OPERATORS SATISFYING A $G_1$ CONDITION

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An operator $T$ on a Hilbert space is said to be $G_1$ if $\|(T - z)^{-1}\| = 1/\text{dist}(z, \sigma(T))$ for $z \notin \sigma(T)$ and completely $G_1$ if, in addition, $T$ has no normal part. Certain results are obtained concerning the spectra of completely $G_1$ operators and of their real parts. It is shown in particular that there exist completely $G_1$ operators having spectra of zero Hausdorff dimension. Some sparseness conditions on the spectrum are given which assure that a $G_1$ operator has a normal part.

1. Introduction. All operators considered in this paper will be bounded (linear) on a Hilbert space $\mathcal{H}$ of elements $x$. For any such operator $T$ it is well-known (and due to Wintner [26]) that

$$\|(I - z)^{-1}\| \geq 1/\text{dist}(z, \sigma(T))$$

for $z \notin \sigma(T)$ and $\|(T - z)^{-1}\| \leq 1/\text{dist}(z, W^-(T))$ for $z \notin W^-(T)$, where $\sigma(T)$ denotes the spectrum of $T$ and $W^-(T)$ denotes the (convex) closure of the numerical range $W(T) = \{(Tx, x) : \|x\| = 1\}$. An operator $T$ is said to be $G_1$ (or to satisfy a $G_1$ condition, or to be of class $G_1$) if

$$\|(T - z)^{-1}\| = 1/\text{dist}(z, \sigma(T)) \quad \text{for} \quad z \notin \sigma(T).$$

For instance, (1.1) holds for operators $T$ which are normal ($T^*T = TT^*$), more generally, for those which are subnormal ($T$ has a normal extension on a larger Hilbert space), and still more generally, for hyponormal operators ($T^*T - TT^* \geq 0$). The inclusions indicated here,

$$\text{normals} \subset \text{subnormals} \subset \text{hyponormals} \subset (G_1),$$

are all proper and, needless to say, the simple stratification (1.2) can be interstitially (and endlessly) refined. In this connection, see the brief survey in Putnam [16].

An operator $T$ will be called completely $G_1$ if $T$ is $G_1$ and if, in addition, $T$ has no normal part, that is, $T$ has no reducing subspace on which it is normal. Similarly, one has corresponding definitions of completely subnormal or completely hyponormal operators. It is well-known that every compact set of the plane is the spectrum of some normal operator. Moreover, necessary and sufficient conditions are known in order that a compact set be the spectrum of a completely subnormal operator (Clancey and Putnam [4]) or of a completely
hyponormal operator (Putnam [15], [17]). On the other hand, no such
conditions are known for the class of completely $G_1$ operators.

It may be noted that if $T$ is $G_1$ and if $\sigma(T)$ is finite, in particular,
if $\mathcal{H}$ is finite-dimensional, then necessarily $T$ is normal. In fact,
Stampfli [20], p. 473, shows that if $T$ is $G_1$ and if $z_0$ is an iso-
lated point of $\sigma(T)$ then $z_0$ is a normal eigenvalue of $T$, that is,
$z_0 \in \sigma_2(T)$, the point spectrum of $T$, and the corresponding eigenvectors
form a reducing space of $T$ on which $T$ is normal. (For some related
results, see also Hildebrandt [8], p. 234, and Luecke [10], p. 631.)
More generally, it was shown by Stampfli ([22], [23]) that if $T$ is
$G_x$ and if $z_0$ is an iso-
lated point of $\sigma(T)$ then $z_0$ is a normal eigenvalue of $\Gamma$, that is,
$z_0 \in \sigma_3(T)$, the point spectrum of $T$, and the corresponding eigenvectors
form a reducing space of $T$ on which $T$ is normal. (For some related
results, see also Hildebrandt [8], p. 234, and Luecke [10], p. 631.)

All of this suggests that a simple necessary and sufficient con-
dition on a compact set in order that it be the spectrum of a
completely $G_1$ operator is not easily obtained. In fact, even such a
condition on a countable compact set in order that it be the spectrum
of a nonnormal operator of class $G_1$ is not known. (A sufficient
condition for normality is that of Luecke [10] mentioned above;
another is given in Theorem 2 below.) Of course, any $G_1$ operator
having a countable spectrum certainly has a normal part. It is thus
clear that a necessary condition on a compact set, $X$, in order that
it be the spectrum of a completely $G_1$ operator is that $X$ be perfect.
In order to describe certain types of sets $X$ occurring below, it will
be convenient to recall the definition of Hausdorff measure.

A "measure function" $h(t)$ is an increasing continuous function
on $0 \leq t < \infty$ satisfying $h(0) = 0$. For a bounded set, $X$, of the
complex plane and a fixed $\delta > 0$ let $\Gamma = \{D_i, D_0, \cdots\}$ be any countable
covering of $X$ by open disks $D_j$ of radius $\delta_j \leq \delta$. Then
$\Lambda_h(X) = \lim_{\delta \to 0} [\inf \{\sum_{j=1}^\infty h(\delta_j)\}]$ exists and is the Hausdorff $h$-measure of $X$. (See
Garnett [5], p. 58; also Carleson [2], Rogers [19].) If $h(t) = t^r$, $r > 0$,
then $\Lambda_h(X)$ is the $r$-dimensional Hausdorff measure of $X$. In par-
ticular, a nonempty set $X$ is said to have Hausdorff dimension $= 0$
if $\Lambda_h(X) = 0$ for all $h = t^r, r > 0$.

