EMBEDDING LATTICES INTO LATTICES OF IDEALS

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OF IDEALS

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A lattice $L$ is transferable iff, whenever $L$ can be embedded in the ideal lattice of a lattice $M$, then $L$ can be embedded in $M$. This concept was introduced by the first author in 1965 who also proved in 1966 that in a transferable lattice there are no doubly reducible elements. In fact, he proved that every lattice can be embedded in the ideal lattice of a lattice containing no doubly reducible elements. In a recent paper of the first two authors, the idea emerged that one should study transferability via classes $K$ of lattices with the property that every lattice is embeddable in the ideal lattice of a lattice in $K$. This approach was used to establish that transferable lattices are semi-distributive. This investigation is carried further in this paper. Our main result shows that every lattice can be embedded in the ideal lattice of a lattice satisfying the two semi-distributive properties and two variants of Whitman's condition.

1. Introduction. It was shown by G. Grätzer ([6], [7]) that every transferable lattice $L$ satisfies the condition

$(X)$ $L$ has no doubly reducible element.

In fact, he proved a stronger result, namely, that every lattice can be embedded in the ideal lattice of a lattice satisfying $(X)$.

In general, if $(P)$ is a lattice-theoretic property which is preserved by sublattices and which satisfies the assertion

$\mathcal{E}(P)$: every lattice can be embedded in the ideal lattice of a lattice satisfying $(P)$,

then $(P)$ is a property of all transferable lattices. In addition to $(X)$, properties of a lattice $L$ for which this assertion is known to hold include

$(SF)$ $L$ is sectionally finite (that is, all principal ideals are finite);

$(SD_\wedge)$ for $a, b, c \in L$, $a \wedge b = a \wedge c$ implies that $a \wedge b = a \wedge (b \vee c)$;

$(SD_\vee)$ for $a, b, c \in L$, $a \vee b = a \vee c$ implies that $a \vee b = a \vee (b \wedge c)$.

That $\mathcal{E}(SF)$ holds is a consequence of P. M. Whitman's embedding
theorem [10] and the observation that the partition lattice on a set $S$ is isomorphic to the ideal lattice of the lattice of all finite partitions of $S$; that $\mathcal{E}(SD_{\wedge})$ and $\mathcal{E}(SD_{\lor})$ hold is the content of a recent paper of G. Grätzer and C. R. Platt [8].

Consider the properties

$$\begin{align*}
(W) \ & \text{for } a, b, c, d \in L, a \land b \leq c \lor d \text{ implies that } [a \land b, c \lor d] \cap \{a, b, c, d\} \neq \emptyset; \\
(W_i) \ & \text{for } a, b, c, d \in L, c \leq a \land b \leq c \lor d \text{ implies that } [a \land b, c \lor d] \cap \{a, b, c, d\} \neq \emptyset; \\
(W_u) \ & \text{for } a, b, c, d \in L, a \land b \leq c \lor d \leq a \text{ implies that } [a \land b, c \lor d] \cap \{a, b, c, d\} \neq \emptyset.
\end{align*}$$

K. Baker and A. W. Hales [2] proved that if a lattice satisfies $(W)$, then so does its ideal lattice. Hence $\mathcal{E}(W)$ fails; however, in this paper, we will show that $\mathcal{E}(W_i)$ and $\mathcal{E}(W_u)$ hold. In fact, we will prove that every lattice can be embedded in the ideal lattice of a lattice satisfying the four properties $(SD_{\lor}), (SD_{\wedge}), (W_i),$ and $(W_u)$ simultaneously. More succinctly, our main result is

**Theorem.** $\mathcal{E}((SD_{\lor}) \land (SD_{\wedge}) \land (W_i) \land (W_u))$ holds.

It follows from the theorem and the preceding remarks that every transferable lattice is sectionally finite and satisfies $(SD_{\lor}), (SD_{\wedge}), (W_i),$ and $(W_u)$. By a result of R. Antonius and I. Rival [1], we conclude:

**Corollary.** Every transferable lattice satisfies $(W)$.

The proof of the theorem is contained in §2. In §3 we shall settle the truth or falsity of $\mathcal{E}(P)$ for most remaining combinations $(P)$ of the above properties. In particular, it will be shown that $\mathcal{E}((SD_{\lor}) \land (SF) \land (X))$ holds and that $\mathcal{E}((SF) \land (SD_{\wedge}))$ and $\mathcal{E}((SF) \land (W_u))$ fail. With these results, we can determine the status of $\mathcal{E}(P)$ for all but two combinations $(P)$ of the properties $(X), (SF), (SD_{\lor}), (SD_{\wedge}), (W_i),$ and $(W_u)$. These two will be given at the end of the paper.

