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Let G be a locally compact group and let m be a left Haar measure on G. For $0 < p < \infty$, let $L^p(G)$ be the usual Lebesgue space of functions f on G for which

$$||f||_p = \left(\int_G |f(x)|^p \, dm(x)\right)^{1/p} < \infty .$$

If T is a linear operator which takes $L^p(G)$, or a subspace of $L^p(G)$, into measurable functions on G, then T is said to be of weak type (p, p) if there exists a positive constant C such that

$$m\{x \in G\colon |Tf(x)| \geqq \alpha\} \leqq C||f||_p^p/\alpha^p \quad \textbf{for} \quad f \in L^p(G),\ \alpha > 0 .$$

We are interested in the translation-invariant operators of weak type (p, p).

To be more precise, for $x \in G$ we define the left and right translation operators L_x and R_x by $L_x f(y) = f(xy)$ and $R_x f(y) = f(yx)$ for functions f on G and $y \in G$. An operator T will be called translation-invariant if T commutes with each R_x: $TR_x = R_x T$ for each $x \in G$. We shall prove the following theorems.

THEOREM 1. *Suppose that the locally compact group G is amenable. If $0 < p < q \leqq 2$ and T is a translation-invariant operator of weak type (p, p) on $L^p(G)$, then T is a bounded linear operator on $L^q(G)$.*

THEOREM 2. *Let G be an arbitrary locally compact group and suppose that $0 < p < 1$. Then T is a translation-invariant operator of weak type (p, p) on $L^p(G)$ if and only if T has the form $\sum_{n=1}^{\infty} a_n L_{x_n}$ for distinct $x_n \in G$ and complex numbers a_n satisfying $|a_n| = 0(n^{-1/p})$.*

To state Theorem 3 we need some additional terminology. For a compact group G, let $\sum$ denote the dual object of G. For $0 < p < \infty$ and a subset E of $\sum$, let $L_E^p (= L_E^p(G))$ denote the closure in $L^p(G)$ of the set of trigonometric polynomials with spectrum in E.

THEOREM 3. *With notation as above, suppose $0 < p < q \leqq 2$ and that T is a translation-invariant operator of weak type (p, p) on L_E^p. Then T is bounded on L_E^q.*

Theorem 1 should be compared with a previous result of M. Cowling [2]. Cowling's result states that if T is a continuous translation-invariant operator between two rearrangement-invariant Banach function spaces on G, then T is automatically bounded on $L^2(G)$. We note that the hypothesis of amenability is necessary to Theorem 1: N. Lohoue has proved that for $1 < p < 2$ there are translation-invariant linear operators bounded on $L^p(\mathrm{SL}(2, R))$ which are not bounded on $L^2(\mathrm{SL}(2, R))$ [5].

Theorem 2 is an analogue of the result of [7] for operators of weak type. For the circle group T, Theorem 2 was established in [8]. But the methods of [8] do not seem to generalize beyond the case of compact G.

Theorem 3 is a partial answer to question (ii) of [6]. We mention that if $2 < q < p = 2m (m = 2, 3, \cdots)$, a translation-invariant operator on L_E^p may fail to be bounded on L_E^q [1].

2. The proofs.

We begin with some preliminaries from probability theory. Our probability space will be the unit interval I equipped with Lebesgue measure, which we shall denote by P.

Fix q with $0 < q \leqq 2$. A complex-valued random variable g on I is said to be q-stable of type $k > 0$ if its characteristic function $\chi_g(z) = \int_I \exp(-i \operatorname{Re}[z\overline{g(t)}])dP(t)$ is equal to $\exp(-k^q |z|^q) (z \in C)$. Now suppose that $\{g_i\}_{i=1}^\infty$ is a sequence of independent q-stable random variables of type 1 defined on I. We shall need the facts that given n and complex numbers $c_1, \cdots, c_n$,

$$(1) \qquad c_1 g_1 + \cdots + c_n g_n \quad \text{is } q\text{-stable of type } \left(\sum_1^n |c_i|^q\right)^{1/q} ,$$

and

$$(2) \qquad \int_I \left|\sum_1^n c_i g_i(t)\right|^p dP(t) = \left(\sum_1^n |c_i|^q\right)^{p/q} \int_I |g_1(t)|^p dP(t) , \quad 0 < p < q .$$

LEMMA 1. *For fixed q with $0 < q \leqq 2$ there exists a decreasing nonnegative function ϕ_q defined on $(0, \infty)$ such that if g is a q-stable random variable of type k on I, then*

$$P\{t \in I: |g(t)| \geqq \alpha\} = \phi_q(\alpha^q/k^q) .$$

Proof. This follows from the fact that g/k is q-stable of type 1 if g is q-stable of type k.