2. THEOREM 1. For any given measure function $h$ there exists
a perfect set $X$ of the complex plane and a completely $G_1$ operator $T$
for which $X = \sigma(T)$ has Hausdorff $h$-measure $= 0$.

It may be noted that, in particular, there exist completely $G_1$
operators with spectra of Hausdorff dimension $= 0$. That the function
$h$ of Theorem 1 be preassigned is an essential requirement however.
In fact, the condition that $\Lambda_h(\sigma(T)) = 0$ for all measure functions
$h$ is sufficient (as well as necessary) in order that $\sigma(T)$ be countable; see Rogers [19], p. 67.

Proof. As in Stampfli ([20], [22]), consider the matrix
\begin{equation}
A = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix},
\end{equation}
acting on a two-dimensional Hilbert space, so that $(A - z)^{-1} =
\begin{pmatrix}
-1/z & -1/z^2 \\
0 & -1/z
\end{pmatrix}$, and hence $||(A - z)^{-1}|| \leq 1/|z| + 1/|z|^2$ for all $z \in \sigma(A) = \{0\}$. Note also that $W(A)(= W^{-}(A)) = \{z: |z| \leq 1/2\}$ and $||A|| = 1$. Then $||(A - z)^{-1}|| \leq (|z| - 1/2)^{-1}$ for $|z| > 1/2$ and clearly there exists a countable set $\alpha = \{z_1, z_2, \ldots\} \subset \{z: 0 < |z| < 1\}$ satisfying $z_n \to 0$ as $n \to \infty$ and such that
\begin{equation}
||(A - z)^{-1}|| \leq 1/\text{dist}(z, \alpha) \quad \text{for} \quad z \neq 0.
\end{equation}

Next, choose a sequence of nonoverlapping open disks $\{D_n, D_2, \ldots\}$, where each $D_n$ has center $z_n$ and is contained in $\{z: 0 < |z| < 1\}$. Let $A_n = a_n A + z_n$, where $0 < a_n < \text{radius } D_n$, so that $||A_n - z_n|| = \text{radius } D_n$ and $\sigma(A_n) = \{z_n\}$. Then, for each $n = 1, 2, \ldots$, choose a countable set $\alpha_n = \{z_{n1}, z_{n2}, \ldots\} \subset D_n$ satisfying $z_{nk} \neq z_n$ and $z_{nk} \to z_n$ as $k \to \infty$ and the inequality $||(A_n - z)^{-1}|| \leq 1/\text{dist}(z, \alpha_n)$ for $z \neq z_n$. Thus, if $T_0 = A$ and $T_1 = \sum \bigoplus A_n$, one sees that
\begin{equation}
T = T_0 \bigoplus T_1 \text{ satisfies } ||(T - z)^{-1}|| \leq 1/\text{dist}(z, \alpha_0, \bigcup \alpha) \end{equation}
for $z \in \sigma(T) = \{0\} \cup \alpha$.

In the next step each of the disks $D_n$ plays the role of the containing disk $\{z: |z| < 1\}$ in the previous construction. Thus, for each $n = 1, 2, \ldots$, one chooses a sequence of nonoverlapping open disks $\{D_{n1}, D_{n2}, \ldots\}$, contained in $D_n$ and clustering at $z_n$, and obtains a new operator $T_n$ for which $T = T_0 \bigoplus T_1 \bigoplus T_n$ satisfies a condition analogous to (2.2) for $T = T_0$ and to (2.3) for $T = T_0 \bigoplus T_1$. Continu-
ation of this process leads to an operator \( T = \sum_{k=0}^{\infty} T_k \) satisfying
\[
||(T - z)^{-1}|| \leq 1/\text{dist}(z, X) \quad \text{for } z \in X,
\]
where \( X \) is the closure of the set of all centers of circles occurring in the above construction. Since \( X \subseteq \sigma(T) \) then, by (2.4), \( \sigma(T) = X \) and \( T \) satisfies (1.1). Moreover, it is clear that \( T \) is a completely \( G_1 \) operator. Further, the inclusions
\[
\{z: z < 1\} \supset \bigcup D_n \cup \{0\} \supset \bigcup D_{nk} \cup \{0, z_1, z_2, \cdots\} \supset \cdots \supset \sigma(T)
\]
show that, for any given measure function \( h \), one can always choose the countable collection of disks \( \{D_n\}, \{D_{nk}\}, \cdots \), in such a way that \( \sigma(T) \) has Hausdorff \( h \)-measure = 0. This completes the proof of Theorem 1.

**Corollary 1.** If \( X \) denotes an arbitrary compact set of the plane and if \( h \) is any measure function, then there exists a perfect set \( P \supset X \) and a completely \( G_1 \) operator \( T \) such that \( P - X \) has Hausdorff \( h \)-measure = 0 and \( \sigma(T) = P \).

