRESENTATIONS OF $Sp(n, C)$

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This paper has its origins in the problem of proving irreducibility or reducibility for principal series representations of certain noncompact, complex, semi-simple groups by Fourier-analytic methods; for example, the abelian methods of Gelfand-Naimark for  $Sl(n, C)$ , and the non commutative (nilpotent) methods of K. Gross for  $Sp(n, C)$ . As is well-known, principal series representations are induced from unitary characters of a parabolic subgroup, the series being termed "nondegenerate" if the parabolic is minimal (i.e., the Borel subgroup) and otherwise "degenerate". Here we consider degenerate principal series for  $Sp(n, C)$  corresponding to maximal parabolic subgroups (more general than the situation studied by Gross) and reduce them with respect to the "opposite" parabolic. Let  $n_1$  denote the complex dimension of the isotropic subspace corresponding to the maximal parabolic, let  $0 < n_1 < n$ , and  $n_0 = n - n_1$ . The resulting reduction is described in terms of the natural representation of the complex orthogonal group  $O(n_1, C)$  acting on the space  $L^2(C^{n_1 \times n_0})$  and the tensor product of  $n_1$  copies of the oscillator representation of  $Sp(n_0, C)$ . In the terminology introduced by R. Howe, this harmonic analysis reduces to the theory of a "dual reductive pair", and any further resolution of the question of irreducibility by these methods will depend upon the study of the oscillator representations for such a dual reductive pair.

We now describe our work in more detail. As a presentation of the complex symplectic group, take

$$\Sigma_n = \{g \in C^{2n \times 2n} : gM_n g' = M_n\},$$

where  $M_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$ ,  $I_n$  is the  $n \times n$  identity matrix, and  $g'$  denotes the transpose of  $g$ . Specify a complete set of conjugacy class representatives of the maximal parabolic subgroups  $H$  in  $\Sigma_n$  (c.f., [9], §8) by defining  $H = Z'SA$ , where the subgroups  $Z$ ,  $S$ , and  $A$  are given below. Let the isotropic subspace of  $C^{2n}$  corresponding to  $H$  have dimension  $n_1$ , with  $0 < n_1 \leq n$  and  $n_0 = n - n_1$ . Then the blocking scheme used in defining  $Z$ ,  $S$ , and  $A$  has diagonal blocks of dimensions  $n_1 \times n_1$ ,  $n_0 \times n_0$ ,  $n_1 \times n_1$ , and  $n_0 \times n_0$  from upper left to lower right.

$$\begin{aligned}
 Z &= \left\{ \begin{pmatrix} I & 0 & 0 & 0 \\ -y' & I & 0 & 0 \\ \eta & x & I & y \\ x' & 0 & 0 & I \end{pmatrix} : \eta - \eta' = yx' - xy' \right\} \\
 S &= \left\{ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & s_{11} & 0 & s_{12} \\ 0 & 0 & I & 0 \\ 0 & s_{21} & 0 & s_{22} \end{pmatrix} : s = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \in \Sigma_{n_0} \right\} \\
 A &= \left\{ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \alpha'^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} : \alpha \in \text{Gl}(n_1, C) \right\}
 \end{aligned}$$

Clearly,  $S$  is isomorphic to  $\text{Sp}(n_0, C)$ ,  $A$  is isomorphic to  $\text{Gl}(n_1, C)$  and elements of  $S$  commute with elements of  $A$ . Also, it is easily shown that  $Z$  and  $Z'$  are normalized by  $SA$  and hence,  $ZSA$  and  $Z'SA$  are semidirect products.

The maximal parabolic subgroup  $H = Z'SA$  gives rise to a degenerate principal series of representations  $T_\chi$  of  $\Sigma_n$  induced from unitary characters  $\chi$  on  $H$ . We shall realize  $T_\chi$  in the Hilbert space  $L^2(Z)$  as follows: Let  $dz$  denote Haar measure on the unimodular group  $Z$ . Denote by  $d_l h$  and  $d_r h$ , respectively, fixed left and right Haar measures on  $H$ , and let  $\delta_H$  be the modular function defined by  $\delta_H(h) = d_l h / d_r h$ . By direct calculation (cf., [3], §6),  $HZ$  is an open subset of  $\Sigma_n$  whose complement is a set of Haar measure zero. Thus, we can extend the positive character  $\delta_H$  and any unitary character  $\chi$  on  $H$  to functions defined almost everywhere on  $\Sigma_n$  by defining  $\delta_H(hz) = \delta_H(h)$  and  $\chi(hz) = \chi(h)$  for any  $hz \in HZ$ . Also, each right coset of  $H$  in  $\Sigma_n$ , except for a set of cosets whose union is a null set, contains a unique element of  $Z$ . It follows that the canonical action of  $\Sigma_n$  on the right coset space  $H \backslash \Sigma_n$  gives rise to an "action" of  $\Sigma_n$  on  $Z$ : for any  $g \in \Sigma_n$  and  $z \in Z$ , let  $z\bar{g}$  be the unique element of  $Z$  such that  $H(z\bar{g}) = Hzg$ , provided that such an element exists. To be specific, denote by  $Z^g$  the subset of  $Z$  such that  $z\bar{g}$  exists, then  $Z^g$  is an open subset of  $Z$  whose complement is a null set. Therefore, if  $f \in L^2(Z)$  then the function  $z \rightarrow f(z\bar{g})$ , for fixed  $g \in \Sigma_n$ , is defined almost everywhere in  $Z$ . Now, the formula ([2], §30) defining the (continuous) unitary representations  $T_\chi$  of  $\Sigma_n$ , which form a (degenerate) principal series is

$$(1.1) \quad T_\chi(g)f(z) = \delta_H(zg)^{-1/2} \chi(zg) f(z\bar{g}) \quad (g \in \Sigma_n, f \in L^2(Z)).$$