2. Proof of the theorem.

**Definition 1.** Let $L$ be a lattice and let $\langle a, b, c, d \rangle$ be an ordered quadruple of elements of $L$. Then we will say that

(i) $\langle a, b, c, d \rangle$ is a $(W_i)$-failure if $c \leq a \land b \leq c \lor d$ and $[a \land b, c \lor d] \cap \{a, b, c, d\} = \emptyset$;
(ii) \( \langle a, b, c, d \rangle \) is a \((W_u)\)-failure if \( a \wedge b \leq c \vee d \leq a \) and \([a \wedge b, c \vee d] \cap \{a, b, c, d\} = \emptyset \);

(iii) \( \langle a, b, c, d \rangle \) is an \((SD_\wedge)\)-failure if \( a \wedge b = a \wedge c = d \) and \( a \wedge (b \vee c) \neq d \);

(iv) \( \langle a, b, c, d \rangle \) is an \((SD_\vee)\)-failure if \( a \vee b = a \vee c = d \) and \( a \vee (b \wedge c) \neq d \);

(v) \( \langle a, b, c, d \rangle \) is a failure if it is any of the above four types of failures.

**Definition 2.** Let \( L \) be a lattice, let \( \langle a, b, c, d \rangle \) be a failure in \( L \), and let \( \varphi \) be a homomorphism from a lattice \( M \) onto \( L \). Then \( \varphi \) repairs \( \langle a, b, c, d \rangle \), or \( \langle a, b, c, d \rangle \) is repaired in \( M \) by \( \varphi \), iff \( \langle a', b', c', d' \rangle \) is never a failure in \( M \) of the same type as \( \langle a, b, c, d \rangle \), for any \( a' \in \varphi^{-1}(a), b' \in \varphi^{-1}(b), c' \in \varphi^{-1}(c), \) and \( d' \in \varphi^{-1}(d) \).

**Lemma 3.** Let \( K, L, \) and \( M \) be lattices, let \( \varphi_1: M \to L \) and \( \varphi_2: L \to K \) be onto homomorphisms, and let \( \langle a, b, c, d \rangle \) be a failure in \( K \). If \( \langle a, b, c, d \rangle \) is repaired in \( L \) by \( \varphi_2 \), then it is repaired in \( M \) by \( \varphi_2 \circ \varphi_1 \).

**Proof.** Each of the four conditions \((SD_\vee), (SD_\wedge), (W_i), \) and \((W_u)\) can be expressed in the form \( P(x, y, z, w) \Rightarrow Q(x, y, z, w) \), where \( P \) and \( Q \) are disjunctions of polynomial equations and hence are preserved under homomorphisms. Since \( \langle a, b, c, d \rangle \) is a failure in \( K \), there exist appropriate \( P \) and \( Q \) such that \( P(a, b, c, d) \) holds but \( Q(a, b, c, d) \) fails. Suppose that \( \langle a, b, c, d \rangle \) is not repaired in \( M \) by \( \varphi_2 \circ \varphi_1 \); then there are elements \( a', b', c', d' \in M \) such that \( \varphi_2 \circ \varphi_1(x') = x \) for \( x \in \{a, b, c, d\} \), \( P(a', b', c', d') \) holds, and \( Q(a', b', c', d') \) fails. Consequently, \( P(\varphi_1(a'), \varphi_1(b'), \varphi_1(c'), \varphi_1(d')) \) holds in \( L \). Since \( \langle a, b, c, d \rangle \) is repaired in \( L \) by \( \varphi_2 \), this implies that \( Q(\varphi_1(a'), \varphi_1(b'), \varphi_1(c'), \varphi_1(d')) \) holds in \( L \). But now \( Q(a, b, c, d) = Q(\varphi_2(\varphi_1(a')), \varphi_2(\varphi_1(b')), \varphi_2(\varphi_1(c')), \varphi_2(\varphi_1(d'))) \) holds in \( K \), a contradiction.

Part of the proof of our theorem involves showing how to repair all failures in a lattice. Before describing the constructions by which this is accomplished, we make some observations.

Denote the lattice of ideals of a lattice \( L \) by \( \mathcal{I}(L) \). Let \( L \) and \( K \) be lattices and let \( \varphi \) be a homomorphism of \( L \) onto \( K \). For \( I \in \mathcal{I}(K) \), consider the set
\[
\varphi^{-1}(I) = \{x \in L \mid \varphi(x) \in I\}.
\]
\( \varphi^{-1}(I) \) is an ideal of \( L \), and hence \( \varphi^{-1} \) is a map of \( \mathcal{I}(K) \) into \( \mathcal{I}(L) \).
which is easily seen to be order preserving and one-to-one. Moreover, since meets of ideals are defined by set intersection, $\varphi^{-1}$ is also meet preserving.