**Proof.** Let \( \{z_1, z_2, \cdots\} \) be any countable subset of \( X \) dense in \( X \). For each \( n = 1, 2, \cdots \), let \( D_n \) be an open disk centered at \( z_n \) and suppose that diam \( D_n \to 0 \) as \( n \to \infty \). Then let \( T_n \) be a completely \( G_1 \) operator having spectrum of Hausdorff \( h \)-measure = 0 and such that \( z_n \in \sigma(T_n) \subseteq D_n \). One need only choose \( T_n \), for instance, to be an appropriate linear function of the operator \( T \) constructed in the proof of Theorem 1. (Note that the \( G_1 \) property is invariant under linear transformations; see Luecke [11], p. 36.) If \( T = \sum \bigoplus T_n \) then, since each \( T_n \) is \( G_1 \), \( \sigma(T) = (\bigcup \sigma(T_n))^- \) and hence, since diam \( D_n \to 0 \) as \( n \to \infty \), \( \sigma(T) = \bigcup \sigma(T_n) \cup X = P \) satisfies the conditions stated in the corollary.

A related result is the following

**Corollary 2.** If \( B \) is any operator and \( h \) is any measure function there exists a completely \( G_1 \) operator \( T \) for which \( B \bigoplus T \) is also \( G_1 \) and \( \sigma(T) \subseteq \partial(\sigma(B)) \cup \beta \) where \( \beta \) has Hausdorff \( h \)-measure = 0.

**Proof.** Choose a sequence of points \( \alpha = \{z_1, z_2, \cdots\} \) in such a way that no \( z_n \) lies in \( \sigma(B) \), dist(\( z_n, \sigma(T) \)) \( \to 0 \) as \( n \to \infty \), and such that
\[
||(B - z)^{-1}|| \leq 1/\text{dist}(z, \alpha) \quad \text{for } z \notin \sigma(B).
\]
Then choose a sequence of open disks \( \{D_1, D_2, \cdots\} \), where \( z_n \) is the center of \( D_n \), satisfying \( D_n \cap \sigma(B) = \emptyset \) and diam \( D_n \to 0 \) as \( n \to \infty \), so that the \( D_n \)'s cluster only on the set \( \partial(\sigma(B)) \). If \( T_1, T_2, \cdots \) are \( G_1 \) operators such that \( z_n \in \sigma(T_n) \subseteq D_n \) and \( \sigma(T_n) \) has Hausdorff \( h \)-measure = 0, then \( T = \sum \bigoplus T_n \).
\[ \sum \oplus T_n \] satisfies the conditions stated in the corollary.

3. Some lemmas. If \( \{A_1, A_2, \cdots \} \) is a decreasing sequence of self-adjoint operators then the \( A_n \) converge strongly to a (self-adjoint) operator \( A \), a result due to Vigier (see Riesz and Sz.-Nagy [18], p. 263). In particular, if each \( A_n \) is an orthogonal projection, so also is \( A \). Further, it is well-known that a projection \( P(P = P^2) \) is orthogonal if and only if \( ||P|| \leq 1 \). We shall need need the following generalization to arbitrary projections \( P_n \) of the above results.

**Lemma 1.** Let \( \{P_1, P_2, \cdots \} \) be a sequence of projections \( (P_n = P^2_n) \) satisfying

\[
P_n P_{n+p} = P_{n+p} \quad (n = 1, 2, \cdots; p = 0, 1, 2, \cdots)
\]

and

\[
\limsup_{n \to \infty} ||P_n|| \leq 1.
\]

Then the \( P_n \) converge strongly as \( n \to \infty \) to an orthogonal projection.

**Proof.** First, let \( P \) denote any projection and let \( t \geq 0 \) satisfy

\[
||P|| \leq 1 + t.
\]

Since \( P^2 = P \), the range of \( P^* \) is orthogonal to the range of \( I - P \) and hence, if \( x \) is arbitrary in \( \mathcal{D} \) and \( y = P^*x \), then \( y = P^*y \perp (I - P)y \).

Since \( Py = y - (I - P)y \), then

\[
||y||^2 + ||(I - P)y||^2 = ||Py||^2 \leq (1 + t)^2 ||y||^2,
\]

and so \( ||(I - P)P^*x||^2 \leq (2t + t^2)||P^*x||^2 \). Consequently,

\[
||P - PP^*|| = ||P^* - PP^*|| \leq t^{1/2}(2 + t)^{1/2}(1 + t),
\]

and hence

\[
||P - P^*|| \leq 2t^{1/2}(2 + t)^{1/2}(1 + t).
\]

Relations (3.2) and (3.5) (with \( P \) replaced by \( P_n \)) imply that \( ||P_n - P^*_n|| \to 0 \) as \( n \to \infty \). Further, if \( Q_n = P_n P_n^* \), also \( ||Q_n - P_n|| \to 0 \) as \( n \to \infty \) (uniformly in \( p \geq 0 \)). Similarly, \( ||Q_n Q_{n+p} - Q_{n+p} Q_n|| \to 0 \) as \( n \to \infty \) (uniformly in \( p \geq 0 \)) and hence also \( ||Q_n(I - Q_{n+p}) - Q_{n+p}^1(I - Q_{n+p})Q_{n+p}^0|| \to 0 \) (uniformly in \( p \geq 0 \)). It follows that there exists a sequence of positive numbers \( \{t_1, t_2, \cdots\} \) with limit 0 for which

\[
A_{n+p} = Q_n - Q_{n+p} + t_n \geq 0 \quad \text{for all} \quad n \geq 1 \quad \text{and} \quad p \geq 0.
\]
If $x$ is arbitrary in $\mathfrak{F}$, then clearly one can choose integers $n = n_k \to \infty$ and $p = p_k \to \infty$ so that $(Q_{n_k} x, x) \to \liminf_{n \to \infty} (Q_n x, x)$ and also $(Q_{n_k+p} x, x) \to \limsup_{n \to \infty} (Q_n x, x)$. Hence, by (3.6),

$$\lim_{n \to \infty} (Q_n x, x)$$

exists, for each $x$ in $\mathfrak{F}$.

