A COMMUTATIVE BANACH ALGEBRA OF FUNCTIONS OF GENERALIZED VARIATION

A. M. RUSSELL
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It is known that the space of functions, anchored at $\alpha$, and having bounded variation form a commutative Banach algebra under the total variation norm. We show that functions of bounded $k$th variation also form a Banach algebra under a norm defined in terms of the total $k$th variation.

1. Introduction. Let $BV_1[a, b]$ denote the space of functions of bounded variation on the closed interval $[a, b]$, and denote the total variation of $f$ on that interval by $V_1(f)$ or $V_1(f; a, b)$. If

$$BV_1[a, b] = \{ f; V_1(f) < \infty, f(a) = 0 \},$$

then it is a well known result that $BV_1[a, b]$ is a Banach space under the norm $\| \cdot \|_1$, where $\| f \|_1 = V_1(f)$. What appears to be less well known is that, using pointwise operations, $BV_1[a, b]$ is a commutative Banach algebra with a unit under $\| \cdot \|_1$ — see for example [1] and Exercise 17.35 of [2].

In [4] it was shown that $BV_k[a, b]$ is a Banach space under the norm $\| \cdot \|_k$, where

$$(1) \quad \| f \|_k = \sum_{s=0}^{k-1} |f^{(s)}(a)| + V_k(f; a, b),$$

and where the definition of $V_k(f; a, b) \equiv V_k(f)$ can be found in [3]. The subspace

$$BV_k^*[a, b] = \{ f; f \in BV_k[a, b], f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0 \}$$

is clearly also a Banach space under the norm $\| \cdot \|_k^*$, where

$$(2) \quad \| f \|_k^* = \alpha_k V_k(f),$$

and $\alpha_k = 2^{k-1}(b - a)^{k-1}(k - 1)!$.

If we define the product of two functions in $BV_k^*[a, b]$ by pointwise multiplication, then we show, in addition, that $BV_k^*[a, b]$ is a commutative Banach algebra under the norm given in (2). It is obvious that $BV_k^*[a, b]$ is commutative, so our main programme now is to show that if $f$ and $g$ belong to $BV_k^*[a, b]$, then so does $fg$, and that

$$V_k(fg) \leq 2^{k-1}(k - 1)! (b - a)^{k-1} V_k(f) V_k(g), \quad k \geq 1.$$
We accept the case $k = 1$ as being known, so restrict our discussion to $k \geq 2$. Because the same procedure does not appear to be applicable to the cases $k = 2$ and $k \geq 3$, we present different treatments for these cases.

In order to achieve the stated results it was found convenient to work with two definitions of bounded $k$th variation, one defined with quite arbitrary subdivisions $a = x_0, x_1, \ldots, x_n = b$ of $[a, b]$, and the other using subdivisions in which all sub-intervals are of equal length. If we call the two classes of functions so obtained $BV_k[a, b]$ and $\overline{BV}_k[a, b]$ respectively, then we show that provided we restrict our functions to being continuous, then these classes are identical. More specifically, if we denote $C[a, b]$, $BV_k[a, b]$, and $\overline{BV}_k[a, b]$ by $C$, $BV_k$ and $\overline{BV}_k$ respectively, then we show that

$$ C \cap BV_k = \overline{BV}_k. $$

2. Notation and preliminaries.

**DEFINITION 1.** We shall say that a set of points $x_0, x_1, \ldots, x_n$ is a $\pi$-subdivision of $[a, b]$ when $a \leq x_0 < x_1 < \cdots < x_n = b$.

**DEFINITION 2.** If $h > 0$, then we will denote by $\pi_h$ a subdivision $x_0, x_1, \ldots, x_n$ of $[a, b]$ such that $a = x_0 < x_1 < \cdots < x_n \leq b$, where $x_i - x_{i-1} = h$, $i = 1, 2, \ldots, n$, and $0 \leq b - x_n \leq h$.