**Lemma 4.** The map $\varphi^{-1}: \mathcal{F}(K) \to \mathcal{F}(L)$ is an embedding if and only if $\varphi$ satisfies the condition

\[
(*) \text{ if } y \in L, \ x_1, x_2 \in K, \text{ and } \varphi(y) \leq x_1 \vee x_2, \text{ then } y \leq y_1 \vee y_2 \text{ for some } y_1, y_2 \in L \text{ satisfying } \varphi(y_1) = x_1, \varphi(y_2) = x_2.
\]

**Proof.** To prove the "if" direction, by the above remarks we need only show that for $I, J \in \mathcal{F}(K)$, $\varphi^{-1}(I \vee J) \subseteq \varphi^{-1}(I) \vee \varphi^{-1}(J)$. Let $x \in \varphi^{-1}(I \vee J)$; then $\varphi(x) \in I \vee J$, so there exist $x_1 \in I, x_2 \in J$ such that $\varphi(x) \leq x_1 \vee x_2$. By $(*)$ there are $y_1, y_2 \in L$ such that $\varphi(y_1) = x_1, \varphi(y_2) = x_2$, and $x \leq y_1 \vee y_2$. But then $y_1 \in \varphi^{-1}(I), y_2 \in \varphi^{-1}(J)$, so $x \in \varphi^{-1}(I) \vee \varphi^{-1}(J)$, as desired.

Conversely, suppose that $\varphi^{-1}$ is an embedding, and let $y \in L$ and $x_1, x_2 \in K$ be such that $\varphi(y) \leq x_1 \vee x_2$. Then $\varphi(y) \leq (x_1) \vee (x_2)$ in $\mathcal{F}(K)$, and since $\varphi^{-1}$ is join preserving we have that $y \in \varphi^{-1}((\varphi(y))) \subseteq \varphi^{-1}((x_1)) \vee \varphi^{-1}((x_2))$. Thus there exist $y_1 \in \varphi^{-1}((x_1)), y_2 \in \varphi^{-1}((x_2))$ such that $y \leq y_1 \vee y_2$. Clearly we may assume that $\varphi(y_1) = x_1$ and $\varphi(y_2) = x_2$.

The next three propositions allow us to repair all failures in a lattice. The constructions used in these results are slight modifications of constructions that have appeared elsewhere; that of Proposition 5 is taken from Theorem 4.4 of H. S. Gaskill, G. Gratzer, and C. R. Platt [5], and that of Propositions 6 and 7 is taken from Theorem 3.1 of T. G. Kucera and B. Sands [9]. We have included Figures 1 and 2 to illustrate the constructions in Propositions 5 and 6 respectively.

**Proposition 5.** Let $x = \langle a, b, c, d \rangle$ be a failure of $(W_i)$ or $(W_o)$ in the lattice $L$. There exists a lattice $L_x$ and a homomorphism $\varphi_x$ of $L_x$ onto $L$ satisfying $(*)$ such that $x$ is repaired in $L_x$ by $\varphi_x$.

**Remark.** One method of repairing failures of $(W_i)$ or $(W_o)$ is already in the literature; namely, the "interval construction" of A. Day [3]. However, it will be crucial for the proof of our main theorem that the homomorphisms we use to repair failures satisfy $(*)$, and it is easy to verify that the homomorphism associated with the interval construction does not enjoy this necessary property.

**Proof of Proposition 5.** Let $Z$ be the integers with their natural order and let $E$ and $O$ denote the sets of even and odd integers,
respectively. Extend \( \mathbb{Z} \) to \( \mathbb{Z}_b = \mathbb{Z} \cup \{ -\infty, \infty \} \) where \(-\infty\) is the least and \(\infty\) the greatest element of \(\mathbb{Z}_b\). Setting \( u = a \wedge b \) and \( v = c \lor d \), we define a subset \( L_x \) of \( L \times \mathbb{Z}_b \) by

\[
L_x = ([b] \times \{ \infty \}) \cup ([d] \times \{ -\infty \}) \cup ([[v] - [d]] \times E) \\
\cup (((L - [v]) \cup [u]) - [b]) \times O).
\]

(Figure 1(c) shows \( L_x \) for the case when \( L \) is the lattice of Figure 1(a) and \( x = \langle a, b, c, d \rangle \). Figure 1(b) shows \( L_x \) as a subset of \( L \times \mathbb{Z}_b \).) It is not hard to verify that each element of \( L \times \mathbb{Z}_b \) that
is not of the form \( \langle y, -\infty \rangle \) for \( y \not\leq d \) has a least upper bound in \( L_x \). This and a dual observation shows that \( L_x \), with the partial order inherited from \( L \times \mathbb{Z}_b \), is a lattice. Also, the projection \( \pi_1 : L \times \mathbb{Z}_b \rightarrow L \) restricts to a homomorphism \( \varphi_x \) of \( L_x \) onto \( L \).