Let us briefly explain the Fourier-analytic reduction of the restriction of  $T_\chi$  to the (opposite) parabolic subgroup  $ZSA$ . Fix  $n_1$  with  $1 \leq n_1 < n$ , fix  $\chi$ , and let  $T = T_\chi$ . We observe that the restriction  $T|_Z$  of  $T$  to  $Z$  is just the right regular representation of  $Z$ . Thus, it is natural to replace  $T$  with the unitarily equivalent representation  $\hat{T} = \mathcal{P}T\mathcal{P}^{-1}$  where  $\mathcal{P}$  is the Plancherel transform of  $L^2(Z)$ , for  $\hat{T}|_Z$  decomposes as a direct integral. The operator  $\mathcal{P}$  maps  $L^2(Z)$  unitarily onto the Hilbert space  $L^2(A, X, dm(\lambda))$ , of  $X$ -valued, square-integrable functions on  $A$ . Here  $A$  is the dual object of  $Z$ ,  $dm(\lambda)$  is the Plancherel measure on  $A$ , and  $X = HS(L^2(C^{n_1 \times n_0}))$  is the Hilbert space of Hilbert-Schmidt operators on  $L^2(C^{n_1 \times n_0})$ . It is also the case that  $\hat{T}|_S$  decomposes as a direct integral, and one can explicitly analyze  $\hat{T}(z\alpha)$  for all  $z\alpha \in ZSA$ . The operators of  $\hat{T}|_{ZSA}$  involve a representation  $\tilde{I}: S \rightarrow \mathcal{Z}(L^2(C^{n_1 \times n_0}))$  which is the tensor product of  $n_1$  copies of the oscillator representation of  $\text{Sp}(n_0, C)$ , as well as a representation  $D: A \rightarrow \mathcal{Z}(L^2(C^{n_1 \times n_0}))$ , in which  $\text{Gl}(n_1, C)$  acts on  $L^2(C^{n_1 \times n_0})$  by generalized dilations. The above results are contained in §2 of this paper.

Let  $\mathcal{A}'(\hat{T}|_{ZSA})$  denote the commuting algebra of  $\hat{T}|_{ZSA}$ . There are sufficiently many operators of  $\hat{T}|_{ZSA}$  which are diagonalizable to force  $\mathcal{A}'(\hat{T}|_{ZSA})$  to be decomposable. Moreover, the components of  $\mathcal{A}'(\hat{T}|_{ZSA})$  are essentially copies of the intersection,  $\mathcal{A}'(\tilde{I}) \cap \mathcal{A}'(D|_{A_1})$  of the commuting algebras of  $\tilde{I}$  and  $D|_{A_1}$  where  $A_1 \cong O(n_1, C)$ . That is, there is an isometric isomorphism of von Neumann algebras, which we exhibit, between  $\mathcal{A}'(\hat{T}|_{ZSA})$  and  $\mathcal{A}'(\tilde{I}) \cap \mathcal{A}'(D|_{A_1})$ . This is the content of §3.

It should be noted that there are two special cases in which complete results are known. The case  $n_1 = n$  is special since  $Z$  is abelian. It is not difficult to show that  $\mathcal{A}'(\hat{T}|_{ZSA})$  is one-dimensional and, hence, for all  $\chi$ ,  $T_\chi$  is already irreducible upon restriction to  $ZSA$ . Also, the irreducibility problem has been completely solved in the case  $n_1 = 1$ ,  $n_0 = n - 1$ , in [3], which may be regarded as the prototype for the general case. There it is proved that  $T_\chi$  is irreducible unless  $\chi$  is the trivial character on  $H$ , in which case,  $T_\chi$  splits into the sum of two irreducible representations of  $\Sigma_n$ . The complete results of [3] rest on the fact that the commuting algebra  $\mathcal{A}'(\tilde{I}) \cap \mathcal{A}'(D|_{A_1})$  is just 2-dimensional when  $n_1 = 1$ . In the general case, this algebra is infinite dimensional and the full analysis of  $\hat{T}$  on all of  $\Sigma_n$  depends upon its explicit description.<sup>1</sup>

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<sup>1</sup> R. Howe's results show that the joint representation of  $\text{Sp}(n_0, C) \times O(n_1, C)$  decomposes continuously. It follows that the commuting algebra is infinite dimensional.

2. The operators  $\hat{T}(z\alpha)$ . In order to analyze  $\hat{T}|_{ZSA}$  we need to introduce the dual object of  $Z$ , the resulting Plancherel transform of  $L^2(Z)$  and the oscillator representation of  $\text{Sp}(n_0, C)$ .

Procedures of Kirillov [5] can be applied to the simply connected, nilpotent lie group  $Z$  to yield the dual object — the set of equivalence classes of irreducible, unitary representations of  $Z$ . The results are given below.

Denote the elements of  $Z$  by  $(x, y, t)$ , where  $t = \eta - yx'$  so that  $t$  is symmetric. In this way  $Z$  is identified with  $V \times V \times A_0$ , where  $V = C^{n_1 \times n_0}$  and  $A_0 = \{t \in C^{n_1 \times n_1} : t = t'\}$ . Multiplication in  $Z$  is now given by

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 - x_1y_2' - y_2x_1').$$

Also, the center of  $Z$  is easily seen to be  $\{0\} \times \{0\} \times A_0$  and the Haar measure of the unimodular group  $Z$  is real Lebesgue measure  $dz = dx \, dy \, dt$  on the Euclidean space  $V \times V \times A_0$ .

For  $\lambda \in A_0$  with  $\text{rank } \lambda = r$ , let  $C(\lambda) \in O(n_1, C)$  be such that  $C(\lambda)\lambda C(\lambda)' = \begin{vmatrix} \lambda_r & 0 \\ 0 & 0 \end{vmatrix}$ , where  $\lambda_r$  is a symmetric, invertible  $r \times r$  matrix.

Also, let  $X = \begin{vmatrix} I_r & 0 \\ 0 & 0 \end{vmatrix} C^{n_1 \times n_0}$ , thought of as a measure space with real Lebesgue measure. Finally, throughout this paper we shall let  $(u|v) = \text{Re tr } uv'$  for all  $u, v \in C^{p \times q}$ .

**THEOREM 2.1.** *Every irreducible, unitary representation of  $Z$  is unitarily equivalent to  $\Pi_{(\alpha, \beta, \lambda)}$  for some choice of  $\alpha, \beta \in V$  and  $\lambda \in A_0$ , where  $\Pi_{(\alpha, \beta, \lambda)}$  is defined as follows:*

- (1) *If  $\lambda = 0$  then  $\Pi_{(\alpha, \beta, \lambda)}$  is 1-dimensional and is given by*

$$\Pi_{(\alpha, \beta, 0)}(x, y, t) = \exp 2\pi i[(\alpha|x) + (\beta|y)]$$

for  $(x, y, t) \in Z$ .