Our next lemma is a result for operators of weak type analogous to Lemma 2 of [4].

LEMMA 2. *Fix p and q with $0 < p < q \leqq 2$. Let T be a linear operator of weak type (p, p) on a subspace S of $L^p(G)$. There exists a positive constant C such that the following holds: If $f(x, y)$ is a continuous function of compact support on $G \times G$ such that $f(\cdot, y) \in S$ for each $y \in G$, then, for $\alpha > 0$,*

$$(3)\qquad \begin{aligned} &m\left\{x \in G: \left(\int_G |Tf(\cdot, y)(x)|^q dm(y)\right)^{1/q} \geqq \alpha\right\} \\ &\leqq C\int_G\left(\int_G |f(x, y)|^q dm(y)\right)^{p/q} dm(x)/\alpha^p . \end{aligned}$$

Proof. For each $n = 1, 2, \cdots$ there exist $m(= m(n))$ pairwise disjoint Borel sets $E_1, \cdots, E_m \subseteqq G$ and continuous compactly-supported functions $k_1, \cdots, k_m \in S$ such that if χ_i is the characteristic function of E_i and if

$$f_n(x, y) = \sum_1^m k_i(x)\chi_i(y) ,$$

then

$$(4)\qquad \begin{aligned} &\text{support } (f_n) \subseteqq K \text{ for some compact } K \subseteqq G \text{ and all } n, \text{ and} \\ &\sup\{|f_n(x, y) - f(x, y)|: (x, y) \in G \times G\} = o(n^{-1}) . \end{aligned}$$

In the following, C will denote a positive constant which is independent of f but may increase from line to line. The hypothesis on T implies that C may be chosen large enough to insure that

$$\begin{aligned} &m\{x \in G: |Tf(\cdot, y)(x) - Tf_n(\cdot, y)(x)| \geqq \alpha\} \\ &\leqq C\int_G |f(x, y) - f_n(x, y)|^p dm(x)/\alpha^p \quad (y \in G, \alpha > 0) . \end{aligned}$$

Integrating this inequality over G with respect to y, applying Fubini's theorem, and taking into account (4), we find that

$$m \times m\{(x, y) \in G \times G: |Tf(\cdot, y)(x) - Tf_n(\cdot, y)(x)| \geqq n^{-1}\} \longrightarrow 0 .$$

It follows that, by passing to a subsequence if necessary, we can assume $Tf_n(\cdot, y)(x) \to Tf(\cdot, y)(x)$ almost everywhere on $G \times G$. Thus, by Fatou's lemma,

$$\underline{\lim} \int_G |Tf_n(\cdot, y)(x)|^q dm(y) \geqq \int_G |Tf(\cdot, y)(x)|^q dm(y) \text{ for almost all } x \in G .$$

Let ϕ_q be the function in Lemma 1 and let $\alpha, \beta > 0$ be arbitrary. Since ϕ_q is decreasing, it follows from the inequality above and another application of Fatou's lemma that

$$\text{(5)}\qquad \begin{aligned} &\int_G \phi_q\left(\beta^q \Big/ \int_G |Tf(\cdot, y)(x)|^q dm(y)\right) dm(x) \\ &\qquad \leqq \underline{\lim} \int_G \phi_q\left(\beta^q \Big/ \int_G |Tf_n(\cdot, y)(x)|^q dm(y)\right) dm(x) . \end{aligned}$$

Fix a number $M > 0$ such that $\phi_q(M^{-q}) > 0$. Then

$$\int_G |Tf(\cdot, y)(x)|^q dm(y) \geqq \alpha^q$$

implies

$$\phi_q\left([\alpha/M]^q \Big/ \int_G |Tf(\cdot, y)(x)|^q dm(y)\right) \geqq \phi_q(M^{-q}) .$$