Before introducing the two definitions of bounded $k$th variation we need the definition and some properties of $k$th divided differences, and for this purpose we refer the reader to [3]. In addition, we make use of the difference operator $\Delta_k$ defined by

$$ \Delta_k f(x) = f(x + h) - f(x), $$

and

$$ \Delta_k^* f(x) = \Delta_k^*[\Delta_k^{-1} f(x)]. $$

**DEFINITION 3.** The total $k$th variation of a function $f$ on $[a, b]$ is defined by

$$ V_k(f; a, b) = \sup_{\pi} \sum_{i=0}^{n-k} (x_{i+k} - x_i) |Q_k(f; x_i, \ldots, x_{i+k})|. $$

If $V_k(f; a, b) < \infty$, we say that $f$ is of bounded $k$th variation on $[a, b]$, and write $f \in BV_k[a, b]$.

**DEFINITION 4.** If $f$ is continuous on $[a, b]$, then we define the total $k$th variation of $f$ on $[a, b]$ (restricted form) by
\[
\tilde{V}_k(f; a, b) = \sup_{\pi_h} \sum_{i=0}^{n-1} \left| \frac{A_i f(x_i)}{h^{k-1}} \right|
\]

If \( \tilde{V}_k(f; a, b) < \infty \), we say that \( f \) is of restricted bounded \( k \)th variation on \([a, b]\), and write \( f \in BV_k[a, b] \).

As before, we will usually write \( V_k(f) \) and \( \tilde{V}_k(f) \) for \( V_k(f; a, b) \) and \( \tilde{V}_k(f; a, b) \) respectively.

We now show that \( C \cap BV_k = BV_k \), and point out at this stage that the restriction to continuous functions is not nearly as severe as it first may appear, because functions belonging to \( BV_k[a, b] \), when \( k \geq 2 \), are automatically continuous — see Theorem 4 of [3].

**Lemma 1.** Let \( I_1, I_2, \ldots, I_n \) be a set of \( n \) adjoining closed intervals on the real line having lengths \( p_1/q_1, p_2/q_2, \ldots, p_n/q_n \) respectively, where \( p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n \) are positive integers. Then it is possible to subdivide the intervals \( I_1, I_2, \ldots, I_n \) into sub-intervals of equal length.

The proof is easy and will be omitted.

**Lemma 2.** If \( k \geq 1 \), then \( C \cap BV_k \subset BV_k \), using abbreviated notation.

**Proof.** This is easy and will not be included.

**Lemma 3.** If \( k \geq 1 \), then \( C \cap BV_k \supset BV_k \).

**Proof.** Let us suppose that \( f \) is continuous, belongs to \( BV_k[a, b] \), but \( f \not\in BV_k[a, b] \). Then for an arbitrarily large number \( K \), and an arbitrarily small positive number \( \varepsilon \), there exists a subdivision \( \pi_1(x_0, x_1, \ldots, x_n) \) of \([a, b]\) such that

\[
S_{x_1} = \sum_{i=0}^{n-k} (x_{i+k} - x_i) |Q_k(f; x_i, x_{i+1}, x_{i+2})| > K + \varepsilon.
\]

If not all the lengths \( (x_{i+1} - x_i) \), \( i = 0, 1, \ldots, n-1 \) are rational, then because \( f \) is continuous we can obtain a subdivision \( \pi_2(y_0, y_1, \ldots, y_n) \) of \([a, b]\) in which all the lengths \( (y_{i+1} - y_i) \), \( i = 0, 1, \ldots, n-1 \) are rational, and such that \( |S_{x_1} - S_{x_2}| < \varepsilon \), \( S_{x_2} \) being the approximating sum of \( V_k(f; a, b) \) corresponding to the \( \pi_2 \) subdivision. Consequently,

\[
S_{x_2} \geq S_{x_1} - |S_{x_1} - S_{x_2}| > K.
\]

In the \( \pi_2 \) subdivision, all sub-intervals have rational length, so we can apply Lemma 1 to obtain a \( \pi_h \) subdivision of \([a, b]\) in which each
sub-interval has length \( h \). If \( S_{x_h} \) is the corresponding approximating sum of \( \bar{V}_k(f; a, b) \), then it follows from Theorem 3 of [3] that

\[
\frac{1}{(k-1)!} S_{x_h} \geq S_{\pi_z} > K,
\]

since for any \( \pi_h \) subdivision, and each \( i = 0, 1, \ldots, n - k \),

\[
\frac{\Delta_h^i f(x_i)}{h^{k-1}} = (k-1)! (x_{i+k} - x_i) Q(f; x_i, \ldots, x_{i+k}).
\]