- (2) *If  $\text{rank } \lambda = r \neq 0$ , then  $\Pi_{(\alpha, \beta, \lambda)} : Z \rightarrow \mathcal{Z}(L^2(X))$  is  $\infty$ -dimensional and is given by*

$$\begin{aligned} \Pi_{(\alpha, \beta, \lambda)}(x, y, t)f(u) = \exp 2\pi i \left[ (\alpha|C(\lambda)') \begin{vmatrix} 0_r & 0 \\ 0 & I \end{vmatrix} C(\lambda)x \right. \\ \left. + (\beta|y) + (\lambda|t - 2C(\lambda)'uy') \right] f\left(u + \begin{vmatrix} I_r & 0 \\ 0 & 0 \end{vmatrix} C(\lambda)x\right) \end{aligned}$$

for  $f \in L^2(X)$  and  $(x, y, t) \in Z$ . Moreover  $\Pi_{(\alpha_1, \beta_1, \lambda_1)}$  and  $\Pi_{(\alpha_2, \beta_2, \lambda_2)}$  are unitarily equivalent if and only if  $\lambda_1 = \lambda_2$ ,  $\alpha_1 = \alpha_2 + \lambda_1 a$ , and  $\beta_1 = \beta_2 + \lambda_1 b$  for some  $a, b \in V$ .

The Plancherel transform of  $L^2(Z)$  does not require the entire dual object of  $Z$ , but only the representations corresponding to

“maximal orbits”. These are the representation  $\Pi_{(0,0,\lambda)}$ , where  $\lambda$  is invertible (i.e.,  $r = n_1$ ). Thus, let  $A = \{\lambda \in A_0; \text{rank } \lambda = n_1\}$  and for  $\lambda \in A$  denote  $\Pi_{(0,0,\lambda)}$  by  $\hat{\lambda}$ , then  $\hat{\lambda}$  acts in the Hilbert space  $L^2(V)$  by the formula

$$(2.2) \quad \hat{\lambda}(x, y, t)f(u) = \exp [2\pi i(\lambda|t - 2uy')]f(u + x)$$

for  $f \in L^2(V)$  and  $z = (x, y, t) \in Z$ .

Let  $\mathcal{S} = \mathcal{S}(V) \times \mathcal{S}(V) \times \mathcal{S}(A_0)$ , where  $\mathcal{S}(V)$  (respectively  $\mathcal{S}(A_0)$ ) is the vector space of all infinitely differentiable, rapidly decreasing functions with domain  $V$  (respectively  $A_0$ ). Then  $\mathcal{S}$  is a dense subspace of  $L^2(Z)$  with which we can state and prove the following results concerning the Plancherel transform  $\mathcal{P}$  of  $L^2(Z)$ .

**THEOREM 2.3.** (1) *The Plancherel measure  $m$  on  $A$  is given by*

$$dm(\lambda) = 2^{2n_0n_1+n_1(n_1-1)} \cdot |\det \lambda|^{2n_0} d\lambda$$

where  $d\lambda$  is the restriction to  $A$  of the Lebesgue measure on  $A_0$ . (2) *The mapping  $f \rightarrow K_f$  defined for  $f \in \mathcal{S}$  by*

$$K_f(x, y, \lambda) = \int_V \int_{A_0} f(x - y, v, t) \exp [2\pi i(\lambda|t - 2vy')] dt dv$$

*extends uniquely to a linear isometry of  $L^2(Z)$  onto  $L^2(V \times V \times A_0, dx dy dm(\lambda))$ . This is the function-valued Plancherel transform. (3) The mapping  $f \rightarrow \hat{f}$  defined weakly for  $f \in \mathcal{S}$  by*

$$\hat{f}(\lambda) = \int_Z f(z) \hat{\lambda}(z) dz$$

*extends uniquely to an isometry  $\mathcal{P}$  of  $L^2(Z)$  onto  $L^2(A, HS(L^2(V))), dm(\lambda)$ . This is the Plancherel transform of  $L^2(Z)$ .*

*Proof.* A computation shows that  $\hat{f}(\lambda)$  is an integral operator with kernel  $K_f$ . The mapping  $f \rightarrow K_f$  is decomposed as in [3] (1.8) into ordinary, partial Fourier transforms

$$\mathcal{F}_2 f(x, y, \lambda) = \int_V f(x, v, \lambda) \exp [-2\pi i(y|v)] dv$$

$$\mathcal{F}_3 f(x, y, \lambda) = 2^{-n_1(n_1-1)/2} \int_{A_0} f(x, y, t) \exp [-2\pi i(\lambda|t)] dt$$

( $f \in \mathcal{S}$ , the factor  $c = 2^{-n_1(n_1-1)/2}$  makes  $\mathcal{F}_3$  an isometry) and a transformation  $\mathcal{B}: L^2(V \times V \times A_0, dx dy d\lambda) \rightarrow L^2(V \times V \times A_0, dx dy dm(\lambda))$  given by

$$\mathcal{B}f(x, y, \lambda) = cf(x - y, 2\lambda y, \lambda).$$

In fact,  $K_f = \mathcal{R}\mathcal{F}_2\mathcal{F}_3^{-1}f$  for all  $f \in \mathcal{S}$ . Now  $dm(\lambda)$  is chosen so that  $\mathcal{R}$  is an isometry and hence  $f \rightarrow K_f$  is an isometry.

To prove (3) and (2), one shows that the mapping  $K_f \rightarrow \hat{f}$ , defined for  $f \in \mathcal{S}$ , extends to a linear isometry of  $L^2(V \times V \times \Lambda_0, dx dy dm(\lambda))$  onto  $L^2(A, HS(L^2(V)), dm(\lambda))$ .