We first claim that \( \varphi_x \) satisfies (*) . Let \( \langle y, t \rangle \in L_x \) and \( x_1, x_2 \in L \) be such that \( y = \varphi_x \langle y, t \rangle \leq x_1 \lor x_2 \). There exist \( i, j \in \mathbb{Z}_b \) such that \( y_1 = \langle x_i, i \rangle \) and \( y_2 = \langle x_j, j \rangle \) are in \( L_x \). If \( y \not\leq d \) then \( t = -\infty \), so \( \langle y, t \rangle \leq y_1 \lor y_2 \), as desired. Also, if \( x_1 \lor x_2 \geq b \) then \( y_1 \lor y_2 = \langle x_1 \lor x_2, \infty \rangle \), whence again \( \langle y, t \rangle \leq y_1 \lor y_2 \). Thus we may assume \( y \leq d \) and \( x_1 \lor x_2 \leq b \), and without loss of generality we have both \( y \) and \( x_1 \) in \( L - ((d] \cup [b]) \), implying that \( t, i \in \mathbb{Z} \). Since, for any \( x \in L \) and \( n \in \mathbb{Z} \), \( \langle x, n \rangle \in L_x \) implies \( \langle x, n \pm 2 \rangle \in L_x \), we may choose \( i \geq t \), and so \( \langle y, t \rangle \in y_1 \lor y_2 \) holds in any case, showing that \( \varphi_x \) satisfies (*).

Secondly, we show that \( x \) is repaired in \( L_x \) by \( \varphi_x \). Let \( a', b', c', d' \in L_x \) be such that \( \varphi_x(a') = a, \varphi_x(b') = b, \varphi_x(c') = c \), and \( \varphi_x(d') = d \). It follows that \( a' = \langle a, i \rangle \) where \( i \in O \), \( c' = \langle c, j \rangle \) where \( j \in E \), \( b' = \langle b, \infty \rangle \), and \( d' = \langle d, -\infty \rangle \). Thus \( a' \land b' = \langle a \land b, i \rangle \) and \( c' \lor d' = \langle c \lor d, j \rangle \), whence if \( a' \land b' \leq c' \lor d' \) we have \( i \leq j \). If \( x \) is failure of \( (W_i) \), assume that \( \langle a', b', c', d' \rangle \) is a failure of \( (W_i) \) in \( L_i \); then \( c' \leq a' \land b' \), yielding \( j \leq i \) and thus \( i = j \), which is impossible since \( i \) is odd and \( j \) is even. Thus \( \langle a', b', c', d' \rangle \) cannot be a failure of \( (W_i) \). Similarly, if \( x \) is a failure of \( (W_u) \), \( \langle a', b', c', d' \rangle \) is not a failure of \( (W_u) \) in \( L_x \). Hence \( x \) is repaired in \( L_x \) by \( \varphi_x \).

**Proposition 6.** Let \( x = \langle a, b, c, d \rangle \) be a failure of \( (SD_v) \) in the lattice \( L \). There exists a lattice \( L_x \) and a homomorphism \( \varphi_x \) of \( L_x \) onto \( L \) satisfying (*) such that \( x \) is repaired in \( L_x \) by \( \varphi_x \).

**Proof.** Let \( \mathbb{Z}_b, E, \) and \( O \) be as in Proposition 5, and set \( p = a \lor (b \land c) \). Define a subset \( L_x \) of \( L \times \mathbb{Z}_b \) by

\[
L_x = ((p] \times \{-\infty\}) \cup ((L - ((p] \cup [b]) \times E)
\cup ((L - ((p] \cup [c]) \times O)).
\]

Figure 2(c) shows \( L_x \) when \( L \) in the lattice of 2(a) and \( x = \langle a, b, c, d \rangle \), and Figure 2(b) shows \( L_x \) as a subset of \( L \times \mathbb{Z}_b \). It is easy to see that \( L_x \) is a join-semilattice, and that each element of \( L \times \mathbb{Z}_b \) that is not of the form \( \langle y, \infty \rangle \) has a greatest lower bound in \( L_x \); whence \( L_x \), with the partial order inherited from \( L \times \mathbb{Z}_b \), is a lattice. Furthermore, the projection \( \pi_1 : L \times \mathbb{Z}_b \rightarrow L \) again restricts to a homomorphism \( \varphi_x \) of \( L_x \) onto \( L \).