Thus \( S_{x_h} > (k - 1)! K \), and this is a contradiction to the assumption that \( f \in \overline{BV}_k[a, b] \). Hence \( f \in \overline{BV}_k[a, b] \), and so \( \overline{BV}_k \subset C \cap BV_k \).

**Theorem 1.** If \( k \geq 1 \), then \( C \cap BV_k = \overline{BV}_k \); and if \( f \) is a continuous function on \([a, b]\), then

\[
(3) \quad \bar{V}_k(f; a, b) = (k - 1)! V_k(f; a, b), \quad k \geq 1.
\]

**Proof.** The first part follows from Lemmas 2 and 3. For the second part we first observe that

\[
(4) \quad \bar{V}_k(f; a, b) \leq (k - 1)! V_k(f; a, b).
\]

Let \( \varepsilon > 0 \) be arbitrary. Then there exists a \( \pi_1 \) subdivision of \([a, b]\) and the corresponding approximating sum \( S_{\pi_1} \) to \( V_k(f; a, b) \) such that

\[
S_{\pi_1} > V_k(f; a, b) - \frac{\varepsilon}{2(k - 1)!}.
\]

If not all the sub-intervals of \( \pi_1 \) have rational lengths, then we can proceed as in Lemma 3 to obtain a \( \pi_h \) subdivision of \([a, b]\) in which all sub-intervals are of equal length \( h \). Then, if \( S_{x_h} \) is the corresponding approximating sum to \( \bar{V}_k(f; a, b) \), we can show that

\[
\frac{1}{(k-1)!} S_{x_h} \geq S_{\pi_1} - \frac{\varepsilon}{2(k - 1)!} > V_k(f; a, b) - \frac{\varepsilon}{(k - 1)!}.
\]

Consequently,

\[
\bar{V}_k(f; a, b) \geq S_{x_h} > (k - 1)! V_k(f; a, b) - \varepsilon,
\]

from which it follows that \( \bar{V}_k(f; a, b) \geq (k - 1)! V_k(f; a, b) \). This inequality together with (4) gives (3).
Lemma 4. If $f$ and $g$ are any two real valued functions defined on $[a, b]$, $h > 0$ and $a \leq x < x + kh \leq b$, then

$$
\Delta^k_h[f(x)g(x)] = f(x + kh)\Delta^k_hg(x) + \left(\begin{array}{c} k \\ 1 \end{array}\right)\Delta^k_hf(x + (k-1)h)\Delta^{k-1}_hg(x) + \cdots
$$

(5)

$$
+ \left(\begin{array}{c} k \\ s \end{array}\right)\Delta^k_hf(x + (k-s)h)\Delta^{k-s}_hg(x) + \cdots + \Delta^k_hf(x)\Delta^k_hg(x)
$$

$$
= \sum_{s=0}^{k} \left(\begin{array}{c} k \\ s \end{array}\right)\Delta^k_hf(x + (k-s)h)\Delta^{k-s}_hg(x), \text{ where } \Delta^s_hg(x) = g(x).
$$

Proof. The proof by induction is straightforward and will not be included.

Lemma 5. If $f$ and $g$ belong to $BV_k[a, b]$, $k \geq 1$, then $fg \in BV_k[a, b]$.