The tensor product of  $n_1$  copies of the oscillator representation of  $S$  occurs naturally in the present setting. Note that  $S$  normalizes  $Z$ . Specifically, for  $s \in S$  and  $z = (x, y, t) \in Z$

$$szs^{-1} = (xs'_{22} - ys'_{21}, ys'_{11} - xs'_{12}, t + xs'_{22}s_{12}x' + ys'_{21}s_{11}y' - xs'_{12}s_{21}y' - ys'_{21}s_{12}x').$$

Let  $\lambda \in A$  and fix  $s \in S$ . The mapping  $z \rightarrow \hat{\lambda}(szs^{-1})$  is an irreducible, unitary representation of  $Z$  acting in  $L^2(V)$  which agrees with  $\hat{\lambda}$  on the center of  $Z$ . Thus, these two irreducible representations are unitarily equivalent, and so, there is a unitary operator  $\tilde{\lambda}(s)$  on  $L^2(V)$  such that

$$\hat{\lambda}(szs^{-1}) = \tilde{\lambda}(s)\hat{\lambda}(z)\tilde{\lambda}(s)^{-1} \quad (z \in Z).$$

For each  $s$ ,  $\tilde{\lambda}(s)$  is unique up to scalar multiples of absolute value 1, and, as we will show,  $\tilde{\lambda}(s)$  can be normalized so that  $s \rightarrow \tilde{\lambda}(s)$  is a unitary representation of  $S$  acting in  $L^2(V)$ . Let us now be more explicit.

Identify  $S$  with  $\Sigma_{n_0}$  and define the following subgroups of  $\Sigma_{n_0}$  using the blocking scheme with two diagonal blocks of size  $n_0 \times n_0$ :

$$M = \left\{ m(b) = \begin{vmatrix} I & b \\ 0 & I \end{vmatrix} : b = b' \right\}$$

$$L = \left\{ l(a) = \begin{vmatrix} a & 0 \\ 0 & a^\vee \end{vmatrix} : a \in \text{Gl}(n_0, \mathbb{C}), a^\vee = (a')^{-1} \right\}.$$

Also, let  $p = \begin{vmatrix} 0 & -I \\ I & 0 \end{vmatrix}$ . The set  $L \cup M \cup \{p\}$  generates  $\Sigma_{n_0}$  ([3], p. 404), so to define  $\tilde{\lambda}$  on  $S$  it is enough to define it on this generating set.

**DEFINITION 2.4.** Given  $\lambda \in A$ , define  $\tilde{\lambda}: L \cup M \cup \{p\} \rightarrow \mathcal{U}(L^2(V))$  by

$$\tilde{\lambda}(l(a))f(u) = |\det a|^{n_1} f(ua) \quad (l(a) \in L)$$

$$\tilde{\lambda}(m(b))f(u) = \exp[-2\pi i(\lambda u | ub)]f(u) \quad (m(b) \in M)$$

$$\tilde{\lambda}(p)f(u) = \gamma(\lambda)|\det 2\lambda|^{n_0} Uf(2\lambda u)$$

where  $U$  is the Fourier transform of  $L^2(V)$  defined for  $f \in L^1(V)$  by

$$Uf(u) = \int_V f(v) \exp [2\pi i(u|v)]dv$$

and  $\gamma(\lambda)$  is a complex number with modulus 1, which will be determined in the proof of Theorem 2.5.

**THEOREM 2.5.** *The mapping  $\tilde{\lambda}$ , defined on  $L \cup M \cup \{p\}$  above, extends uniquely to be a continuous unitary representation of  $\Sigma_{n_0}$  (and hence  $S$ ) acting in  $L^2(V)$  which satisfies*

$$\hat{\lambda}(szs^{-1}) = \tilde{\lambda}(s)\hat{\lambda}(z)\tilde{\lambda}(s)^{-1}$$

for all  $s \in S$  and  $z \in Z$ .

*Proof.* It is easy to verify that the restrictions of  $\tilde{\lambda}$  to  $L$ ,  $M$ , and  $LM$  are continuous unitary representations of these groups. Now apply Lemma 1 of [3]. To prove that condition (2) of Lemma 1 is satisfied, we use the following: Observe that  $m(I)pm(I) = pm(-I)p$ . Let  $m = m(I)$ , then  $m^{-1} = m(-I)$ . From the definitions of the operators  $\tilde{\lambda}(m)$  and  $\tilde{\lambda}(p)$ , a computation shows that

$$\begin{aligned} &\tilde{\lambda}(m)\tilde{\lambda}(p)\tilde{\lambda}(m)\hat{\lambda}(z)\tilde{\lambda}(m)^{-1}\tilde{\lambda}(p)^{-1}\tilde{\lambda}(m)^{-1} \\ &= \hat{\lambda}(mpmzm^{-1}p^{-1}m^{-1}) \\ &= \hat{\lambda}(pm^{-1}pzp^{-1}mp^{-1}) \\ &= \tilde{\lambda}(p)\tilde{\lambda}(m)^{-1}\tilde{\lambda}(p)\hat{\lambda}(z)\tilde{\lambda}(p)^{-1}\tilde{\lambda}(m)\tilde{\lambda}(p)^{-1} \end{aligned}$$

and hence  $Y_1(\lambda) = \tilde{\lambda}(p)^{-1}\tilde{\lambda}(m)\tilde{\lambda}(p)^{-1}\tilde{\lambda}(m)\tilde{\lambda}(p)\tilde{\lambda}(m) \in \mathcal{A}'(\hat{\lambda})$ , the commut- ing algebra of  $\hat{\lambda}$ . This is true regardless of the value of  $\gamma(\lambda)$  with  $|\gamma(\lambda)| = 1$ . Thus, letting

$$Y_2(\lambda) = [\gamma(\lambda)^{-1}\tilde{\lambda}(p)]^{-1}\tilde{\lambda}(m)[\gamma(\lambda)^{-1}\tilde{\lambda}(p)]^{-1}\tilde{\lambda}(m)[\gamma(\lambda)^{-1}\tilde{\lambda}(p)]\tilde{\lambda}(m),$$

we have  $Y_2(\lambda) = \gamma(\lambda)Y_1(\lambda) \in \mathcal{A}'(\hat{\lambda})$ . But  $\hat{\lambda}$  is irreducible so  $\mathcal{A}'(\hat{\lambda})$  is 1-dimensional and hence  $Y_2(\lambda) = c(\lambda)I$  for some unique  $c(\lambda) \in \mathbb{C}$  with  $|c(\lambda)| = 1$ . Define  $\gamma(\lambda) = c(\lambda)$  then  $Y_1(\lambda) = I$  and it follows that

$$\tilde{\lambda}(m)\tilde{\lambda}(p)\tilde{\lambda}(m) = \tilde{\lambda}(p)\tilde{\lambda}(m^{-1})\tilde{\lambda}(p)$$

which is condition (2) of Lemma 1 of [3].