To show that \( \varphi_x \) satisfies (*), let \( \langle y, t \rangle \in L_x \) and \( x_1, x_2 \in L \) be such that \( y = \varphi_x \langle y, t \rangle \leq x_1 \lor x_2 \). There exist \( i, j \in \mathbb{Z}_b \) such that \( y_1 = \langle x_i, i \rangle \) and \( y_2 = \langle x_j, j \rangle \) are in \( L_x \). If \( y \leq p \), then \( t = -\infty \), so \( \langle y, t \rangle \leq
y₁ ∨ y₂. On the other hand, if y ≤ p, then without loss of generality we may let x₁ ≤ p, and both t and i are in Z. We may now choose i such that \( x₁, i \in L_x \) and \( t ≤ i \), and thus \( y, t \leq y₁ \lor y₂ \) follows in either case, proving that \( φ_x \) satisfies (*)

To show that \( x \) is repaired in \( L_x \) by \( φ_x \), let \( a', b', c', d' \in L_x \) be such that \( φ_x(a') = a, φ_x(b') = b, φ_x(c') = c \), and \( φ_x(d') = d \). Then \( b' = \langle b, i \rangle \) where \( i \in O, c' = \langle c, j \rangle \) where \( j \in E \), and since \( a ≤ p \), \( a' = \langle a, -\infty \rangle \). Thus \( a' \lor b' = \langle a \lor b, i \rangle \) and \( a' \lor c' = \langle a \lor c, j \rangle \). Since \( i \) is odd and \( j \) is even, \( a' \lor b' \neq a' \lor c' \), so \( \langle a', b', c', d' \rangle \) is not a failure of \( (SD_j) \) in \( L_x \).

**Proposition 7.** Let \( x = \langle a, b, c, d \rangle \) be a failure of \( (SD_j) \) in the lattice \( L \). There exists a lattice \( L_x \) and a homomorphism \( φ_x \) of \( L_x \) onto \( L \) satisfying (*) such that \( x \) is repaired in \( L_x \) by \( φ_x \).

**Proof.** Let \( p = a \land (b \lor c) \), and define a subset \( L_x \) of \( L \times Z_b \) by

\[
L_x = ([p] \times \{\infty\}) \cup ((L - ([p] \cup [b])) \times E) \\
\cup ((L - ([p] \cup [c])) \times O)
\]
This construction is just the dual of the one in the previous proposition, so $L_x$ is a lattice and we have the natural homomorphism $\varphi_x$ of $L_x$ onto $L$. An argument dual to that in Proposition 6 shows that $x$ is repaired in $L_x$ by $\varphi_x$, so we need only show that $\varphi_x$ satisfies (*). Let $\langle y, t \rangle \in L$ and $x_1, x_2 \in L$ be such that $\varphi_x(\langle y, t \rangle) \leq x_1 \lor x_2$. There exist $i, j \in Z$ such that $y_1 = \langle x_1, i \rangle$ and $y_2 = \langle x_2, j \rangle$ are in $L_x$. If $x_1 \lor x_2 \geq p$ then $y_1 \lor y_2 = \langle x_1, i \rangle \lor \langle x_2, j \rangle = \langle x_1 \lor x_2, \infty \rangle \geq \langle y, t \rangle$; therefore, we assume $x_1 \lor x_2 \neq p$, which implies that $t, i, j \in Z$. Now we can choose $y_1, y_2 \in L_x$ such that $\langle x_1, i \rangle \in L_x$ and $i \geq t$, whence $\langle y, t \rangle \leq y_1 \lor y_2$ follows.

Before continuing with the proof of the theorem, we recall the following construction.

**Definition 8.** Let $(L_i | i \in I)$ be a family of lattices, let $L$ be a lattice, and let $\varphi_i : L_i \to L$ be a lattice homomorphism for each $i \in I$. Form the direct product $\Pi(L_i | i \in I)$, and consider the subset

$$K = \{ x \in \Pi(L_i) | \varphi_i(x(i)) = \varphi_j(x(j)) \text{ for all } i, j \in I \}.$$ 

Then $K$ is a sublattice of $\Pi(L_i)$, and is called the pullback of the family $(\varphi_i | i \in I)$. Letting $\pi_i : K \to L_i$ be the restriction of the projection of $\Pi(L_i)$ onto $L_i$, we have $\pi_i \circ \varphi_i = \varphi_j \circ \pi_j$ for all $i, j \in I$; hence there is a natural homomorphism $\varphi = \varphi_i \circ \pi_i$ of $K$ into $L$. If $\varphi_i$ is onto for all $i \in I$, then $\varphi$ is onto.

