ON THE SIGNATURE OF GRASSMANNIANS

PATRICK SHANAHAN
ON THE SIGNATURE OF GRASSMANNIANS

Patrick Shanahan

1. Introduction. Let \( G_{n,k} \) denote the manifold of linear subspaces of \( \mathbb{R}^n \) of dimension \( k > 0 \). Then \( G_{n,k} \) is compact and has dimension \( k(n-k) \). When \( n \) is even \( G_{n,k} \) is orientable and we may consider the topological invariant \( \text{Sign}(G_{n,k}) \). The cohomology algebra of \( G_{n,k} \) over \( \mathbb{R} \) was determined by Borel in [3] and thus in principle the problem of computing \( \text{Sign}(G_{n,k}) \) is a problem in linear algebra. In practice this is very awkward, and it is the purpose of this paper to compute this invariant by a simpler method:

**THEOREM.** The signature of \( G_{n,k} \) is zero except when \( n \) and \( k \) are even and \( k(n-k) \equiv 0 \pmod{8} \). In this case (with a conventional orientation)

\[
\text{Sign}(G_{n,k}) = \begin{bmatrix} \frac{n}{4} \\ \frac{k}{4} \end{bmatrix}.
\]

**REMARK.** When \( n \) is odd, \( G_{n,k} \) is nonorientable and \( \text{Sign}(G_{n,k}) \) is not defined; however, for odd \( n \) \( \text{Sign}(\widetilde{G}_{n,k}) = 0 \), where \( \widetilde{G}_{n,k} \) is the orientation covering of \( G_{n,k} \).

2. The Atiyah-Bott formula. We recall a few definitions. Let \( X \) be a compact orientable manifold of dimension \( 4l \). The signature of \( X \) is defined by

\[
\text{Sign}(X) = \dim H^+ - \dim H^-,
\]

where \( H^{2l}(X; \mathbb{R}) = H^+ \oplus H^- \) is a decomposition of the middle-dimensional cohomology of \( X \) into subspaces on which the cup-product form \( B(x, y) = \langle x \cup y, X \rangle \) is positive definite and negative definite, respectively. When \( \dim X \) is not divisible by 4 one defines \( \text{Sign} X = 0 \).

More generally, let \( f: X \to X \) be a mapping of \( X \) into itself. When the decomposition of \( H^{2l}(X; \mathbb{R}) \) is invariant under \( f \) one defines

\[
\text{Sign}(f) = \text{tr} f^* | H^+ - \text{tr} f^* | H^-,
\]

where \( f^*: H^{2l}(X; \mathbb{R}) \to H^{2l}(X; \mathbb{R}) \) is the homomorphism induced by \( f \). \( \text{Sign}(f) \) is then independent of the choice of \( H^+ \) and \( H^- \). When \( f \) is homotopic to the identity mapping one obviously has \( \text{Sign}(f) = \text{Sign}(X) \).
Now suppose that $X$ is an oriented Riemannian manifold. If $f: X \to X$ is an orientation preserving isometry, then at each isolated fixed point $p$ of $f$ the differential $df_p: T_pX \to T_pX$ is an orthogonal transformation with determinant 1. Let $\theta_1(p), \ldots, \theta_{2l}(p)$ be the $2l$ rotation angles associated with the eigenvalues of $df_p$. When the fixed point set of $f$ consists of isolated points one has the formula of Atiyah and Bott ([1], p. 473):

$$\text{Sign}(f) = (-1)^l \sum_{p \text{ fixed}} \prod_{v=1}^{2l} \text{ctn} \left( \frac{\theta_v(p)}{2} \right).$$

We will apply this formula to a certain mapping $f: G_{n,k} \to G_{n,k}$.

**Remark.** When $f$ is an element of a compact group acting on $X$ (and this will be the situation in our application) the formula above is also a consequence of the $G$-signature theorem of Atiyah and Singer. (See [1], p. 582 or [6], §18.)

For simplicity of notation we confine our attention to the case $n = 2s$, $k = 2r$; the remaining cases can be dealt with by minor adjustments in the argument.

Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation which rotates the $i$th coordinate plane $P_i = \text{span}\{e_{2i-1}, e_{2i}\}$ ($i = 1, 2, \ldots, s$) through the angle $\alpha_i$, where $0 < \alpha_i < \pi$. The transformation $F$ induces a smooth mapping $f: G_{n,k} \to G_{n,k}$ which is clearly homotopic to the identity mapping. If $P_I$ denotes the $k$-plane

$$P_I = P_{i_1} \oplus \cdots \oplus P_{i_r},$$

where $I = (i_1, \ldots, i_r)$ is a multi-index with $i_1 < i_2 < \cdots < i_r$ and $1 \leq i_r \leq s$, then $f(P_I) = P_I$.

**Proposition 2.1.** If the angles $\alpha_i$ are all distinct, then the points $P_I \in G_{n,k}$ are the only fixed points of $f$.

**Proof.** Let $W$ be a $k$-dimensional linear subspace of $\mathbb{R}^n$ not equal to any $P_i$. By regarding $W$ as the row space of a matrix in reduced row echelon form one sees that there exists a $v \in W$ whose orthogonal projections $v_i$ on $P_i$ are nonzero for at least $r + 1$ indices $i$.

If $F(W) = W$, the vectors $v, F(v), \ldots, F^k(v)$ all belong to $W$, and hence there is a nontrivial relation

$$\sum_{v=0}^{v=k} a_v F^v(v) = 0. $$

But this implies
for all $i$. Writing $X_3- = \cos (\alpha, ) + i \sin (\alpha, )$ it follows that the $k$-degree polynomial $q(x) = a_0 + a_1x + \cdots + a_kx^k$ has zeros $\lambda_i$ and $\bar{\lambda}_i$ for each of the $r+1$ indices $i$ for which $v_i$ is nonzero. Since the $\alpha_i$ are all distinct, the coefficients $a_i$ must all be zero, which contradicts our assumption. Thus when $F(W) = W$, the subspace $W$ must coincide with one of the subspaces $P_i$.

3. The Normal angles $\theta,(p)$. We wish to show that with respect to an appropriate metric on $G_{n,k}$ the mapping $f$ is an isometry, and then compute the normal angles $\theta,(p)$ at the fixed points $p$ of $f$. We begin with some remarks about the differentiable structure on $G_{n,k}$.

The smooth structure on $G_{n,k}$ may be defined by identifying $G_{n,k}$ with the left coset space $G/H$, where $G = O(n)$ is the orthogonal group and $H = O(k) \times O(n-k)$ is the closed subgroup of orthogonal transformations which take span $\{e_i, \ldots, e_k\}$ into itself. The space $O(n)$ may be regarded as the space of orthogonal $n \times n$ matrices (and hence as a subspace of $R^{n^2}$), or, equivalently, as the space of orthonormal $n$-frames $a = (a_1, \ldots, a_n)$ in $R^n$. We denote the image of an element $a \in G$ under the natural projection $\pi: G \to G/H$ by $\bar{a}$, and the image of a tangent vector $v \in T_aG$ under $d\pi: T_aG \to T_{\bar{a}}G/H$ by $\bar{v}$.

The elements of the tangent space $T_aG$ are determined by smooth curves passing through the identity matrix $e$. By differentiating the relation $aa^t = e$ one obtains the usual identification of $T_aG$ with the space of skew-symmetric $n \times n$ matrices. As a basis for $T_aG$ we may take the set $\{b_{rs} \mid r < s\}$ of matrices $b_{rs}$ having $-1$ in column $s$, row $r$, $1$ in column $r$, and row $s$, and $0$ everywhere else. The ordering $\{b_{12}, b_{13}, b_{23}, b_{14}, b_{24}, \ldots\}$ then defines a standard orientation for $G$. More generally, the system of matrices $\{ab_{rs}\}$ may be taken as a basis for the tangent space $T_aG$ at an arbitrary $a \in G$.

To obtain an oriented basis for the tangent space $T_{\bar{a}}G/H$ we simply restrict ourselves to vectors in $T_aG$ which are orthogonal, as vectors in $R^{n^2}$, to $T_{\bar{a}}(aH)$. It is easily shown that the vectors $ab_{ij}$ with $1 \leq i \leq k$ and $k + 1 \leq j \leq n$ provide such a system. The coherence of the orientations will follow from the proof of Proposition 3.1. Note that even when $a$ and $a'$ represent the same coset in $G/H$, the bases $\{ab_{ij}\}$ and $\{a'b_{ij}\}$ will in general be different bases.