An argument like that in Riesz and Sz.-Nagy [18], p. 263, shows that $\|A_n x\|^2 = (A_n x, A_n x) \leq (A_p x, x)(A_p x, A_p x)$ and hence, by (3.7) and the definition of $A_n$ in (3.6), $(Q_n - Q_{n+p}) x \to 0$ (strongly) as $n \to \infty$ (uniformly in $p \geq 0$), so that $Q = s\lim_{n \to \infty} Q_n$ exists and is self-adjoint. Since $\|Q_n - P_n\| \to 0$, then $s\lim_{n \to \infty} P_n = Q$ is an orthogonal projection and the proof of Lemma 1 is complete.

**Lemma 2.** Let $T$ be a $G_1$ operator and suppose that $z_0 \in \sigma(T)$. In addition, suppose that there exists a sequence of circles $C_n = \{z: |z - z_0| = r_n\}$, $n = 1, 2, \ldots$, lying in the resolvent set of $T$, and for which $r_n > r_1 > \cdots \to 0$ and

$$r_n/\text{dist}(C_n, \sigma(T)) \to 1 \quad \text{as} \quad n \to \infty.$$  

If each $C_n$ is positively oriented and if $P_n$ denotes the projection

$$P_n = -(2\pi i)^{-1} \int_{C_n} (T - z)^{-1} dz \quad (n = 1, 2, \cdots),$$

then $P_n \to P$ (strongly), where $P$ is an orthogonal projection commuting with $T$, and

$$T - z_0) P = 0.$$  

**Proof.** That the $P_n$ satisfy (3.1) follows from a computation similar to that in Riesz and Sz.-Nagy [18], p. 419. In addition, it is clear that

$$||P_n|| \leq (2\pi)^{-1} \left( \max_{z \text{ on } C_n} ||(T - z)^{-1}|| \right) 2\pi r_n \leq r_n/\text{dist}(C_n, \sigma(T)),$$

so that (3.8) implies (3.2). Thus, by Lemma 1, $P_n \to P$ (strongly), where $P$ is an orthogonal projection. Since $P_n T = TP_n$, then also $PT = TP$. Relation (3.10) follows from the limit relation $r_n \to 0$ and an estimate of $(T - z_0) P = -(2\pi i)^{-1} \int_{C_n} (z - z_0) (T - z)^{-1} dz$ similar to that of (3.11).

**Lemma 3.** Let $T$ be an arbitrary operator and suppose that $z_0 \in \sigma_p(T)$. In addition, suppose that there exist $z_n \in \sigma(T)$ such that $z_n \to z_0$ and $|z_n - z_0| ||(T - z_n)^{-1}|| \to 1$ as $n \to \infty$. Then $z_0$ is a normal eigenvalue of $T$. 

Proof. The result was given in Putnam [14] and, before this, implicitly in Stampfli [21] (cf. Stampfli's remark in [24], p. 135). A variation appears earlier in Sz.-Nagy and Foias [25], p. 93. See also Hildebrandt [8], p. 234.

REMARK. Let $T$ be $G_1$. It is clear from Lemma 3 that if $z_0 \in \sigma_p(T)$ and if

$$z_n \notin \sigma(T), \quad z_n \longrightarrow z_0 \quad \text{and} \quad \text{dist}(z_n, \sigma(T))/|z_n - z_0| \longrightarrow 1$$

as $n \longrightarrow \infty$, then $z_0$ is a normal eigenvalue of $T$. In Lemma 2, it is assumed only that $z_0$ is in $\sigma(T)$ but not necessarily in $\sigma_p(T)$. On the other hand, the condition (3.8) for such a $z_0$ is clearly much stronger than (3.12). Since $T$ commutes with $P$, relation (3.10) implies that if $P \neq 0$ then necessarily $z_0$ is a normal eigenvalue of $T$.

If only $z_0 \in \sigma(T)$ is assumed, it may be noted that (3.12) may hold for a completely $G_1$ operator, so that, in particular, $z_0 \notin \sigma_p(T)$. For example, let $T$ be a completely $G_1$ operator as constructed in the proof of Theorem 1, so that $T$ has the form $T = \sum \oplus (b_n A + w_n)$, where $b_n > 0$ and $A$ is given by (2.1). If $s = \sup \text{Re} \sigma(T)$, then there exists some $z_0 \in \sigma(T)$ with $s = \text{Re} z_0$, and hence (3.12) holds with, say, $z_n = z_0 + c_n$, where $0 < c_n \rightarrow 0$.

