THE PROJECTIVITY OF $\text{Ext}(T, A)$ AS A MODULE OVER $E(T)$

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Let A and T be abelian groups. Then Ext (T, A) can be considered as a right module over E(T), the ring of endomorphisms of T. In this paper necessary and sufficient conditions are developed for Ext (T, A) to be E(T)-projective whenever T is reduced torsion and A is reduced.

In this paper A and T will be abelian groups and Ext (T, A) will be considered as a right E(T)-module. (See [5].) We consider the question of when Ext (T, A) is a projective E(T)-module. Theorems 1 and 2 provide necessary and sufficient conditions for Ext (T, A) to be E(T)-projective whenever T is reduced torsion and A is reduced. It is interesting to note (Theorem 3) that if B is any reduced group, a necessary condition for Ext (B, A) to be E(B)-projective is that Ext (B, A) \approx Ext (T(B), A). Hence if Ext (B, A) is E(B)-projective, Ext (B, A) \approx Ext (T(B), A) and Ext (T(B), A) may be considered as an E(T(B))-module, where T(B) is, of course, reduced torsion.

We shall employ the following notations and conventions: The word "group" will always mean "abelian group." We reserve the letter T for a torsion group, and in this case, T_p will be the p-primary component of T. For an arbitrary group A, T_p (A) is the p-primary component of the torsion part of A. For a ring R and a left R-module M, r_p (M) will refer to the rank of M as defined in [4], hd_R (M) and id_R (M) will refer, respectively, to the homological and injective dimensions of M as defined in [6]. An isomorphism of R-modules M and N will be denoted by: M \cong N. Other notations will follow [2]. Importantly, whenever we speak of Ext (T, A) as a right E(T)-module we may assume without loss of generality that A is reduced as a group. Finally, if A \cong \mathbb{Z} \oplus A', and if a \in A, we will write, conveniently, when defining an endomorphism \alpha of A: \alpha (v) = a, \alpha = 0 otherwise. We mean, more precisely, that: \alpha (v) = a, \alpha |_{A'} = 0. We now state our main theorems:

**Theorem 1.** Let T be a reduced p-primary group and let A be a reduced group. Then Ext (T, A) is a projective right E(T)-module if and only if either Ext (T, A) = 0, or all of the following conditions hold:

( i ) T is bounded, with minimal annihilator p^k, say.
(ii) $A[p^k]$ is either zero or is a direct sum of cyclic groups of order $p^k$.

(iii) If $D$ is a divisible hull of $T(A)$ and $E$ is a divisible hull of $A/T(A)$, and if

$$\max \left\{ r_p\left(\frac{D}{T(A)}\right),\ r_p\left(\frac{E}{A/T(A)}\right)\right\} = m,$$

$m$ an infinite cardinal, then $T$ is either finite, or in a decomposition of $T$ into cyclic groups, there are at least $m$ summands isomorphic to $Z(p^k)$.

**Theorem 2.** Let $T$ be a reduced torsion group and let $A$ be a reduced group. Then $\text{Ext}(T, A)$ is a projective $E(T)$-module if and only if for every $p$, $\text{Ext}(T_p, A)$ is a projective $E(T_p)$-module.

**Theorem 3.** Let $A$ and $B$ be reduced groups. Then a necessary condition for $\text{Ext}(B, A)$ to be $E(B)$-projective is that $\text{Ext}(B, A) \cong \text{Ext}(T(B), A)$.

**Proofs of the theorems.** The proof of Theorem 1 will require numerous preliminary results. We postpone its proof. Theorem 2 follows easily from Lemmas 1 and 2 below. We now prove Theorem 3:

**Proof of Theorem 3.** Since $B$ is reduced, it is easily verified that $E(B)$ is reduced. Now, from the $Z$-exact sequence: $0 \rightarrow T(B) \rightarrow B \rightarrow B/T(B) \rightarrow 0$, we obtain the $Z$-exact sequence: $0 \rightarrow \text{Ker } i^* \rightarrow \text{Ext}(B, A) \rightarrow \text{Ext}(T(B), A) \rightarrow 0$. Since $\text{Ker } i^*$ is a subgroup of $\text{Ext}(B/T(B), A)$, and since $B/T(B)$ is torsionfree, $\text{Ker } i^*$ is divisible. (See [2].) Since $\text{Ext}(T(B), A)$ is reduced (see [2]), it follows that $\text{Ker } i^*$ is the maximal divisible subgroup of $\text{Ext}(B, A)$. Now, since $E(B)$ is reduced as a group, any free $E(B)$-module is reduced as a group. So if $\text{Ext}(B, A)$ is to be $E(B)$-projective, we must have $\text{Ker } i^* = 0$.

We will now aim at proving Theorem 1.