These facts all have simple interpretations in terms of curves in $O(n)$ and $G_{n,k}$. For example, the tangent vector $ab_{ij}$ may be viewed as the infinitesimal motion of the $k$-plane span $\{a_i, \ldots, a_k\}$
towards its orthogonal complement obtained by rotating the vector 
$a_i$ toward complementary vector $a_j$.

**Proposition 3.1.** There is a unique Riemannian metric on $G_{n,k}$
for which the standard bases \{ab_{ij}\} are all orthonormal. The
mapping $f: G_{n,k} \rightarrow G_{n,k}$ is an orientation preserving isometry with
respect to this metric. Moreover, the system of normal angles \{$\theta_\nu(p)$\}
is the same at each fixed point $p$ of $f$.

**Proof.** To prove the first assertion it will be enough to show
that for arbitrary $n$-frames $a$ and $a'$ in $SO(n)$ the matrix of transition
between the bases \{ab_{ij}\} and \{a'b_{ij}\} is orthogonal. Let $a' = ah$, where
$h \in O(k) \times O(n - k)$. Then $a' b_{ij} = a' b_{ij} h^{-1} = ah b_{ij} h^{-1}$.

Let $hb_{ij} h^{-1} = \sum_{\nu, \mu} q_{ij,\nu,\mu} b_{\nu,\mu}$. Clearly $q = [q_{ij,\nu,\mu}]$ is the required
transition matrix. Writing

$$h = \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}, \quad E \in O(k), \; F \in O(n - k),$$

we obtain $q_{ij,\nu,\mu} = e_{\nu,\mu} f_{\nu,\mu}$, that is, $q = E \otimes F$. Hence

$$\sum_{i,j} q_{ij,\nu,\mu} q_{ij,\nu',\mu'} = \sum_{i,j} e_{\nu,\mu} f_{\nu,\mu} e_{\nu',\mu'} f_{\nu',\mu'}$$

$$= \sum_{i,j} e_{\nu,\mu} e_{\nu',\mu'} f_{\nu,\mu} f_{\nu',\mu'} = \delta_{\nu,\mu} \delta_{\mu,\mu'},$$

which proves that $qq' = e$. Moreover, it follows from $\det q = (\det E)^{n-k}(\det F)^k = 1$ that the various bases are coherently oriented.

To see that $f$ is an isometry it is enough to observe that

$$df_\alpha(ab_{ij}) = \overline{F(a)b_{ij}}.$$

Finally, let $p = \bar{a}$ be any fixed point of $f$. We will compare
the normal angles at $\bar{a}$ with those $\bar{e}$.

Denoting $F(e)$ by $c$ we have

$$df_\bar{a}(b_{ij}) = \overline{cb_{ij}} = \overline{cb_{ij} c^{-1}},$$

since $c \in O(k) \times O(n - k)$. On the other hand, $f(\bar{a}) = \bar{a}$ implies that
$F(a) = ah$ for some $h \in O(k) \times O(n - k)$. Thus $ca = ah$ and hence

$$df_\bar{a}(ab_{ij}) = \overline{F(a)b_{ij}} = \overline{ab_{ij} a^{-1} ca}.$$

Writing out the matrices $D$ and $D'$ of $df_\bar{a}$ and $df_\bar{a}$ with respect
to the appropriate bases we have

$$\overline{cb_{ij} c^{-1}} = df_\bar{a}(b_{ij}) = \sum_{\nu,\mu}^{} d_{ij,\nu,\mu} b_{\nu,\mu},$$

$$\overline{cab_{ij} a^{-1} c^{-1} a} = df_\bar{a}(ab_{ij}) = \sum_{\nu,\mu}^{} d'_{ij,\nu,\mu} ab_{\nu,\mu}.$$
Let \( ab_\varphi a^{-1} = \sum_{i,j} m_{ij} b_{ij} \), and \( m = [m_{ij}] \). Then (2) becomes
\[
\sum_{i,j} m_{ij} b_{ij} = \sum_{i,j} d'_{ij} m_{ij} b_{ij}.
\]
Substituting (1) we obtain
\[
\sum_{i,j} m_{ij} b_{ij} = \sum_{i,j} d'_{ij} m_{ij} b_{ij}
\]
for each \( i \) and \( j \). Thus \( md = d'm \). Since \( m \) is nonsingular this means that \( d' \) is similar to \( d \), and hence the normal angles of \( f \) at \( p \) are the same as those at \( \vec{e} \).