With $\beta = \alpha/M$ in (5) it follows that

$$\begin{aligned} &m\left\{x \in G: \int_G |Tf(\cdot, y)(x)|^q dm(y) \geqq \alpha^q\right\} \\ &\qquad \leqq [\phi_q(M^{-q})]^{-1} \underline{\lim} \int_G \phi_q\left((\alpha/M)^q \Big/ \int_G |Tf_n(\cdot, y)(x)|^q dm(y)\right) dm(x) , \end{aligned}$$

and so (3) will be established when we show

$$\text{(6)}\qquad \begin{aligned} &\underline{\lim} \int_G \phi_q\left(\beta^q \Big/ \int_G |Tf_n(\cdot, y)(x)|^q dm(y)\right) dm(x) \\ &\qquad \leqq C\beta^{-p} \int_G \left(\int_G |f(x, y)|^q dm(y)\right)^{p/q} dm(x) . \end{aligned}$$

To this end, suppose that $h_1, \cdots, h_m$ are functions in S and that $g_1, \cdots, g_m$ are independent q-stable random variables on I of type 1. For each $t \in I$ we have

$$m\left\{x \in G: \left|\sum_1^m g_i(t) Th_i(x)\right| \geqq \beta\right\} \leqq C\beta^{-p} \int_G \left|\sum_1^m g_i(t) h_i(x)\right|^p dm(x) .$$

Integrating this over I, using Fubini's theorem, and recalling (2), we find that

$$\text{(7)}\qquad \begin{aligned} &\int_G P\left\{t \in I: \left|\sum_1^m g_i(t) Th_i(x)\right| \geqq \beta\right\} dm(x) \\ &\qquad \leqq C\beta^{-p} \int_G \left(\sum_1^m |h_i(x)|^q\right)^{p/q} dm(x) . \end{aligned}$$

For fixed $x \in G$, (1) implies that $\sum_1^m g_i(t) Th_i(x)$ is symmetric q-stable of type $(\sum_1^m |Th_i(x)|^q)^{1/q}$. Thus Lemma 1 and (7) yield

$$\int_G \phi_q\left(\beta^q \Big/ \sum_1^m |Th_i(x)|^q\right) dm(x) \leqq C\beta^{-p} \int_G \left(\sum_1^m |h_i(x)|^q\right)^{p/q} dm(x) .$$

Now (6) follows from (4) and the representation

$$f_n(x, y) = \sum_1^m k_i(x)\chi_i(y) .$$

LEMMA 3. *Fix p and q with $0 < p < q \leqq 2$. Let S be a subspace of $L^p(G)$ such that $R_x S \subseteq S$ for each $x \in G$ and let T be a translation-invariant operator of weak type (p, p) on S. There exists a positive constant C such that the following holds: Fix a compact symmetric $K \subseteq G$ and a nonvoid compact set $U \subseteq G$. Suppose u is a compactly supported con tinuous function such that $u = 1$ on KKU. Suppose $h \in S$ is a continuous function supported in K such that*

$$u \cdot (R_y h) \in S , \quad y \in G . \tag{8}$$

Then

$$\left(\int_K |Th(y)|^q dm(y)\right)^{p/q} \leqq C \int_G |u(x)|^p dm(x) \left(\int_G |h(y)|^q dm(y)\right)^{p/q} / m(U) .$$

Proof. Let $V = (KU)^{-1}$. By the translation-invariance of T we have, for arbitrary $x \in G$,

$$\int_V |T(u(\cdot)h(\cdot y))(x)|^q dm(y) = \int_V |T(u(\cdot y^{-1})h(\cdot))(xy)|^q dm(y) . \tag{9}$$

Since $y \in V$ implies $u(\cdot y^{-1}) = 1$ on the support of h, it follows that the latter integral is

$$\int_V |Th(xy)|^q dm(y) = \int_G |Th(y)\chi_V(x^{-1}y)|^q dm(y) . \tag{10}$$

Here χ_V denotes the characteristic function of the set V. Now if $x \in U$, then $\chi_V(x^{-1}y) = 1$ as long as $y \in K = K^{-1}$. Thus, for $x \in U$,

$$\int_K |Th(y)|^q dm(y) \leqq \int_G |Th(y)\chi_V(x^{-1}y)|^q dm(y) .$$

Together with (9) and (10) this gives

$$\left(\int_K |Th(y)|^q dm(y)\right)^{1/q} \leqq \left(\int_G |T(u(\cdot)h(\cdot y))(x)|^q dm(y)\right)^{1/q}$$

if $x \in U$. It follows that

$$m\left\{x \in G: \left(\int_G |T(u(\cdot)h(\cdot y))(x)|^q dm(y)\right)^{1/q} \right. \\ \left. \geqq \left(\int_K |Th(y)|^q dm(y)\right)^{1/q}\right\} \geqq m(U) . \tag{11}$$