**Proposition 9.** For any lattice $L$, there exists a lattice $L^*$ and a homomorphism $\varphi^*$ of $L^*$ onto $L$ satisfying (*) that repairs all failures in $L$.

**Proof.** Let $\mathcal{F}(L)$ be the set of all failures in $L$. From Propositions 5, 6, and 7, we obtain a family $(L_x | x \in \mathcal{F}(L))$ of lattices and a family $(\varphi_x : L_x \to L | x \in \mathcal{F}(L))$ of onto homomorphisms satisfying (*) such that for each $x \in \mathcal{F}(L)$, $x$ is repaired in $L_x$ by $\varphi_x$. Let $L^*$ be the pullback of $\{ \varphi_x | x \in \mathcal{F}(L) \}$ and let $\varphi^*$ be the natural homomorphism of $L^*$ onto $L$. Then by Lemma 3, $\varphi^*$ repairs all failures in $L$. To show that $\varphi^*$ satisfies (*), let $p \in L^*$ and $u, v \in L$ be such that $\varphi^*(p) \leq u \lor v$. Letting $p_x$ denote the $x$th component of $p$, for each $x \in \mathcal{F}(L)$, we have that $\varphi_x(p_x) = \varphi^*(p) \leq u \lor v$. For each $x \in \mathcal{F}(L)$, since $\varphi_x$ satisfies (*), there exist $u_x, v_x \in L_x$ such that $\varphi_x(u_x) = u$, $\varphi_x(v_x) = v$, and $p_x \leq u_x \lor v_x$ in $L_x$. Then the elements $u = (u_x | x \in \mathcal{F}(L))$ and $v = (v_x | x \in \mathcal{F}(L))$ are clearly in $L^*$; moreover $\varphi^*(u) = u$, $\varphi^*(v) = v$, and $p \leq u \lor v$.

Finally, we are in a position to prove our main result.
**Theorem 10.** Every lattice can be embedded in the ideal lattice of a lattice satisfying \((SD\lor), (SD\land), (W_l), \text{and} (W_u)\).

**Proof.** Let \(L\) be a lattice. Set \(L_0 = L\), and inductively let \(L_{n+1} = L_n^*\) and \(\varphi_n^*: L_n^* \rightarrow L_n\), \(n \geq 0\), be the lattice and homomorphism of Proposition 9. Let \(L_\infty\) be the inverse limit of the system of lattices \((L_n \mid n < \omega)\) and homomorphisms \((\varphi_n^* \mid n < \omega)\), and let \(\varphi_\infty^*: L_\infty \rightarrow L_n\) be the natural projection for each \(n\).

We first claim that \(L_\infty\) satisfies \((SD\lor), (SD\land), (W_l), \text{and} (W_u)\). Again note that each of these four conditions is expressible in the form \(P(x, y, z, w) \Rightarrow Q(x, y, z, w)\), where \(P\) and \(Q\) are disjunctions of polynomial equations. Let \(\langle a, b, c, d \rangle\) be a failure in \(L_\infty\); then there exist appropriate \(P\) and \(Q\) such that \(P(\varphi_\infty^*(a), \varphi_\infty^*(b), \varphi_\infty^*(c), \varphi_\infty^*(d))\) holds but \(Q(\varphi_\infty^*(a), \varphi_\infty^*(b), \varphi_\infty^*(c), \varphi_\infty^*(d))\) fails for some \(m \in \omega\). Therefore \(\langle \varphi_m^*(a), \varphi_m^*(b), \varphi_m^*(c), \varphi_m^*(d) \rangle\) is a failure in \(L_m\). But by construction \(\langle \varphi_{m+1}^*(a), \varphi_{m+1}^*(b), \varphi_{m+1}^*(c), \varphi_{m+1}^*(d) \rangle\) is not a failure in \(L_{m+1}\), which contradicts Lemma 3. Thus there can be no failures in \(L_\infty\); that is, \(L_\infty\) satisfies \((SD\lor), (SD\land), (W_l), \text{and} (W_u)\).