Just as the representation  $\tilde{\lambda}$  of  $S$  arises from intertwining operators  $\tilde{\lambda}(s)$  between  $\hat{\lambda}$  and  $z \rightarrow \hat{\lambda}(szs^{-1})$ , a representation  $D$  of  $A$  arises from intertwining operators  $D(\alpha)$  between  $(\alpha\lambda\alpha')^\wedge$  and  $z \rightarrow \hat{\lambda}(\alpha^{-1}z\alpha)$ . For  $\alpha \in A$  and  $z = (x, y, t) \in Z$ , we have

$$\alpha^{-1}z\alpha = (\alpha'x, \alpha'y, \alpha't\alpha),$$

and from formula (2.2), the representations  $(\alpha\lambda\alpha')^\wedge$  and  $z \rightarrow \hat{\lambda}(\alpha^{-1}z\alpha)$



are easily seen to agree on the center of  $Z$ . Hence, these two irreducible representations of  $Z$  are unitarily equivalent and, in fact, the operator  $D(\alpha)$  given by

$$D(\alpha)f(u) = |\det \alpha|^{n_0} f(\alpha'u) \quad (f \in L^2(V))$$

intertwines them. Also, the fact that

$$\begin{aligned} \tilde{\chi}(s)D(\alpha)^{-1}(\alpha\lambda\alpha')^\wedge(z)D(\alpha)\tilde{\chi}(s)^{-1} \\ = \hat{\lambda}(s\alpha^{-1}z\alpha s^{-1}) = \hat{\lambda}(\alpha^{-1}zs s^{-1}\alpha) \\ = D(\alpha)^{-1}(\alpha\lambda\alpha')^\wedge(s)(\alpha\lambda\alpha')^\wedge(z)(\alpha\lambda\alpha')^\wedge(s)^{-1}D(\alpha) \end{aligned}$$

suggests that  $D(\alpha)$  may also intertwine  $\tilde{\chi}$  and  $(\alpha\lambda\alpha')^\wedge$  and, indeed, this is the case. We summarize these facts in the next theorem.

**THEOREM 2.6.** *The mapping  $D: A \rightarrow \mathcal{Z}(L^2(V))$  is a continuous unitary representation of  $A$  which satisfies*

- (1)  $D(\alpha)^{-1}(\alpha\lambda\alpha')^\wedge(z)D(\alpha) = \hat{\lambda}(\alpha^{-1}z\alpha)$
- (2)  $D(\alpha)\tilde{\chi}(s)D(\alpha)^{-1} = (\alpha\lambda\alpha')^\wedge(s)$

for all  $\lambda \in A$ ,  $\alpha \in A$ ,  $s \in S$ , and  $z \in Z$ .

We omit the proof of the above theorem since it is fairly straightforward (cf., [3], Theorem 2), however, we make the following observation related to the proof. For each invertible symmetric matrix  $\lambda$  there exists  $\beta \in \text{Gl}(n_1, C)$  such that  $\lambda = \beta\beta'$ . Consequently, the action of  $A$  on  $A$  defined by  $\alpha \cdot \lambda = \alpha\lambda\alpha'$  is transitive, and from this follow two important facts: First, the function  $\gamma$  defined on  $A$  in the proof of Theorem 2.5 is *constant*. Secondly, we have the

**COROLLARY.** *Let  $\lambda \in A$  and let  $I \in A$  denote the  $n_1 \times n_1$  identity matrix, then  $\tilde{\chi}$  is unitarily equivalent to  $\tilde{I}$ .*

View  $L^2(V)$  as the tensor product  $\bigotimes^{n_1} L^2(C^{n_0})$  by defining

$$f_1 \otimes f_2 \otimes \cdots \otimes f_{n_1}(u) = f_1(u_1)f_2(u_2) \cdots f_{n_1}(u_{n_1}),$$

where  $u_i$  is the  $i$ th row of  $u$  and  $f_i \in L^2(C^{n_0})$ . By inspection of  $\tilde{I}$  and comparison with Theorem 2 of [3], one sees that  $\tilde{I}$  is a tensor product of  $n_1$  copies of the oscillator representation  $\tilde{I}$  of [3].

We may now compute the operators  $\hat{T}(z_0s\alpha)$  for  $z_0s\alpha \in ZSA$ . Recall that the formula (1.1) for  $T(g)$  involves the action of  $g$  on  $Z$  given by  $H(z\bar{g}) = Hz\bar{g}$ . If  $g = z_0s\alpha$  then the action becomes  $zz_0s\alpha = \alpha^{-1}s^{-1}zz_0s\alpha$ . Also,  $\delta_H(zg) = \delta_H(zz_0s\alpha) = \delta_H((s\alpha)(\alpha^{-1}s^{-1}zz_0s\alpha)) = \delta_H(s\alpha)$ , since  $\delta_H(hz)$  is defined to be  $\delta_H(h)$  for any  $hz \in HZ$ . Furthermore,  $\delta_H(s\alpha) = \delta_H(\alpha)$  because  $S$  has no nontrivial characters. Similarly,  $\chi(zz_0s\alpha) = \chi(\alpha)$  for any unitary character  $\chi$  on  $H$ . Thus, (1.1) becomes

$$(2.7) \quad T(z_0 s \alpha) f(z) = \delta_H(\alpha)^{-1/2} \chi(\alpha) f(\alpha^{-1} s^{-1} z z_0 s \alpha)$$

for  $f \in L^2(Z)$ . Now let  $f \in L^1(Z) \cap L^2(Z)$  and  $\hat{f} = \mathcal{P}f$ , then  $\hat{f} \in \mathcal{H} = L^2(A, HS(L^2(V)), dm(\lambda))$  and  $\mathcal{P}(L^1(Z) \cap L^2(Z))$  forms a dense subspace of  $\mathcal{H}$ . The next theorem determines the transformed representation  $\hat{T}|_{zSA}$ .

**THEOREM 2.8.** For  $f \in L^1(Z) \cap L^2(Z)$  and  $z s \alpha \in ZSA$ ,

$$\hat{T}(z s \alpha) \hat{f}(\lambda) = \delta_H(\alpha)^{1/2} \chi(\alpha) \tilde{\lambda}(s) D(\alpha) \hat{f}(\alpha^{-1} \cdot \lambda) D(\alpha)^{-1} \tilde{\lambda}(s)^{-1} \hat{\lambda}(z)^{-1}$$

for almost every  $\lambda \in A$ .