**Lemma 1.** Let $M = \Pi_{i \in I} M_i$ where each $M_i$ is an $R_i$-module, $R = \Pi_{i \in I} R_i$, and $M$ is an $R$-module via the coordinatewise action of $\Pi_{i \in I} R_i$. Then $M$ is $R$-projective (resp. injective) if and only if $M_i$ is $R_i$-projective (resp. injective) for all $i \in I$.

**Proof.** The proof is easy and is omitted.
LEMMA 2. Let $F: \text{Ab} \times \text{Ab} \to \text{Ab}$ be either of the functors $\text{Hom}$ or $\text{Ext}$. Then:

(i) If $A = \bigoplus_{i \in I} A_i$ where the $A_i$ are fully invariant subgroups of $A$, then $F(A, B) \simeq \prod_{i \in I} F(A_i, B)$.

(ii) If $B = \prod_{i \in I} B_i$ where the $B_i$ are fully invariant subgroups of $B$, then $F(A, B) \simeq \prod_{i \in I} F(A, B_i)$.

Proof. The isomorphism in (i) is given by: $F(A, B) \simeq \prod_{i \in I} F(A_i, B)$ where, for $f \in F(A, B)$, $\psi(f) = [f \alpha_i]_{i \in I}$ where $\alpha_i \in E(A)$ is defined by: $\alpha_i |_{A_i} = 1_{A_i}, \alpha_i = 0$ otherwise. It is easily verified that $\psi$ is an $E(A)$-homomorphism.

The isomorphism for (ii) is similar.

Lemma 3 computes the injective dimension over $E(T)$ of $\text{Ext}(T, A)$ when $T$ is torsion and $A$ is torsionfree:

LEMMA 3. Let $T$ be torsion and let $A$ be torsionfree. Suppose $S$ is the set of primes for which $A$ is $p$-divisible. Then:

(i) $\text{id}_{E(T)}(\text{Ext}(T, A)) = 0$ if and only if for every prime $p \in S$, $T_p$ is either bounded or has an unbounded basic subgroup.

(ii) Otherwise, $\text{id}_{E(T)}(\text{Ext}(T, A)) = 1$.

Proof. If $D$ is a divisible hull of $A$, then $D/A$ is torsion and $\text{Hom}(T, D/A) \simeq \text{Ext}(T, A)$. By Lemma 2, it suffices to prove the result in the case in which $T$ is a $p$-group, and we may assume $(D/A)_p \neq 0$, since otherwise $\text{Ext}(T, A) = 0$. Assuming this, we note that by [8, Lemma 2], $\text{Hom}(T, D/A)$ is $E(T)$-injective if and only if $T$ is $E(T)$-flat. From [9] we know that this holds if and only if the condition (i) of the lemma holds (where $T$ is a $p$-group, and $p \in S$). Otherwise, from [1], we know $T$ has dimension one as an $E(T)$-module, and if we take a projective resolution of $T$ and dualize it, applying [8, Lemma 2] again, we obtain an injective resolution for $\text{Hom}(T, D/A)$, establishing part (ii) of the lemma.

LEMMA 4. Let $A$ be a reduced group with $T_p(A)$ unbounded. Then if $M$ is a right $E(A)$-module with $\text{Hom}_Z(A, Z(p^\infty)) \subseteq M$, then $M$ is not $E(A)$-projective.

Proof. We will show that there is no $E(A)$-monic map $\phi$:

$$0 \longrightarrow \text{Hom}(A, Z(p^\infty)) \longrightarrow \bigoplus_{b \in B} E(A)_b$$

for any indexing set $B$. This will complete the proof. Consider
\( I = \{1, 2, 3, \ldots \} \). Since \( T_p(A) \) is reduced and unbounded, for each \( i \in I \), we may choose \( \nu_i \in T_p(A) \) with the property that \( (\nu_i) \) is a cyclic summand of \( T_p(A) \) and such that \( O(\nu_i) < O(\nu_{i+1}) \), \( i = 1, 2, 3, \ldots \). Say \( (\nu_i) = Z(p^{n_i}) \) for \( i = 1, 2, 3, \ldots \). Now, let \( h_i \in \text{Hom}(A, Z(p^{\omega})) \) be defined by:

\[
h_i(\nu_i) = \frac{1}{p^{n_i}}, \quad h_i = 0 \text{ otherwise}.
\]

Let:

\[
\psi(h_i) = \alpha_{b_{j_i}} + \alpha_{b_{2j_i}} + \cdots + \alpha_{b_{k_{j_i}}}
\]

where \( \alpha_{b_{j_i}} \in E(A)_{b_{j_i}} \) for all \( j = 1, 2, \ldots, k_i \). Define \( \beta_i \in E(A) \) by:

\[
\beta_i(\nu_i) = 0, \quad \beta_i = 1 \text{ otherwise}.
\]

Then the computation:

\[
0 = \psi(0) = \psi(h_i \beta_i) = \alpha_{b_{j_i}} \beta_i + \alpha_{b_{2j_i}} \beta_i + \cdots + \alpha_{b_{k_{j_i}}} \beta_i
\]

shows that \( \alpha_{b_{j_i}} \beta_i = 0 \) for all \( j = 1, 2, \ldots, k_i \), and hence that \( \alpha_{b_{j_i}} = 0 \), except possibly on \( \nu_i \), for all \( j = 1, 2, \ldots, k_i \) and for all \( i = 1, 2, 3, \ldots \).