Further, note that it is possible that $T$ is $G_1$ with $z_0 \in \sigma_p(T)$ and that there exist circles $C_n = \{z: |z - z_0| = r_n\}$, $n = 1, 2, \ldots$, lying in the resolvent set of $T$ and satisfying $r_1 > r_2 > \cdots \rightarrow 0$ and for which the projections $P_n$ of (3.9) are orthogonal and converge strongly to an orthogonal projection $P \neq 0$, but for which $z_0$ is not a normal eigenvalue of $T$. Thus, (3.10) need not hold if (3.8) is not assumed, even though the other hypotheses of Lemma 2 are retained.

A simple example is obtained by considering the construction of Stampfli ([20], [22]), with

$$T = A \oplus N,$$

where $A$ is given by (2.1) and $N$ is normal with spectrum $\alpha^-$. Here $\alpha$ is defined as in the beginning of the proof of Theorem 1 and, in particular, (2.2) holds. Clearly, for $z_0 = 0$, there exist circles $C_n = \{z: |z| = r_n\}$ lying in the resolvent set of $T$ with $r_1 > r_2 > \cdots \rightarrow 0$. It is seen that each $P_n$ is an orthogonal projection. Further, if $A$ acts on the two-dimensional space $\mathcal{H}_0$ then $P_n \rightarrow P$ (strongly), where $P$ is the projection of $\mathcal{H}_0$ onto $\mathcal{H}_0$. Although $z_0 \in \sigma_p(T)$, it is clear that $z_0$ is not a normal eigenvalue of $T$.

The above procedure can be modified so as to yield a completely $G_1$ operator $T$. One need only consider the operator $T$ constructed
in the proof of Theorem 1 above where the numbers $z_1, z_2, \ldots$, and the first sequence of disks $\{D_1, D_2, \ldots\}$, with $D_n = \{z : |z - z_n| < r_n\}$, are chosen so that $(0, t) \cap \bigcup_{n=1}^{\infty} (|z_n| - r_n, |z_n| + r_n) \neq (0, t)$ for all $t > 0$. This enables one to choose circles $C_n$ as in the preceding paragraph and to proceed in a manner similar to that described there.

4. Theorem 2. Let $T$ be $G_x$ and suppose that $\sigma(T)$ is not a perfect set and that for each $z_0 \in \sigma(T)$ there exists a sequence of circles $C_n = \{z : |z - z_0| = r_n\}$, $n = 1, 2, \ldots$, lying in the resolvent of $T$ for which $r_1 > r_2 > \cdots \to 0$ and (3.8) holds. Then

(4.1) $T$ is normal if $\sigma(T)$ is countable,

and

(4.2) $T = T_1 \oplus T_2$ if $\sigma(T)$ is not countable,

where $T_1$ is normal with $\sigma(T_1) = \alpha^-$ and $\alpha$ a countable set, and where $\sigma(T_2)$ is perfect and $\sigma(T_2) \cap \alpha = \emptyset$.

Proof. Since $\sigma(T)$ is not perfect, $\sigma(T)$ contains a nonempty (countable) set, $S_0$, of isolated points. Hence, as noted earlier, $T$ has a normal part $N_0$ corresponding to these points with $\sigma(N_0) = S_0$. In case $S_0 = \sigma(T)$, the proof is complete. Otherwise, as will be assumed, $T = N_0 \oplus A_0$, where $\sigma(A_0) \cap S_0 = \emptyset$, and we let $S_1$ denote the (countable) set of isolated points of the first derivative, $\sigma'(T)$, of $\sigma(T)$. If $S_1$ is empty the proof is over and so we can suppose that $S_1 \neq \emptyset$. It follows from (3.10) of Lemma 2 that each point $z_0$ of $S_1$ either corresponds to a normal eigenvalue (if $P \neq 0$), or, if $P = 0$, can simply be ignored. Thus, at the end of the second stage we have $T = N_1 \oplus A_1$ where $\sigma(N_1) = S_0^+ \cup S_1^-$ and, if $A_1$ is present, $\sigma(A_1) \cap (S_0 \cup S_1) = \emptyset$. One then repeats this process. It should be noted that for $n = 0, 1, 2, \ldots$, $S_n = \sigma^{(n)}(T) - \sigma^{(n+1)}(T)$, where $\sigma^{(n)}(T)$ denotes the $n$th derived set of $\sigma(T) \equiv \sigma^{(0)}(T)$. If for any positive integer $n$, $S_n$ is empty, the process terminates. In addition, if $\sigma(T) = \bigcup_{n=0}^{\infty} S_n$, the process also terminates, and, of course, implies that $T$ is normal and that $\sigma(T)$ is countable. Otherwise, the process continues via transfinite induction as noted below.

The $\nu$th derived set of $\sigma(T)$ can be defined, in the manner of Cantor using transfinite induction, for any ordinal $\nu$; see Kamke [9], p. 127. It follows from a transfinite induction argument ([9], pp. 132–133) that there is a least ordinal $\gamma$, where $0 \leq \text{cardinality of } \gamma \leq \aleph_0$, with the property that $\sigma^{(\gamma)}(T) = \sigma^{(\alpha)}(T)$ for all ordinals $\alpha \geq \gamma$. In particular, if $\sigma^{(\gamma)}(T)$ is not empty then it is perfect. It follows (cf. [9], p. 133) that if $\sigma(T)$ is countable, then $\sigma^{(\gamma)}(T)$ is empty and,
by the process described in the preceding paragraph, (4.1) is established. If \( \sigma(T) \) is not countable then \( \sigma^{(r)}(T) \) is perfect and so (4.2) holds with the properties described in Theorem 2.