**Proposition 3.2.** At each fixed point \( p \) of \( f: G_{2s,2r} \to G_{2s,2r} \) the normal angles \( \{\theta(p)\} \) are the \( 2r(s - r) \) angles \( \{\alpha_j \pm \alpha_i\} \) with \( 1 \leq i \leq r \) and \( r + 1 \leq j \leq s \).

**Proof.** It is enough to compute the matrix \( m \) of \( df \) relative to the basis \( \{b_{ij}\} \). Since \( c = F(e) \in O(k) \times O(n - k) \),
\[
df(b_{ij}) = F(e)b_{ij} = cb_{ij} c^{-1}
\]
for \( 1 \leq i \leq r \) and \( r + 1 \leq j \leq s \). Hence, as above, we have
\[
m_{i,j'} = e_{i,i'} e_{j,j'}.
\]
It follows that \( m \) is a sum of disjoint \( 4 \times 4 \) blocks
\[
\begin{bmatrix}
cos(\alpha_j)B - \sin(\alpha_j)B \\
\sin(\alpha_j)B & \cos(\alpha_j)B
\end{bmatrix}
\]
where \( B = \begin{bmatrix} \cos(\alpha_i) & -\sin(\alpha_i) \\ \sin(\alpha_i) & \cos(\alpha_i) \end{bmatrix} \). Each such block is the image of the matrix \( e^{i\alpha_j}B \) under the standard monomorphism \( U(2) \to SO(4) \). Since the eigenvalues of \( e^{i\alpha_j}B \) are \( e^{i(\alpha_j \pm \alpha_i)} \), the proposition follows.

**4. Computation of the signature.** We apply the Atiyah-Bott formula to the mapping \( f: G_{n,k} \to G_{n,k} \) described above. Since \( f \) is homotopic to the identity mapping we obtain
\[
\text{Sign}(G_{n,k}) = (-1)^I \sum_{p \in I} \prod_{i \in I} \cotn(\alpha_j \pm \alpha_i)/2.
\]
Here \( I = (i_1, \ldots, i_r) \) is the multi-index which corresponds to the fixed point \( P_I = P_{i_1} \oplus \cdots \oplus P_{i_r} \) and \( J \) is the complementary multi-index.

With the aid of the formula for the cotangent of a sum the right-hand side may be written in the form
\[
\sum_{p \in I} \prod_{j \in J} \frac{1 - x_j x_i}{x_j - x_i}
\]
where $x_v = \cot^2(\alpha_v/2)$. Since the formula is true for all systems of distinct angles between 0 and $\pi$ (noninclusive), it is true in particular when the angles $\alpha_1, \alpha_2, \cdots$ are taken between 0 and $\pi/2$ and the angles $\alpha_3, \alpha_4, \cdots$ are chosen to be their supplements.

Consider first the case $s$ even, $r$ even. Then the indicated choice of angles gives

\[
x_v = x_i^{-1}, \quad x_i = x_i^{-1}, \quad \ldots, \quad x_s = x_i^{-1}.
\]

For such a choice most of the terms in the sum vanish, since if there exists an $i \in I$ for which $x_j = x_i^{-1}$ for some $j \in J$, then

\[
(1 - x_jx_i)(x_j - x_i)^{-1} = (1 - x_i^{-1}x_i)(x_i^{-1} - x_i)^{-1} = 0.
\]

The only terms which survive are those for which no $x_i^{-1}$ can be an $x_j$; for such $I$, the factors may be grouped in pairs of the form

\[
[(1 - x_jx_i)(x_j - x_i)^{-1}][(1 - x_jx_i^{-1})(x_j - x_i^{-1})^{-1}] = 1,
\]

and to evaluate the sum we need only count the number of such multi-indices $I$. Since these are precisely those multi-indices which are a disjoint union of pairs (odd, odd + 1) the sum in question is $\binom{s/2}{r/2}$.

If $s$ is even and $r$ is odd, some $x_i^{-1}$ must be an $x_j$; thus in this case no terms survive and the sum is 0.