Proof. The result for $k = 1$ is well known, so we assume that $k \geq 2$, in which case $f$ and $g$ are continuous in $[a, b]$. Consequently, in view of Theorem 1, there will be no loss of generality in working with equal sub-intervals of $[a, b]$. Using (5) we have, suppressing the "$h$" in "$\Delta^k_h$",

$$
\frac{\Delta^k_h[f(x)g(x)]}{h^{k-1}} = \frac{f(x + kh)\Delta^k_hg(x)}{h^{k-1}} + \cdots + \left(\begin{array}{c} k \\ s \end{array}\right)\frac{\Delta^k_hf(x + (k-s)h)\Delta^{k-s}_hg(x)}{h^s} + \cdots + \frac{\Delta^{k-1}_hf(x + h)\Delta^1_hg(x) + \Delta^k_hf(x)}{h^{k-1}}g(x).
$$

(6)

It follows from Theorem 4 of [3] that

$$
\frac{\Delta^s_hf(x + (k-s)h)}{h^s}, \ s = 0, 1, \ldots, k - 1
$$

is uniformly bounded. Hence we can conclude from (6) that $fg \in BV_k[a, b]$ by summing over any $\pi_k$ subdivision of $[a, b]$, and noting that $f$ and $g$ belong to $BV_k[a, b] \subset BV_{k-1}[a, b] \subset \cdots \subset BV_1[a, b]$ — see Theorem 10 of [3]. Since $fg$ is continuous it follows from Theorem 1 that $fg \in BV_k[a, b]$.

3. Main results. We now make an application of Theorem 1 to obtain a relationship between $V_{k-1}(f)$ and $V_k(f)$ when $f \in BV_k^*[a, b]$.

Theorem 2. If $f \in BV_k^*[a, b]$, $k \geq 2$, then

$$
V_{k-1}(f) \leq (k - 1)(b - a)V_k(f),
$$

(7)
or
\[ \bar{V}_{k-1}(f) \leq (b - a) \bar{V}_k(f). \]

**Proof.** It follows from Theorem 10 of [3] that \( f \in BV_{k-1}^*[a, b] \), so \( V_{k-1}(f) < \infty \). We now establish the inequality. Since \( f \in BV_{k-1}^*[a, b] \), \( f^{(k-1)}(a) = 0 \). Hence for any \( \varepsilon > 0 \), we can choose a \( \pi_h \) subdivision of \([a, b]\) such that

\[ \left| \frac{A_h^{k-1}f(a)}{h^{k-1}} \right| < \frac{\varepsilon}{(b-a)}. \]

There is no loss of generality in choosing such a subdivision in view of Theorem 3 of [3] which tells us that the approximating sums for total \( k \)th variation are not decreased by the addition of extra points of subdivision. Accordingly, let \( a = x_0, x_1, \ldots, x_n \leq b \) be a \( \pi_h \) subdivision of \([a, b]\) with property (8). Then, suppressing the "\( h \)" in "\( A_h^{k-1} \)" and "\( A_h^k \)" , we obtain

\[
\sum_{i=0}^{n-k+1} |A_h^{k-1}f(x_i)| = \sum_{i=0}^{n-k+1} \left| \sum_{s=1}^{i} [A_h^{k-1}f(x_s) - A_h^{k-1}f(x_{s-1})] + A_h^{k-1}f(x_0) \right|
\]

\[
= \sum_{i=0}^{n-k+1} \left| \sum_{s=1}^{i} A_h^k f(x_{s-1}) + A_h^{k-1}f(x_0) \right|
\]

\[
\leq \sum_{i=0}^{n-k+1} \left| \sum_{s=1}^{i} A_h^k f(x_{s-1}) \right| + \sum_{i=0}^{n-k+1} \left| A_h^{k-1}f(x_0) \right|
\]

\[
\leq n \sum_{i=1}^{n-k} |A_h^k f(x_{s-1})| + n |A_h^{k-1}f(x_0)|
\]

\[
\leq (b - a) \sum_{i=1}^{n-k} \left| \frac{A_h^k f(x_{s-1})}{h} \right| + (b - a) \left| \frac{A_h^{k-1}f(x_0)}{h} \right| .
\]

Therefore, dividing both sides by \( h^{k-2} \), we obtain

\[
\sum_{i=0}^{n-k+1} \left| \frac{A_h^{k-1}f(x_i)}{h^{k-2}} \right| \leq (b - a) \sum_{i=1}^{n-k} \left| \frac{A_h^k f(x_{s-1})}{h^{k-1}} \right| + (b - a) \left| \frac{A_h^{k-1}f(x_0)}{h^{k-1}} \right|
\]

\[
\leq (b - a) \bar{V}_k(f) + \varepsilon ,
\]

from which it follows that

\[ \bar{V}_{k-1}(f) \leq (b - a) \bar{V}_k(f). \]

Consequently, using (2) we obtain

\[ V_{k-1}(f) \leq (k - 1)(b - a)V_k(f), \]

as required.