5. Theorem 3. Let \( T \) be \( G \). Suppose that for every \( \varepsilon > 0 \) there exists a countable covering of \( \sigma(T) \) by open disks \( D_n = \{ z : |z - z_n| < r_n \} \), \( n = 1, 2, \ldots \), with the properties that, for each \( n \), \( D_n \cap \sigma(T) \neq \emptyset \) and \( C_n = \{ z : |z - z_n| = r_n \} \) lies in the resolvent set of \( T \), and that

(5.1) \[ \sum_n \left( \frac{r_n}{d_n} - 1 \right)^{1/2} < \varepsilon, \quad \text{where} \quad d_n = \text{dist}(C_n, \sigma(T)) \quad (\leq r_n), \]

and

(5.2) \[ \sum_n r_n < \varepsilon. \]

Then \( T \) is normal.

Proof. Let \( \varepsilon > 0 \) be fixed. In view of the Heine-Borel theorem it may be suppose that the covering of Theorem 3 is finite, say \( \{ D_1, \ldots, D_N \} \), and that \( D_n \not\subset D_m \) for \( n \neq m \). For \( n = 1, \ldots, N \), define

\[ P_n = -(2\pi i)^{-1} \int_{C_n} (T - z)^{-1} \, dz, \]

where the \( C_n \) are regarded as positively oriented, so that, by an estimate similar to that of (3.11), \( ||P_n|| \leq r_n/d_n \). (Note that in the present case, \( D_n \cap \sigma(T) \neq \emptyset \) but it is not assumed as in Lemma 2 that the center of \( C_n \) is in \( \sigma(T) \).) Next, if \( t_n = r_n/d_n - 1 \) then \( ||P_n|| \leq 1 + t_n \) (cf. (3.3)). It follows from (3.5) with \( P \) and \( t \) replaced by \( P_n \) and \( t_n \) that

(5.3) \[ ||P_n - P_n^\circ|| \leq \text{const}(r_n/d_n - 1)^{1/2} \quad (n = 1, \ldots, N), \]

provided, say, \( 0 < \varepsilon \leq 1/2 \), as will be assumed. Thus, in view of (5.1).

(5.4) \[ \sum_{n=1}^N ||P_n - P_n^\circ|| \leq \text{const} \varepsilon. \]

Next, consider any pair or circles, say \( C_1 \) and \( C_2 \). It will be shown that if \( D_1 \cap D_2 \neq \emptyset \) then either one circle, say \( C_2 \), can be discarded or it can be deformed into a rectifiable simple closed curve \( C_2' \) lying in the resolvent set of \( T \) and with the properties that

(5.5) \[ P_2 = P_{C_2'} = -(2\pi i)^{-1} \int_{C_2'} (T - z)^{-1} \, dz \]

and

(5.6) \[ \text{int} C_2' \subset D_2 \quad \text{and} \quad D_1 \cap \text{int} C_2' = \emptyset. \]
To see this, note first that \( \sigma(T) \cap \{z: r_1 - d_1 < |z - z_1| < r_1 + d_1\} = \emptyset \). If \( D_2 \subset \{z: |z - z_1| < r_1 + d_1\} \), then \( D_2 \cap \sigma(T) \subset D_1 \cap \sigma(T) \) and so \( C_2 \) can be discarded. Also, in case \( D_1 \cap \{z: |z - z_1| \leq r_1 - d_1\} = \emptyset \), then, since \( D_2 \not\subset D_1 \), \( C_2 \) can be deformed into \( C'_2 \) so as to satisfy both (5.5) and (5.6). The remaining possibility is that

\[
D_2 \cap \{z: |z - z_1| \leq r_1 - d_1\} \neq \emptyset \quad \text{and} \quad D_2 \not\subset \{z: |z - z_1| < r_1 + d_1\}.
\]

It may be supposed, however, that \( \{z: |z - z_1| < r_1 + d_1\} \not\subset D_2 \) since, otherwise, \( D_1 \cap \sigma(T) \subset D_2 \cap \sigma(T) \) and \( C_1 \) can be discarded. Consequently, \( r_2 > d_1 \) and \( d_2 < 2(r_1 - d_1) \), so that \( r_2/d_2 > d_1/2(r_1 - d_1) = 1/2(r_1/d_1 - 1)^{-1} \). Hence, \( r_2/d_2 > 1/2\varepsilon \), in view of, and in contradiction to (5.1) (with \( \varepsilon \leq 1/2 \)).