*Proof.* For every  $\lambda \in A$ ,  $\hat{T}(z_0 s \alpha) \hat{f}(\lambda) = \mathcal{P} T(z_0 s \alpha) \mathcal{P}^{-1} \mathcal{P} f(\lambda) = \mathcal{P} T(z_0 s \alpha) f(\lambda) = \int_Z T(z_0 s \alpha) f(z) \hat{\lambda}(z) dz = \int_Z \delta_H(\alpha)^{-1/2} \chi(\alpha) f(\alpha^{-1} s^{-1} z z_0 s \alpha) \hat{\lambda}(z) dz$

$$(1) = \delta_H(\alpha)^{-1/2} \chi(\alpha) \int_Z f(z) \hat{\lambda}(s \alpha z \alpha^{-1} s^{-1} z_0^{-1}) d(s \alpha z \alpha^{-1} s^{-1} z_0^{-1})$$

$$(2) = \delta_H(\alpha)^{-1/2} \chi(\alpha) \int_Z f(z) \hat{\lambda}(s \alpha z \alpha^{-1} s^{-1} z_0^{-1}) \delta_H(\alpha) dz$$

$$(3) = \delta_H(\alpha)^{1/2} \chi(\alpha) \left[ \int_Z f(z) \tilde{\lambda}(s) \hat{\lambda}(\alpha z \alpha^{-1}) \tilde{\lambda}(s)^{-1} dz \right] \hat{\lambda}(z_0)^{-1}$$

$$(4) = \delta_H(\alpha)^{1/2} \chi(\alpha) \tilde{\lambda}(s) \left[ \int_Z f(z) D(\alpha) (\alpha^{-1} \cdot \lambda) \hat{\lambda}(z) D(\alpha)^{-1} dz \right] \tilde{\lambda}(s)^{-1} \hat{\lambda}(z_0)^{-1}$$

$$= \delta_H(\alpha)^{1/2} \chi(\alpha) \tilde{\lambda}(s) D(\alpha) \left[ \int_Z f(z) (\alpha^{-1} \cdot \lambda) \hat{\lambda}(z) dz \right] D(\alpha)^{-1} \tilde{\lambda}(s)^{-1} \hat{\lambda}(z_0)^{-1},$$

which gives the theorem. Equation (1) is a change of variables, (2) is the fact  $d(s \alpha z \alpha^{-1} s^{-1}) = \delta_H(\alpha) dz$  (cf., [6], II. 7), (3) is an application of Theorem 2.5, and (4) is from Theorem 2.6 (1). The formula in the Theorem is said to hold “almost everywhere” since there may be a null set in  $A$  where the right-hand-side is not in  $HS(L^2(V))$ .

**3. The commuting algebra of  $\hat{T}|_{zSA}$ .** We seek necessary and sufficient conditions for  $B \in \mathcal{L}(\mathcal{H})$ , the bounded linear operators on the Hilbert space  $\mathcal{H} = L^2(A, HS(L^2(V)), dm(\lambda))$ , to be in the commuting algebra  $\mathcal{A}'(\hat{T}|_{zSA})$ . Suppose  $B \in \mathcal{A}'(\hat{T}|_{zSA})$ . Then, in particular,  $B$  commutes with  $\hat{T}(z)$  and  $\hat{T}(s)$  for all  $z \in Z$  and  $s \in S$ . We will first see what conditions on  $B$  these facts impose. Then we will obtain additional conditions from the fact that  $B$  commutes with  $\hat{T}(\alpha)$  for  $\alpha \in A$ .

Realize  $HS(L^2(V))$  as  $L^2(V) \bar{\otimes} L^2(V)$ . From the observation that  $\mathcal{H} = L^2(A, L^2(V) \bar{\otimes} L^2(V), dm(\lambda)) = \int_A L^2(V) \bar{\otimes} L^2(V) dm(\lambda)$  is a direct integral of Hilbert spaces, we have the notions of decomposable and diagonalizable operators ([7], I. 3). From Theorem 2.8, it is clear that  $\hat{T}(z)$  and  $\hat{T}(s)$  are decomposable operators for  $z \in Z$  and  $s \in S$  and can be denoted:

$$\hat{T}(z) = \int_A I \otimes \hat{\lambda}(z) dm(\lambda)$$

$$\hat{T}(s) = \int_A \tilde{\lambda}(s) \otimes \tilde{\lambda}(s) dm(\lambda).$$

To show from this that  $B$  is decomposable requires the following technical lemma of Stone-Weierstrass type.

Let  $X$  be a locally compact,  $\sigma$ -compact, Hausdorff space with positive Borel measure  $\mu$ , which is finite on compact sets. Let  $H$  be a separable Hilbert space. For  $a \in L^\infty(X, \mu)$ , let  $M(a)$  denote the diagonalizable operator on  $L^2(X, H, \mu)$  given by  $M(a)f(x) = a(x)f(x)$ .

LEMMA 3.1. *Let  $\mathcal{A}$  be a subalgebra of  $C(X) \cap L^\infty(X, \mu)$  over  $C$  such that*

- (a)  $1 \in \mathcal{A}$ ,
- (b)  $a \in \mathcal{A}$  implies  $\bar{a} \in \mathcal{A}$ ,
- (c)  $\mathcal{A}$  separates points of  $X$ .

If  $B \in \mathcal{L}(L^2(X, H, \mu))$  and  $BM(a) = M(a)B$  for all  $a \in \mathcal{A}$ , then  $BM(a) = M(a)B$  for all  $a \in L^\infty(X, \mu)$ .

THEOREM 3.2. *Suppose  $B \in \mathcal{A}'(\hat{T}|_{ZSA})$ . Then*

- (a)  $B$  is decomposable.
- (b) *There exists a mapping  $\lambda \rightarrow B(\lambda)$  of  $A$  into  $\mathcal{L}(L^2(V))$ , defined a.e.  $[m]$ , such that  $B = \int_A B(\lambda) \otimes Idm(\lambda)$ .*
- (c)  $B(\lambda) \in \mathcal{A}'(\tilde{\lambda})$  for almost every  $\lambda \in A$ .