On the other hand, Lemma 2 $\left(\text{with } f(x, y) = u(x)h(xy) \text{ and } \alpha = \left(\int_K |Th(y)|^q dm(y)\right)^{1/q}\right)$ implies that the LHS of (11) is

$$\leqq C\int_G\left(\int_G|u(x)h(xy)|^q dm(y)\right)^{p/q} dm(x) \Big/ \left(\int_K|Th(y)|^q dm(y)\right)^{p/q} .$$

That is,

$$m(U) \leqq C\int_G|u(x)|^p dm(x)\left(\int_G|h(y)|^q dm(y)\right)^{p/q} \Big/ \left(\int_K|Th(y)|^q dm(y)\right)^{p/q} ,$$

which completes the proof of the lemma.

Proof of Theorem 1. Let h be any continuous compactly-supported function on G, and let K be any compact symmetric subset of G containing the support of h. A characteristic property of amenable groups [3] implies that there exists a compact subset U of G with $m(KKU)/m(U) < 2$. It follows that there exists a continuous compactly-supported function u on G with $u = 1$ on KKU and $\int_G|u(x)|^p dm(x)/m(U) < 2$. Taking $S = L^p(G)$ in Lemma 3 (it is obvious that (8) is satisfied) we conclude that

$$\left(\int_K|Th(y)|^q dm(y)\right)^{p/q} \leqq 2C\left(\int_G|h(y)|^q dm(y)\right)^{p/q} .$$

Since K can be any compact symmetric subset of G containing the support of h, it follows that $\|Th\|_q^p \leqq 2C\|h\|_q^p$. Since h is an arbitrary continuous compactly-supported function on G, the theorem follows.

Proof of Theorem 3. We apply Lemma 3 with $S = L_E^p$ and $K = U = G$. Then $u = 1$ on G and so (8) is satisfied for any continuous $h \in S$. Since such h are dense in L_E^q, Theorem 3 follows immediately from the conclusion of Lemma 3.

To establish Theorem 2 we require two more lemmas.

LEMMA 4. *Let G be a locally compact group. Let $V \subseteq G$ be a measurable set with $0 < m(V) \leqq 1$, and fix r with $0 < r < 1$. Given a positive number C_1 there exists another positive number C_2 such that if F is a nonnegative measurable function on G satisfying*

$$(12) \qquad m\left\{x \in G\colon \int_G F(y)\chi_V(y^{-1}x)dm(y) \geqq \alpha\right\} \leqq C_1/\alpha^r \quad (\alpha > 0) ,$$

then

$$\int_G F(y)dm(y) \leqq C_2 .$$

Proof. Choose nonnegative measurable functions F_n on G with $F_n \uparrow F$ and $\int_G F_n(x)dm(x) = a_n < \infty$. Write

$$H(x) = F*\chi_V(x) = \int_G F(y)\chi_V(y^{-1}x)dm(y)$$

and, similarly, $H_n = F_n^* \chi_V$. Then $H_n \leqq H$, so $m\{x: H_n(x) \geqq \alpha\} \leqq C_1/\alpha^r$ by hypothesis. Also $H_n \leqq a_n$, so

$$a_n m(V) = \int_G F_n * \chi_V(x) dm(x) = \int_G H_n(x) dm(x) = \int_0^{a_n} m\{x: H_n(x) \geqq \alpha\} d\alpha$$
$$\leqq \int_0^{a_n} C_1 \alpha^{-r} d\alpha = C_1 a_n^{1-r}/(1-r) .$$

Thus

$$a_n \leqq [C_1/m(V)(1-r)]^{1/r} = C_2 ,$$

and so

$$\int_G F(y) dm(y) \leqq C_2$$

also.