Repeated applications of the above argument show that the circles \( C_1, \cdots, C_N \) may be replaced by rectifiable simple closed curves, say, \( \gamma_1, \cdots, \gamma_M(M \leq N) \), where each \( \gamma_i \) is some \( C_j \) or some \( C'_j \), and where \( \text{int} \gamma_n \cap \text{int} \gamma_m = \emptyset \) for \( m \neq n \) and \( \sigma(T) \subset \bigcup_{n=1}^{M} \text{int} \gamma_n \). It is seen from relations corresponding to (5.5) and (5.6) that \( \sum_{n=1}^{M} P_n = I \), where \( P_n = -(2\pi i)^{-1} \int_{\gamma_n} (T - z)^{-1} dz \), and hence that \( \sum' P_n = I \) where the prime denotes that the summation is over a subset of \( \{1, \cdots, N\} \). As a result, we revert to the original notation and suppose without loss of generality, that

\[
(5.7) \quad I = \sum P_n \quad \left( \sum' = \sum \right).
\]

It is now easy to complete the proof of Theorem 3. For,

\[
(5.8) \quad T = TI = \sum TP_n = \sum z_n P_n + \sum (T - z_n)P_n.
\]

But \( ||(T - z_n)P_n|| \leq r_n ||P_n|| \leq r_n(r_n/d_n) < r_n(1 + \varepsilon^2) \), the last inequality by (5.1). Since \( \varepsilon \leq 1/2 \), (5.2) shows that \( \sum ||(T - z_n)P_n|| \geq 2\varepsilon \). Also, \( \sum z_n P_n = \sum z_n P^*_n + \sum (z_n(P_n - P^*_n)) \) and, by (5.4), \( \sum ||z_n(P_n - P^*_n)|| \leq (\max |z_n|) \text{const} \varepsilon \). Since each \( D_n \) contains part of \( \sigma(T) \) it is clear from (5.2) that \( \max |z_n| \leq ||T|| + 2\varepsilon \leq ||T|| + 1 \), and so, by (5.8),

\[
(5.9) \quad T = \sum z_n P^*_n + A, \quad \text{where} \quad ||A|| \leq \text{const} \varepsilon.
\]

Hence, \( T^*T = \sum z_n T^*P^*_n + T^*A = \sum z_n[z_n P^*_n - (T^* - \bar{z}_n)P^*_n] + T^*A \). But \( ||T^*A|| \leq \text{const} \varepsilon \) and, as above, \( \sum ||z_n(T^* - \bar{z}_n)P^*_n|| \leq (\max |z_n|)2\varepsilon \), and so another application of (5.4) yields \( ||T^*T|| - \sum |z_n|^2 ||P_n|| < \text{const} \varepsilon \). A similar argument yields the same inequality with \( T \) and \( T^* \) interchanged, hence \( T \) is normal, and the proof is complete.

**Remarks.** It is readily seen that Theorem 3 implies the assertion of Theorem 2 when \( \sigma(T) \) is countable. We do not know whether the hypothesis of Theorem 2 implies that \( T \) is normal even when \( \sigma(T) \)
is not countable, in which case Theorem 2 would imply Theorem 3. The hypothesis (3.8) of Theorem 2 is of course a "sparseness" condition on $\sigma(T)$ and, conceivably, is restrictive enough to imply normality of $T$. In the same vein, we do not know whether the condition (5.2) in the hypothesis of Theorem 3 is essential, although, of course, at least a boundedness restriction must be placed on the $r_n$'s of (5.1). (Note that if $C_r$ is the circle with center at $z = 0$ and radius $r$ then $r/dist(C_r, \sigma(T)) \to 1$ as $r \to \infty$.) It is clear, of course, that (5.2) alone is not enough, since this condition amounts only to requiring that $\sigma(T)$ be of one-dimensional Hausdorff measure 0.

It may be noted that there exist uncountable sets, corresponding to $\sigma(T)$, for which (3.8) holds. To see this, one need only modify the construction of the standard Cantor set so that the length of each removed complementary open interval is a fraction sufficiently close to 1 of the length of the (closed) interval from which it was removed.

6. Real parts of $G_1$ operators. If $T$ is $G_1$ then, as was shown in Putnam [13], p. 509,

$$ (6.1) \quad \text{Re} \sigma(T) \subset \sigma(\text{Re} T). $$

For another proof, see Berberian [1], where it is also shown that, if $\sigma(T)$ is connected,

$$ (6.2) \quad \text{Re} \sigma(T) = \sigma(\text{Re} T). $$

That (6.2) need not hold in general, however, can be deduced from the example of Stampfli of (3.13) above, simply by choosing the sequence $\{z_1, z_2, \cdots\}$ so that, for instance, $\text{Re} z_n \neq \pm 1/2$ for all $n$. Then $\text{Re} \sigma(T)$ consists of 0 and the real parts of the $z_n$'s while $\sigma(\text{Re} T) = \text{Re} \sigma(T) \cup \{\pm 1/2\}$. A consideration of the operator $T$ constructed in Theorem 1, where now the disks $D_n$ are chosen so that $\text{Re} z \neq \pm 1/2$ for $z \in D_n(n = 1, 2, \cdots)$, shows that (6.1) may hold properly also if $T$ is completely $G_1$.