**Corollary.** Let \( p \) be an integer such that \( 1 \leq p < k \). If \( f \in BV_p^*[a, b] \), then \( f \in BV_p^*[a, b] \), and
\[ V_p(f) \leq p(p + 1) \cdots (k - 1)(b - a)^{k-p} V_k(f) , \]
or\[
\bar{V}_p(f) \leq (b - a)^{k-p} \bar{V}_k(f) .
\]

**Proof.** The proof follows from repeated applications of (7), and Theorem 10 of [3].

We now proceed to obtain a relationship between \( V_k(fg) \), \( V_k(f) \) and \( V_k(g) \) when \( f \) and \( g \) belong to \( BV^*_\mathcal{B}[a, b] \). It appears convenient to treat the cases \( k = 2 \), and \( k \geq 3 \) separately, so we begin by considering \( k = 2 \).

**Theorem 3.** If \( f \) and \( g \) belong to \( BV^*_\mathcal{B}[a, b] \), then \( fg \in BV^*_\mathcal{B}[a, b] \), and
\[
V_2(fg) \leq V_2(f)V_1(g) + V_1(f)V_2(g) \\
\leq 2(b - a)V_2(f)V_2(g) .
\]

**Proof.** There is no loss of generality in considering \( \pi_k \) subdivisions of \([a, b]\). Let \( a = x_0, x_1, \ldots, x_n \) be such a subdivision. Then, noting that \( f(a) = 0 = g(a) \) when \( f, g \in BV^*_\mathcal{B}[a, b] \), and writing \( f(x_{i+1}) - f(x_i) = \Delta f(x_i) \), we obtain for \( i \geq 1 \),
\[
\Delta^2 f(x_i)g(x_i) = \Delta[\Delta f(x_i)g(x_i)] \\
= \Delta[f(x_{i+1})\Delta g(x_i) + (\Delta f(x_i))g(x_i)] \\
= \Delta \left( \sum_{s=0}^i \Delta f(x_s) \right) \Delta g(x_i) + \Delta f(x_i) \sum_{s=0}^{i-1} \Delta g(x_s) \\
\sum_{s=0}^i \Delta(\Delta f(x_s)\Delta g(x_s)) + \sum_{s=0}^{i-1} \Delta(\Delta f(x_s)\Delta g(x_s)) \\
= \sum_{s=0}^i [\Delta f(x_{s+1})\Delta^2 g(x_i) + \Delta^2 f(x_s)\Delta g(x_i)] \\
+ \sum_{s=0}^{i-1} [\Delta f(x_{i+1})\Delta^2 g(x_s) + \Delta^2 f(x_i)\Delta g(x_s)] .
\]

Therefore, noting that the last summation in (12) is zero when \( i = 0 \), we have
\[
\sum_{i=0}^{n-2} |\Delta^2 f(x_i)g(x_i)| \leq \sum_{i=0}^{n-2} [ |\Delta f(x_i)| + \cdots + |\Delta f(x_{i+1})| ] |\Delta^2 g(x_i)| \\
+ \sum_{i=0}^{n-2} [ |\Delta^2 f(x_i)| + \cdots + |\Delta^2 f(x_{i+1})| ] |\Delta g(x_i)| \\
+ \sum_{i=0}^{n-2} |\Delta f(x_{i+1})| [ |\Delta^2 g(x_i)| + \cdots + |\Delta^2 g(x_{i-1})| ] \\
+ \sum_{i=1}^{n-2} |\Delta^2 f(x_{i+1})| [ |\Delta g(x_i)| + \cdots + |\Delta g(x_{i-1})| ] ,
\]
which after some re-arrangement is equal to
\[
\left( \sum_{i=1}^{n-1} |\Delta f(x_i)| \right) \left( \sum_{i=0}^{n-2} |\Delta^2 g(x_i)| \right) + \left( \sum_{i=0}^{n-2} |\Delta^2 f(x_i)| \right) \left( \sum_{i=0}^{n-2} |\Delta g(x_i)| \right).
\]
Therefore, dividing by \( h \), and using Definition 4, we observe that \( fg \in BV^*_2[a, b] \), and obtain
\[
\tag{13}
\bar{V}_2(fg) \leq \bar{V}_1(f) \bar{V}_2(g) + \bar{V}_2(f) \bar{V}_1(g),
\]
or
\[
\tag{14}
V_2(fg) \leq V_1(f) V_2(g) + V_2(f) V_1(g),
\]
using Theorem 1.