LEMMA 5. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions on G having the same distribution function $F(\alpha) = m\{x \in G: |f_n(x)| \geqq \alpha\} (\alpha > 0)$. Fix p with $0 < p < 1$. Then if $\alpha > 0$ we have*

$$m\left\{x \in G: \sum_1^{\infty} n^{-1/p} f_n(x) \geqq \alpha\right\} \leqq C \|f_1\|_p^p/\alpha^p , \tag{14}$$

where C is a constant depending only on p.

Proof. Let C denote a positive constant depending only on p, but which may increase from line to line. Fix $\alpha > 0$. For $n = 1, 2, \cdots$ let χ_n be the characteristic function of the set

$$\{x \in G: f_n(x) > \alpha n^{1/p}\}$$

and let χ_n' be the characteristic function of $\{x \in G: f_n(x) \leqq \alpha n^{1/p}\}$. We will establish (14) by estimating separately the two quantities

$$\begin{aligned} & m\left\{x \in G: \sum_1^{\infty} n^{-1/p} f_n(x) \chi_n(x) \geqq \alpha\right\} \quad \text{and} \\ & m\left\{x \in G: \sum_1^{\infty} n^{-1/p} f_n(x) \chi_n'(x) \geqq \alpha\right\} . \end{aligned} \tag{15}$$

We have

$$m\left\{x \in G: \sum_1^{\infty} n^{-1/p} f_n(x) \chi_n(x) \geqq \alpha\right\} \leqq \sum_1^{\infty} m\{x \in G: f_1(x) > \alpha n^{1/p}\}$$
$$= \alpha^{-p} \sum_1^{\infty} \alpha^p n m\{x \in G: \alpha n^{1/p} < f_1(x) \leqq \alpha(n+1)^{1/p}\} \leqq \alpha^{-p} \|f_1\|_p^p .$$

To estimate (15) we begin by writing $H(\lambda) = F(\lambda^{1/p})$, so that

$||f_n||_p^p = -\int_0^\infty \lambda dH(\lambda)$ for each n. Then

$$\int_G \sum_1^\infty (n+1)^{-1/p} f_n(x) \chi_n'(x) dm(x) = \sum_1^\infty (n+1)^{-1/p} \int_{\{f_n(x) \leqq \alpha n^{1/p}\}} f_n(x) dm(x)$$

$$(16) \qquad = -\sum_1^\infty (n+1)^{-1/p} \int_0^{n\alpha^p} \lambda^{1/p} dH(\lambda) \leqq -\int_1^\infty y^{-1/p} \int_0^{y\alpha^p} \lambda^{1/p} dH(\lambda) dy$$

$$= -\int_1^\infty y^{-1/p} \int_0^{\alpha^p} \lambda^{1/p} dH(\lambda) dy - \int_{\alpha^p}^\infty \lambda^{1/p} \int_{\lambda/\alpha^p}^\infty y^{-1/p} dy dH(\lambda) .$$

Now (15) is

$$\leqq C\alpha^{-1} \int_G \sum_1^\infty (n+1)^{-1} f_n(x) \chi_n'(x) dm(x) ,$$

so, by (16), it suffices to establish

$$(17) \qquad -\alpha^{-1} \int_1^\infty y^{-1/p} \int_0^{\alpha^p} \lambda^{1/p} dH(\lambda) dy \leqq C||f_1||_p^p/\alpha^p$$

and

$$(18) \qquad -\alpha^{-1} \int_{\alpha^p}^\infty \lambda^{1/p} \int_{\lambda/\alpha^p}^\infty y^{-1/p} \ \mathrm{dy} \ \ dH(\lambda) \leqq C||f_1||_p^p/\alpha^p .$$

For (17) we note that

$$-\int_0^{\alpha^p} \lambda^{1/p} dH(\lambda) = \int_{\{f_1(x) \leqq \alpha\}} f_1(x) dm(x)$$

and

$$\alpha^{-1} \int_{\{f_1(x) \leqq \alpha\}} f_1(x) dm(x) \leqq \alpha^{-p} \int_{\{f_1(x) \leqq \alpha\}} f_1^p(x) dm(x) .$$

Since $\int_1^\infty y^{-1/p}\, dy < \infty$, this establishes (17). On the other hand

$$\int_{\lambda/\alpha^p}^\infty y^{-1/p} dy = (p^{-1} - 1) \lambda^{1-1/p} \alpha^{1-p} .$$

Thus

$$-\alpha^{-1} \int_{\alpha^p}^\infty \lambda^{1/p} \int_{\lambda/\alpha^p}^\infty y^{-1/p} dy dH(\lambda) \leqq -C\alpha^{-p} \int_{\alpha^p}^\infty \lambda dH(\lambda) \leqq C||f_1||_p^p/\alpha^p .$$

This is (18) and so the proof of the lemma is complete.