It is known that (6.2) always holds for hyponormal operators; see Putnam [12], p. 46. In view of certain known results concerning the spectra of completely subnormal and completely hyponormal operators one has the following

**Theorem 4.** Let $T$ have the rectangular form $T = H + iJ$ and let $X$ be a compact subset of the real line. Then:

(i) $X$ is the spectrum of $H = \text{Re} T$ for some completely subnormal $T$ if and only if $X$ is the closure of an open subset of the real line;
(ii) \( X \) is the spectrum of \( H = \text{Re} \ T \) for some completely hyponormal \( T \) if and only if, for every open interval \( I \), \( \text{meas}_r(X \cap I) > 0 \) whenever \( X \cap I \neq \emptyset \), where \( \text{meas}_r \) denotes linear Lebesgue measure.

Proof of (i). First, let \( X \) be the closure of an open set of real numbers, so that \( X = \bigcup I_n \). Since the unilateral shift \( V \) is subnormal and \( \sigma(V) \) is the closed unit disk (see, e.g., Halmos [7]), one need only put \( T = \sum \bigoplus (a_n V + b_n) \) where \( a_n, b_n \) are real, \( a_n > 0 \), and \( I_n = (-a_n + b_n, a_n + b_n) \). Clearly, \( X \subset \sigma(T) \), while the reverse inclusion follows from the fact that each term \( a_n V + b_n \) is \( G_2 \).

Conversely, suppose that \( H = \text{Re} \ T \) where \( T \) is completely subnormal and let \( X = \text{int} \sigma(H) \). It will be shown that \( X = \sigma(H) \). If \( X \neq \sigma(H) \), then there exists some \( c \in \sigma(H) - X \) and an open interval \( I_c \) containing \( c \) such that \( \sigma(H) \cap I_c \) has no interior. In view of (6.2), there exists an open disk \( D \) intersecting \( \sigma(T) \) for which \( Y = \sigma(T) \cap D \) is nowhere dense and has a connected complement. Hence \( C(Y) = \{Y\} \), by Lavrentiev's theorem (cf. Gamelin [5], p. 48), and hence \( T \) has a normal part with spectrum \( Y \); see Clancey and Putnam [4]. Thus, \( T \) is not completely subnormal, a contradiction.

Proof of (ii). First, suppose that \( X \cap I \) has positive linear measure whenever \( I \) is an open interval and \( X \cap I \neq \emptyset \). Let \( T = H + iJ \) on \( \mathcal{H} = L^2(X) \), where \( (Hx)(t) = tx(t) \) and \( (Jx)(t) = -i \int_s^t (s-t)^{-1} x(s) \, ds \), the integral regarded as a Cauchy principal value. Then \( T \) is completely hyponormal, \( \sigma(T) = X \times [-1, 1] \), and \( \text{Re} \sigma(T) = X \); cf. Clancey and Putnam [3], p. 452.

Next, suppose that \( H = \text{Re} \ T \) where \( T \) is completely hyponormal. Then \( \sigma(T) \cap D \) has positive planar measure whenever \( D \) is an open disk for which \( \sigma(T) \cap D \) is not empty; see Putnam [15], p. 324. Since \( T \) satisfies (6.2), it is clear that \( \sigma(H) \cap I \) has positive linear measure whenever \( I \) is an open interval for which \( \sigma(H) \cap I \) is not empty. This completes the proof of Theorem 4.

As was noted in §1, a necessary and sufficient condition on a compact set of the plane in order that it be the spectrum of a completely \( G_1 \) operator is not known. Also, we do not have an analogue of Theorem 4. However, it is possible to prove the following

**Theorem 5.** In order that a compact set \( X \) of the real line be the spectrum of the real part of a completely \( G_1 \) operator \( T \) it is necessary that \( X \) be uncountable (equivalently, that \( X \) contain a perfect set).
Proof. In view of (6.1) it is clear that if \( T \) is any \( G_x \) operator and if \( X = \sigma(\text{Re} T) \) then \( \sigma(T) \) is contained in the set consisting of all lines \( \{ z : \text{Re} z = c \} \) where \( c \in X \). Further, since \( T \) of the theorem is completely \( G_1 \), then \( \{ z : \text{Re} z = c \} \cap \sigma(T) \) is empty whenever \( c \) is an isolated point of \( X \), as can be seen from (6.1) and Stampfli’s result ([22], [23]) mentioned in §1. Consequently, \( \sigma(T) \) is contained in the union of lines \( \{ z : \text{Re} z = c \} \) where \( c \in X' \), the first derived set of \( X \). As above, no point of \( \sigma(T) \) can lie on \( \{ z : \text{Re} z = c \} \) if \( z \) is an isolated point of \( X' \), that is if \( c \not\in X'' \). It follows as in the proof of Theorem 2 that if \( \gamma \) is the least ordinal (necessarily of finite or denumerable cardinality) with the property that \( X^{(\gamma)} = X^{(\gamma+1)} \) then necessarily \( \sigma(T) \) is contained in the union of lines \( \{ z : \text{Re} z = c \} \) with \( c \in X^{(\gamma)} \). Consequently, \( X^{(\gamma)} \neq \emptyset \), hence is perfect, and the proof of Theorem 5 is complete.

REMARKS. In Theorem 5 it is possible that \( X \) contains some isolated points. One need only consider the example mentioned at the beginning of this section illustrating that (6.1) may be a proper inclusion with \( T \) completely \( G_1 \). We do not know whether the condition of Theorem 5 on \( X \) is also sufficient, that is, whether any uncountable compact set of the real line must be the spectrum of the real part of some completely \( G_1 \) operator.

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