When $s$ is odd $x_s$ is not the inverse of any other $x_v$. For even $r$ the contributing multi-indices are then exactly as in the first case, giving a value of $\binom{(s-1)/2}{r/2}$ for the sum. For odd $r$ the contributing multi-indices are obtained from those already mentioned by adjoining the index $s$. The extra factors then occur in pairs of the form

\[
[(1 - x_jx_i)(x_j - x_i)^{-1}][(1 - x_j^{-1}x_s)(x_j^{-1} - x_s)^{-1}] = 1,
\]

giving a sum of $\binom{(s-1)/2}{(r-1)/2}$.

As for the sign preceding the sum, $(-1)^t = (-1)^{s-r} = 1$ for those cases in which the sum is nonzero.

This completes the proof of the theorem stated at the beginning of the paper.

5. Further remarks.

1. A similar argument may be used to compute the signature of the complex Grassmannian $G_{s,k}(C)$ of complex $k$-dimensional sub-
spaces of $C^*$. The normal angles at a fixed point in this case have the form $\alpha_j - \alpha_i$.

One obtains

$$\text{Sign}(G_{n,k}(C)) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } k(n-k) \text{ even} \\ \left\lceil \frac{k}{2} \right\rceil & \text{if } k(n-k) \text{ odd} \\ 0 & \text{if } k(n-k) \text{ odd} \end{cases}$$

(For a different approach to the computation of $\text{Sign} G_{n,k}(C)$ see Connolly and Nagano [4] (their formula contains a minor error due to a counting mistake).) [Added in proof; see also Mong [5]].

2. The same line of argument used here to compute the signature of $G_{n,k}$ may be used to compute the Euler characteristic $E(G_{n,k})$. The Lefschetz fixed point theorem is used in place of the theorem of Atiyah and Bott, and instead of computing the normal angles $\theta_i(p)$ one need only determine the fixed-point indices $\text{Ind}_p(f)$. Since $f$ is an isometry, these must necessarily be 1. One obtains

$$E(G_{n,k}) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } k(n-k) \text{ even} \\ \left\lceil \frac{k}{2} \right\rceil & \text{if } k(n-k) \text{ odd} \\ 0 & \text{if } k(n-k) \text{ odd} \end{cases}$$

3. The assumption that the angles $\alpha_i$ used in the definition of the transformation $F$ are all distinct was necessary to obtain a mapping $f$ with isolated fixed points. When coincidences $\alpha_{i_1} = \alpha_{i_2} = \cdots$ are permitted the fixed point sets become submanifolds of $G_{n,k}$ of positive dimension. The $G$-signature theorem of Atiyah and Singer (see [2] or [6]) may then be used to obtain information about the normal bundles of these submanifolds.

References


Received October 3, 1978. This paper was written while the author was a visitor at the Mathematical Institute, Oxford.

**College of the Holy Cross**
**Worcester, MA 01610**
Somesh Chandra Bagchi and Alladi Sitaram, *Spherical mean periodic functions on semisimple Lie groups* 241
Billy Joe Ball, *Quasicompactifications and shape theory* 251
Maureen A. Bardwell, *The o-primitive components of a regular ordered permutation group* 261
Peter W. Bates and James R. Ward, *Periodic solutions of higher order systems* 275
Jeroen Bruijning, *A characterization of dimension of topological spaces by totally bounded pseudometrics* 283
Thomas Farmer, *On the reduction of certain degenerate principal series representations of SP(n, C)* 291
Richard P. Jerrard and Mark D. Meyerson, *Homotopy with m-functions* 305
James Edgar Keesling and Sibe Mardešić, *A shape fibration with fibers of different shape* 319
Guy Loupias, *Cohomology over Banach crossed products. Application to bounded derivations and crossed homomorphisms* 333
Rainer Löwen, *Symmetric planes* 367
Alan L. T. Paterson, *Amenable groups for which every topological left invariant mean is invariant* 391
Calvin R. Putnam, *Operators satisfying a G1 condition* 413
Melvin Gordon Rothenberg and Jonathan David Sondow, *Nonlinear smooth representations of compact Lie groups* 427
Werner Rupp, *Riesz-presentation of additive and σ-additive set-valued measures* 445
A. M. Russell, *A commutative Banach algebra of functions of generalized variation* 455
Judith D. Sally, *Superregular sequences* 465
Patrick Shanahan, *On the signature of Grassmannians* 483