To complete the proof we employ (7) with \( k = 2 \).

We are now in a position to consider the general case \( k \geq 3 \) for which we adopt a different procedure. When \( k \geq 3 \) we make use of the fact that \( f^{(k-2)} \in BV^*_2[a, b] \), and consequently exists throughout \([a, b]\), and is in fact absolutely continuous in that interval.

**Theorem 4.** Let \( f \) and \( g \) belong to \( BV^*_k[a, b] \) when \( k \geq 3 \). Then \( fg \in BV^*_k[a, b] \), and
\[
\tag{14}
\bar{V}_k(fg) \leq 2^{k-1}(b - a)^{k-1} \bar{V}_k(f) \bar{V}_k(g),
\]
or
\[
\tag{15}
V_k(fg) \leq 2^{k-1}(b - a)^{k-1}(k - 1)! \, V_k(f) V_k(g).
\]

**Proof.** We first observe from Lemma 5 that \( fg \in BV^*_k[a, b] \). It follows from Theorems 2 and 8 of [5] that
\[
\bar{V}_k(fg) = \bar{V}_k((fg)^{(k-2)})
\]
\[
= \bar{V}_2 \left( \sum_{s=0}^{k-2} \binom{k-2}{s} f^{(k-s-2)} g^{(s)} \right)
\]
\[
\leq \sum_{s=0}^{k-2} \binom{k-2}{s} \bar{V}_2(f^{(k-s-2)} g^{(s)})
\]
\[
\leq 2(b - a) \sum_{s=0}^{k-2} \binom{k-2}{s} \bar{V}_2(f^{(k-s-2)}) \bar{V}_s(g^{(s)}), \text{ using (11)}
\]
\[
= 2(b - a) \sum_{s=0}^{k-2} \binom{k-2}{s} \bar{V}_{k-s}(f) \bar{V}_{s+2}(g)
\]
\[
\leq 2(b - a) \sum_{s=0}^{k-2} \binom{k-2}{s} (b - a)^s \bar{V}_k(f) \cdot (b - a)^{s+2} \bar{V}_k(g),
\]
using (10)
$$= 2(b - a)^{k-1} V_k(f) V_k(g) \sum_{s=0}^{k-2} \left( \begin{array}{c} k - 2 \\ s \end{array} \right)$$

$$= 2^{k-1}(b - a)^{k-1} V_k(f) V_k(g),$$ as required for (14).

To obtain (15) we employ (3).

Combining Theorems 3 and 4 gives

**Theorem 5.** If $f$ and $g$ belong to $BV^*_k[a, b]$, $k \geq 1$, then $fg \in BV^*_k[a, b]$, and

$$V_k(fg) \leq \alpha_k V_k(f) V_k(g),$$

where $\alpha_k = 2^{k-1}(k - 1)! (b - 1)^{k-1}$.

Our final theorem is now apparent.

**Theorem 6.** If $k$ is a positive integer, then $BV^*_k[a, b]$ is a commutative Banach algebra under the norm $\| \cdot \|_k^*$, where

$$\|f\|_k^* = \alpha_k V_k(f),$$

and $\alpha_k = 2^{k-1}(k - 1)! (b - a)^{k-1}$.

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Received June 1, 1978.

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