Next we prove that the homomorphism \(\varphi_\infty^*\) of \(L_\infty\) onto \(L\) satisfies (*) Let \(x \in L_\infty\) and \(u_0, v_0 \in L = L_0\) be such that \(\varphi_\infty^*(x) \leq u_0 \lor v_0\). Then \(\varphi_\infty^*(x) = \varphi_\infty^*(\varphi_1(x)) \leq u_0 \lor v_0\), and since \(\varphi_n^*\) satisfies (*) there exist \(u_1, v_1 \in L_1\) such that \(\varphi_\infty^*(u_1) = u_0, \varphi_\infty^*(v_1) = v_0\), and \(\varphi_1(x) \leq u_1 \lor v_1\). Proceeding by induction, assume that we have \(u_n, v_n \in L_n\) such that \(\varphi_n(x) \leq u_n \lor v_n\). Then \(\varphi_n(x) = \varphi_n^*(\varphi_{n+1}(x)) \leq u_n \lor v_n\), and since \(\varphi_n^*\) satisfies (*) there exist \(u_{n+1}, v_{n+1} \in L_{n+1}\) such that \(\varphi_n^*(u_{n+1}) = u_n\), \(\varphi_n^*(v_{n+1}) = v_n\), and \(\varphi_{n+1}(x) \leq u_{n+1} \lor v_{n+1}\). Now let \(u = \langle u_n \mid n < \omega \rangle\) and \(v = \langle v_n \mid n < \omega \rangle\). It follows that \(u, v \in L_\infty, \varphi_\infty(u) = u, \varphi_\infty(v) = v\), and \(x \leq u \lor v\), whence \(\varphi_\infty^*\) satisfies (*).

From Lemma 4, \(\mathcal{L}(L)\) is embedded in \(\mathcal{L}(L_\infty)\). Since \(L\) is embedded in \(\mathcal{L}(L)\), the theorem is proved.

As mentioned earlier, we have the following corollary.

**Corollary 11.** Every transferable lattice satisfies \((W)\).

**Remark.** The use of homomorphisms, pullbacks, and inverse limits to repair failures stems from a proof in a recent paper of A. Day, namely, the proof (see Theorem 3.2 in \([4]\)) that every lattice is a bounded homomorphic image of a lattice satisfying \((W)\).

3. Additional results. In this section we investigate the status of \(\mathcal{E}(P)\) for most other combinations \((P)\) of the properties defined
in the introduction. First, we shall indicate how certain techniques in a paper of G. Grätzer and C. R. Platt [8] can be modified so as to prove that $\mathcal{E}((SD_\vee) \& (SF) \& (X))$ holds.

Let $L$ be a lattice. It has already been observed that there is a lattice $K$ satisfying $(SF)$ such that $L$ is embeddable in $\mathcal{E}(K)$. Hence we need only show that for every lattice $K$ satisfying $(SF)$ there is a lattice $M$ satisfying $(SD_\vee)$, $(SF)$, and $(X)$ such that $\mathcal{E}(K)$ is embeddable in $\mathcal{E}(M)$.

Let $K$ be a lattice satisfying $(SF)$. In [8], Grätzer and Platt construct a lattice $L(K_T)$ satisfying $(SD_\vee)$ such that $K$ can be embedded in $\mathcal{E}(L(K_T))$. From Lemma 3 and their proof it is clear that they in fact embed $\mathcal{E}(K)$ in $\mathcal{E}(L(K_T))$. The lattice $L(K_T)$ consists of certain subsets (called closed subsets) of $K \times \mathbb{Z}$, ordered by inclusion.

Now we replace $\mathbb{Z}$ by $\omega$, and consider the set $L_f(K_\pi)$ of all finitely generated closed subsets of $K \times \omega$, that is, all closed subsets which are closures of finite subsets of $K \times \omega$. Since $K$ satisfies $(SF)$, each element of $L_f(K_\pi)$ is finite. Hence $L_f(K_\pi)$, ordered by inclusion, is a lattice; in fact $L_f(K_\pi)$ is embeddable in $\mathcal{E}(K)$ and therefore satisfies $(SD_\vee)$. Furthermore, $L_f(K_\pi)$ is sectionally finite. Next, it can be proved as in [8] that $\mathcal{E}(K)$ is embeddable in $\mathcal{E}(L_f(K_\pi))$, and moreover the image under this embedding of each ideal in $K$ is a nonprincipal ideal of $L_f(K_\pi)$. Therefore (G. Grätzer [7]) the elements of $L_f(K_\pi)$ may all be “split” to yield a lattice $M$ satisfying $(X)$ such that $\mathcal{E}(K)$ is embeddable in $\mathcal{E}(M)$. It is easy to see that $M$ will still satisfy $(SD_\vee)$ and $(SF)$. Thus we have:

**Theorem 12.** $\mathcal{E}((SD_\vee) \& (SF) \& (X))$ holds.

In contrast to the above, we now establish two negative results.