Proof of Theorem 2. The "if" part of Theorem 2 is an immediate consequence of Lemma 5. So suppose T is a translation-invariant operator of weak type (p, p) on $L^p(G)$ $(0 < p < 1)$, and we will show that T has the form $\sum_{n=1}^\infty a_n L_{x_n}$, $|a_n| = 0(n^{-1/p})$. Fix q with $0 < p <$

$q \leqq 2$. We will begin by showing that T is "locally bounded" on $L^q(G)$.

Let U and V be neighborhoods of the identity in G with U relatively compact, V symmetric, $V^2 \subseteq U$, and $m(V) \leqq 1$. Let u be a continuous function with compact support satisfying $u(x) = 1$ for $x \in U$, and let h be an arbitrary continuous function with support contained in V. According to Lemma 2, where we take $S = L^p(G)$ and $f(x, y) = u(x)h(xy)$, we have

$$\begin{aligned} &m\left\{x \in G\colon \left(\int_G |T(u(\cdot)h(\cdot y))(x)|^q dm(y)\right)^{1/q} \geqq \beta\right\} \\ &\qquad \leqq C \int_G |u(x)|^p dm(x) \left(\int_G |h(y)|^q dm(y)\right)^{p/q} \Big/ \beta^p \quad (\beta > 0) . \end{aligned} \tag{19}$$

Since T is translation-invariant,

$$\int_V |T(u(\cdot)h(\cdot y))(x)|^q dm(y) = \int_V |T(u(\cdot y^{-1})h(\cdot))(xy)|^q dm(y) .$$

Since $V^2 \subseteq U$, V is symmetric, and h is supported in V, it follows that $u(\cdot y^{-1})$ is equal to 1 on the support of h as long as $y \in V$. Thus the last integral is equal to

$$\int_V |Th(xy)|^q dm(y) = \int_G |Th(y)|^q \chi_V(y^{-1}x) dm(y) ,$$

where we have used $V = V^{-1}$. Thus

$$\int_G |Th(y)|^q \chi_V(y^{-1}x) dm(y) \leqq \int_G |T(u(\cdot)h(\cdot y))(x)|^q dm(y) .$$

With (19) (where we substitute α for β^q) we have

$$\begin{aligned} &m\left\{x \in G\colon \int_G |Th(y)|^q \chi_V(y^{-1}x) dm(y) \geqq \alpha\right\} \\ &\qquad \leqq C \int_G |u(x)|^p dm(x) \left(\int_G |h(y)|^q dm(y)\right)^{p/q} / \alpha^{p/q} . \end{aligned}$$

Taking $r = p/q$, $C_1 = C\int_G |u(x)|^p dm(x)$, and $F(y) = |Th(y)|^q$ in Lemma 4, we see that $\|h\|_q^q \leqq 1$ implies $\|Th\|_q^q \leqq C_2$ for some fixed positive number C_2 and any continuous h supported in V. It follows that

$$\|Th\|_q^q \leqq C_2 \|h\|_q^q \tag{20}$$

holds for any measurable h supported in V. (Thus T is "locally bounded" on $L^q(G)$.)

If $0 < p < q < 1$, it follows from (20), from the translation-invariance of T, and from the subadditivity of $\|\cdot\|_q^q$ that T is actually bounded on $L^q(G)$. Now the theorem in [6] shows that T has the form $\sum_1^\infty a_n L_{x_n}$ for distinct $x_n \in G$ and numbers a_n satisfying $\sum_1^\infty |a_n|^q < \infty$. Using the fact that T is actually of weak type (p, p), it is easy to see that

$$\text{card}\{n: |a_n| \geqq \alpha\} = 0(\alpha^{-p}) \quad (\alpha > 0) .$$

Thus if $\{|a_n^*|\}_{n=1}^\infty$ is a decreasing rearrangement of the sequence $\{|a_n|\}_{n=1}^\infty$, it follows that $|a_n^*| = 0(n^{-1/p})$. This completes the proof of Theorem 2.

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