*Proof.* (a) Let  $\mathcal{A} = \{a \rightarrow \exp [2\pi i(\lambda|t)]: t \in A_0\}$ .  $\mathcal{A}$  is a subalgebra over  $C$  of  $C(A) \cap L^\infty(A, m)$ , which satisfies the conditions of Lemma 3.1. Furthermore, if  $a \in \mathcal{A}$  then the associated diagonalizable operator  $M(a) \in \mathcal{L}(\mathcal{H})$ , given by  $M(a)\hat{f}(\lambda) = a(\lambda)\hat{f}(\lambda)$ , is an operator of the representation  $\hat{T}|_Z$ . In fact, if  $a(\lambda) = \exp [2\pi i(\lambda|t)]$  then  $M(a) = \hat{T}(z)$ , where  $z = (0, 0, t)$ . Therefore  $BM(a) = M(a)B$  for every  $a \in \mathcal{A}$  and the lemma implies that this holds for every  $a \in L^\infty(A, m)$ . Since  $\{M(a): a \in L^\infty(A, m)\}$  is exactly the set of diagonalizable operators on  $\mathcal{H}$ ,  $B$  must be decomposable ([7], I. 3.2).

(b) Since  $B$  is decomposable, there exists an essentially bounded mapping  $\lambda \rightarrow B_\lambda$ , defined a.e.  $[m]$  with values in  $\mathcal{L}(L^2(V) \otimes L^2(V))$ , such that  $B = \int_A B_\lambda dm(\lambda)$ .

Since  $B$  commutes with  $\hat{T}(z) = \int_A I \otimes \hat{\lambda}(z) dm(\lambda)$  for every  $z \in Z$ ,  $B_\lambda$  commutes with  $I \otimes \hat{\lambda}(z)$  except for  $\lambda$  in an  $m$ -null set  $N_z$ .  $Z$  is separable; let  $\{z_i: i \in \mathcal{I}\}$  be a countable dense subset of  $Z$  and let  $N = \bigcup_{i \in \mathcal{I}} N_{z_i}$ . Then  $m(N) = 0$  and for  $\lambda \in N^c$ ,  $B_\lambda$  commutes with  $I \otimes \hat{\lambda}(z_i)$  for all  $i \in \mathcal{I}$ . Thus we have two continuous maps,  $z \rightarrow$

$B_\lambda(I \otimes \hat{\lambda}(z))$  and  $z \rightarrow (I \otimes \hat{\lambda}(z))B_\lambda$ , which agree on a dense subset of  $Z$ . It follows that  $B_\lambda \in \mathcal{A}'(I \otimes \hat{\lambda})$  for all  $\lambda \in N^c$ . Since  $\hat{\lambda}$  is irreducible, for each  $\lambda \in N^c$  there exists  $B(\lambda) \in \mathcal{L}(L^2(V))$  such that  $B_\lambda = B(\lambda) \otimes I$  ([1], VI. 3.14). Therefore,  $B = \int_A B(\lambda) \otimes Idm(\lambda)$ .

(c) Since  $B$  commutes with  $\hat{T}(s) = \int_A \tilde{\lambda}(s) \otimes \tilde{\lambda}(s) dm(\lambda)$  for every  $s \in S$ ,  $B(\lambda) \otimes I$  commutes with  $\tilde{\lambda}(s) \otimes \tilde{\lambda}(s)$  a.e.  $[m]$ . Just as in the proof of (b), since  $S$  is separable there exists an  $m$ -null set  $N$  such that for every  $\lambda \in N^c$  and every  $s \in S$ ,

$$(B(\lambda) \otimes I)(\tilde{\lambda}(s) \otimes \tilde{\lambda}(s)) = (\tilde{\lambda}(s) \otimes \tilde{\lambda}(s))(B(\lambda) \otimes I).$$

It follows that  $B(\lambda)\tilde{\lambda}(s) \otimes \tilde{\lambda}(s) = \tilde{\lambda}(s)B(\lambda) \otimes \tilde{\lambda}(s)$  and hence  $B(\lambda)\tilde{\lambda}(s) = \tilde{\lambda}(s)B(\lambda)$ . Thus,  $B(\lambda) \in \mathcal{A}'(\tilde{\lambda})$  for almost every  $\lambda$ .

Continue to suppose that  $B \in \mathcal{A}'(\hat{T}|_{ZSA})$  so that  $B$  satisfies (a), (b), and (c) of Theorem 3.2. We will now make use of the condition that  $B\hat{T}(\alpha) = \hat{T}(\alpha)B$  for all  $\alpha \in A$ . Recall that  $\alpha$  denotes both an element

of  $Gl(n_1, C)$  and the corresponding element  $\begin{vmatrix} \alpha & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \alpha'^{-1} & 0 \\ 0 & 0 & 0 & I \end{vmatrix}$  of  $A$ .

Concerning the transitive action of  $A$  on  $A$  given by  $\alpha \cdot \lambda = \alpha\lambda\alpha'$ , let  $A_1$  be the stability subgroup of  $A$  at  $I \in A$ . That is,  $A_1 = \{\alpha \in A: \alpha\alpha' = I\}$ , which can be identified with  $O(n_1, C)$ . Let  $p: A \rightarrow A$  be the projection  $p(\alpha) = \alpha \cdot I = \alpha\alpha'$ . We will need to know that a measurable set  $N$  is an  $m$ -null set in  $A$  if and only if  $p^{-1}(N)$  is a null set in  $A$  with respect to Haar measure. This result can be obtained by first showing that  $d\eta(\lambda) = |\det \lambda|^{-(n_1+1)}d\lambda$  is an  $A$ -invariant measure on  $A$ . It follows (as in [1], V. 3) that  $N$  is an  $\eta$ -null set in  $A$  if and only if  $p^{-1}(N)$  is a null set in  $A$ . Since  $m$  and  $\eta$  are clearly equivalent we have the needed result. We are now able to prove

**THEOREM 3.3.** *If  $B \in \mathcal{A}'(\hat{T}|_{ZSA})$  then there exists a mapping  $\lambda \rightarrow B(\lambda)$  of  $A$  into  $\mathcal{L}(L^2(V))$  which is weakly continuous and satisfies:*

- (a)  $B = \int_A B(\lambda) \otimes Idm(\lambda)$ ;
- (b)  $B(\alpha\alpha') = D(\alpha)B(I)D(\alpha)^{-1}$  for all  $\alpha \in A$ ;
- (c)  $B(I) \in \mathcal{A}'(D|_{A_1})$ ;
- (d)  $B(\lambda) \in \mathcal{A}'(\tilde{\lambda})$  for all  $\lambda \in A$ .