**Lemma 13.** If a lattice $L$ satisfies $(SF)$ and $(SD_\wedge)$, then $\mathcal{E}(L)$ satisfies $(SD_\wedge)$.

**Proof.** Let $L$ satisfy $(SF)$ and $(SD_\wedge)$, and let $A, B, C \in \mathcal{E}(L)$ satisfy $A \cap B = A \cap C$. Let $p \in A \cap (B \vee C)$. There exist $b \in B$ and $c \in C$ such that $p \leq b \vee c$. By $(SF)$, there exist largest elements $b_0 \in B$ and $c_0 \in C$ such that $b_0, c_0 \leq b \vee c$. Since $p \in A$, $p \wedge b_0 \in A \cap B$ and $p \wedge c_0 \in A \cap C = A \cap B$. Thus the element $q = (p \wedge b_0) \vee (p \wedge c_0) \in A \cap B$, and by the choice of $b_0, c_0 \leq q \leq b_0$. Hence $p \wedge c_0 \leq p \wedge b_0$; by symmetry we have that $p \wedge b_0 = p \wedge c_0$. Since $L$ satisfies $(SD_\wedge)$, $p \wedge b_0 = p \wedge (b_0 \vee c_0) = p \wedge (b \vee c) = p$. We conclude that $p \in A \cap B$, and so $A \cap (B \vee C) = A \cap B$, showing that $\mathcal{E}(L)$ satisfies $(SD_\wedge)$.
**Corollary 14.** \( ((SF) \& (SD_A)) \) fails.

**Lemma 15.** If a lattice \( L \) satisfies \( (SF) \) and \( (W_u) \), then \( \mathcal{J}(L) \) satisfies \( (W_u) \).

**Proof.** Let \( L \) satisfy \( (SF) \) and \( (W_u) \), and suppose that \( \langle A, B, C, D \rangle \) is a \( (W_u) \)-failure in \( \mathcal{J}(L) \). Then there exists an element \( x \in A \cap B \) such that \( x \in C \) and \( x \in D \), and an element \( b \in B \) such that \( b \supseteq x \) and \( b \in C \cup D \). Since \( L \) satisfies \( (SF) \), there exists a largest element \( x_0 \in A \cap B \) such that \( x_0 \subseteq b \); note that \( x_0 \supseteq x \) and so \( x_0 \in C, x_0 \in D \). Since \( x_0 \in A \cap B \subseteq C \cup D \), there exist \( c \in C, d \in D \) such that \( x_0 \subseteq c \cup d \). However, \( b \in C \cup D \), so \( b \not\subseteq c \cup d \). Finally, since \( A \supset C \cup D \), we may choose \( a \in A \) such that \( a \supset c \cup d \). But now \( a \land b \supseteq (c \cup d) \land b \supseteq x_0 \), and by the maximality of \( x_0 \) we obtain that \( a \land b = x_0 \). Hence the quadruple \( \langle a, b, c, d \rangle \) is a failure of \( (W_u) \) in \( L \), contradicting the hypothesis. We conclude that \( \mathcal{J}(L) \) satisfies \( (W_u) \).

**Corollary 16.** \( ((SF) \& (W_u)) \) fails.

To end this paper we ask two questions that are still open:
(i) Does \( ((SF) \& (W_l)) \) hold?
(ii) Does \( ((SF) \& (W_l) \& (SD_v)) \) hold?

**References**


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Ralph Alexander, *Metric averaging in Euclidean and Hilbert spaces* ........ 1
B. Aupetit, *Une généralisation du théorème de Gleason-Kahane-Żelazko pour les algèbres de Banach* .................................................. 11
Lung O. Chung, Jiang Luh and Anthony N. Richoux, *Derivations and commutativity of rings. II* .......................................................... 19
Lynn Harry Erbe, *Integral comparison theorems for third order linear differential equations* .............................................................. 35
Robert William Gilmer, Jr. and Raymond Heitmann, *The group of units of a commutative semigroup ring* ........................................... 49
George Grätzer, Craig Robert Platt and George William Sands, *Embedding lattices into lattices of ideals* .................................................. 65
Raymond D. Holmes and Anthony Charles Thompson, *n-dimensional area and content in Minkowski spaces* ........................................ 77
Harvey Bayard Keynes and M. Sears, *Modelling expansion in real flows* . . 111
Taw Pin Lim, *Some classes of rings with involution satisfying the standard polynomial of degree 4* .......................................................... 125
Garr S. Lystad and Albert Robert Stralka, *Semilattices having bialgebraic congruence lattices* ............................................................... 131
Theodore Mitchell, *Invariant means and analytic actions* ....................... 145
Daniel M. Oberlin, *Translation-invariant operators of weak type* ............. 155
Raymond Moos Redheffer and Wolfgang V. Walter, *Inequalities involving derivatives* ................................................................. 165
Eric Schechter, *Stability conditions for nonlinear products and semigroups* ................................................................. 179
Jan Søreng, *Symmetric shift registers* ................................................. 201
Toshiji Terada, *On spaces whose Stone-Čech compactification is Oz* ........ 231
Richard Vrem, *Harmonic analysis on compact hypergroups* ................... 239