*Proof.* From Theorem 3.2, we have a weakly measurable mapping  $\lambda \rightarrow B_1(\lambda)$  such that  $B = \int_A B_1(\lambda) \otimes Idm(\lambda)$ . The major part of

this proof is to show that there is an equivalent mapping which is weakly continuous. By Theorem 2.8, if  $\alpha \in A$  and  $\hat{f} \in \mathcal{S}(L^1(Z) \cap L^2(Z))$  then

$$\hat{T}(\alpha)\hat{f}(\lambda) = \delta_H(\alpha)^{1/2}\chi(\alpha)D(\alpha)\hat{f}(\alpha^{-1} \cdot \lambda)D(\alpha)^{-1}$$

for almost every  $\lambda \in A$ , where  $\alpha^{-1} \cdot \lambda = \alpha^{-1}\alpha'^{-1}$ . The condition that  $B\hat{T}(\alpha) = \hat{T}(\alpha)B$  for all  $\alpha \in A$  is seen to be equivalent to

$$(1) \quad \text{For all } \alpha \in A, B_i(\lambda) = D(\alpha)^{-1}B_i(\alpha\lambda\alpha')D(\alpha) \text{ a.e. } [m(\lambda)].$$

Consider the weakly measurable mapping  $\alpha \rightarrow D(\alpha)^{-1}B_i(\alpha\alpha')D(\alpha)$  of  $A$  into  $\mathcal{L}(L^2(V))$ . For fixed  $\alpha \in A$ , (1) implies

$$(2) \quad D(\alpha\beta)^{-1}B_i(\alpha\beta\beta'\alpha')D(\alpha\beta) = D(\beta)^{-1}B_i(\beta\beta')D(\beta) \text{ a.e. } [d\beta],$$

since a null set in  $A$  pulls back under  $p^{-1}$  to a null set in  $A$ . Fix  $\phi, \psi \in L(V)$  and define the measurable, essentially bounded function  $w: A \rightarrow \mathbb{C}$  by

$$w(\beta) = (D(\beta)^{-1}B_i(\beta\beta')D(\beta)\phi, \psi),$$

where  $(\cdot, \cdot)$  is the inner product of  $L(V)$ . Then, by (2),  $w(\beta)d\beta$  defines a left invariant Borel measure on  $A$ . By uniqueness of Haar measure,  $w$  must be almost everywhere constant. Moreover, if this number is denoted  $w_{\phi, \psi}$  then, by application of the Riesz representation theorem to the bilinear form  $(\phi, \psi) \rightarrow w_{\phi, \psi}$ , there exists a unique  $L \in \mathcal{L}(L^2(V))$  such that

$$(3) \quad D(\beta)^{-1}B_i(\beta\beta')D(\beta) = L \text{ a.e. } [d\beta].$$

Consider the weakly measurable maps  $\beta \rightarrow B_i(\beta\beta')$  and  $\beta \rightarrow D(\beta)LD(\beta)^{-1}$  of  $A$  into  $\mathcal{L}(L^2(V))$ . Since  $A_1 = \{\alpha \in A: \alpha\alpha' = I\}$ , the first map is constant on left cosets of  $A_1$ . The second map is continuous (with respect to either the strong or the weak operator topology of  $\mathcal{L}(L^2(V))$ ). Also, (3) implies that the two maps coincide almost everywhere. We can conclude that  $\beta \rightarrow D(\beta)LD(\beta)^{-1}$  is both continuous and constant on left cosets of  $A_1$ . Because of this fact, the mapping  $\beta\beta' \rightarrow D(\beta)LD(\beta')$  is well-defined on  $A$ . It is also continuous since  $\beta \rightarrow D(\beta)LD(\beta)^{-1}$  is continuous and  $p$  is open. Define  $B(\beta\beta') = D(\beta)LD(\beta')$ , then  $B(\beta\beta')$  and  $B_i(\beta\beta')$  differ only on a "strip" set of measure zero in  $A$ , which projects to a null set in  $A$ . Thus,  $\lambda \rightarrow B(\lambda)$  is a continuous mapping such that (a) holds.

Parts (b) and (c) follow immediately from the definition of  $B(\lambda)$ . To prove (d), recall from Theorem 3.2 (c) that  $B(\lambda) \in \mathcal{A}'(\tilde{\lambda})$  for almost every  $\lambda \in A$ . In particular,  $B(\lambda_0) \in \mathcal{A}'(\tilde{\lambda}_0)$  for some  $\lambda_0 = \beta\beta' \in A$ . Let  $\lambda \in A$ , then  $\lambda = \alpha\lambda_0\alpha' = \alpha\beta\beta'\alpha'$  for some  $\alpha \in A$ . Now apply the definition of  $B(\lambda)$  along with Theorem 2.6 (2).

We now have the main theorem.

**THEOREM 3.4.** *The mapping*

$$B(I) \longrightarrow B = \int_A D(\beta)B(I)D(\beta)^{-1} \bar{\otimes} \text{Idm}(\beta\beta')$$

is an isomorphism of von Neumann algebras from  $\mathscr{A}'(D|_{A_1}) \cap \mathscr{A}'(\hat{I})$  onto  $\mathscr{A}'(\hat{T}|_{zSA})$ .

*Proof.* The mapping in the theorem makes sense because the condition  $B(I) \in \mathscr{A}'(D|_{A_1})$  guarantees that  $\beta\beta' \rightarrow D(\beta)B(I)D(\beta)^{-1} \bar{\otimes} I$  is well-defined. It is straightforward to verify separately that  $B \in \mathscr{A}'(\hat{T}|_z)$ ,  $B \in \mathscr{A}'(\hat{T}|_s)$ , and  $B \in \mathscr{A}'(\hat{T}|_A)$ . Also, using properties of decomposable operators, it is easy to show that  $B(I) \rightarrow B$  is an isometric, \*-algebra isomorphism. The fact that it is surjective is proved in Theorem 3.3.

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