Suppose \( \alpha_{b_{j_i}}(\nu_i) = t_{j_i} \). Then \( t_{j_i} \in T_p(A) \), and not all \( t_{j_i} \) are zero for a fixed \( i \), where \( j = 1, 2, \ldots, k_i \). By defining \( \delta_i \in E(A) \) by:

\[
\delta_i(\nu_i) = p^{n_i-n_{i-1}} - \nu_i
\]

\[
\delta_i = 0 \text{ otherwise}
\]

for \( i = 2, 3, 4, \ldots \), the computation:

\[
\psi(h_i \delta_i) = \psi(h_{i-1}) = \alpha_{b_{1i-1}} + \alpha_{b_{2i-1}} + \cdots + \alpha_{b_{k_{i-1}i-1}}
\]

\[
= \psi(h_{i}) \delta_i = \alpha_{b_{1i}} \delta_i + \alpha_{b_{2i}} \delta_i + \cdots + \alpha_{b_{k_{i}} \delta_i}
\]

shows that we may assume \( k_i = k_2 = \cdots = k \), say, and that:

\[
\alpha_{b_{j_i}} = \alpha_{b_{j_i}} \delta_i \text{ for all } j = 1, 2, \ldots, k
\]

Now, since not all \( t_{j_i} \) are zero for a fixed \( i \), where \( j = 1, 2, \ldots, k \), assume that:

\[
\alpha_{b_{j_1}}(\nu_i) = t_{s_1} \neq 0 \text{ where } s \in \{1, 2, \ldots, k\}
\]

From this, we easily obtain the relations:

\[
t_{s_1} = p^{n_{s_2}-n_1} t_{s_2}
\]

\[
t_{s_2} = p^{n_{s_3}-n_2} t_{s_3}
\]

\[
t_{s_3} = p^{n_{s_4}-n_3} t_{s_4}
\]

\[
\vdots
\]

\[
\text{...}
\]
However, this is a contradiction, since the subgroup of $A$ generated by the $t_i$, $i = 1, 2, 3, \ldots$ is isomorphic to $\mathbb{Z}(p^\infty)$, and $A$ was assumed to be a reduced group.

**Corollary 1.** Let $T$ be a reduced torsion group. Then the following statements are equivalent:

(i) $T$ is a projective left $E(T)$-module.
(ii) $\text{Hom}(T, \mathbb{Q}/\mathbb{Z})$ is a projective right $E(T)$-module.
(iii) Every $p$-primary component of $T$ is bounded.

**Proof.** In [7] it is shown that a torsion group $T$ is a projective left $E(T)$-module $\iff\exists\gamma T_p$ is bounded for all $p$.

Now, to prove the equivalence of (i) and (ii), we note first that by Lemma 1 we may assume that $T$ is $p$-primary. Let $T$ be bounded with minimal annihilator $p^k$ and let $\nu$ generate a cyclic summand of $T$ of order $p^k$. Then $T \cong (\nu) \oplus T'$, the isomorphism being one of abelian groups. Hence:

$$\text{Hom}(T, (\nu)) \overset{E(T)}{\cong} \text{Hom}(T, \mathbb{Z}(p^\infty)),$$

is seen to be an $E(T)$ direct summand in $E(T)_{E(T)}$. If $T$ is not bounded, then Lemma 4 completes the proof.

**Corollary 2.** Let $T$ be a torsion group. Then $\text{Hom}(T, \mathbb{Q}/\mathbb{Z})$ is a projective right $E(T)$-module if and only if for every prime $p$, $T_p$ is either bounded or has an abelian group summand isomorphic to $\mathbb{Z}(p^\infty)$.

**Proof.** We may assume that $T$ is $p$-primary. If $T$ is reduced, the result follows from Corollary 1. If $T$ is not reduced, then $T = \mathbb{Z}(p^\infty) \oplus T'$ for some group $T'$, and it is clear that $\text{Hom}(T, \mathbb{Z}(p^\infty))$ is an $E(T)$-direct summand in $E(T)_{E(T)}$.

**Corollary 3.** Let $A$ be a torsion-free group of finite rank, and let $T$ be a torsion group. Further, let $S$ be the set of primes for which $A$ is $p$-divisible. Then $\text{Ext}(T, A)$ is a projective right $E(T)$-module if and only if for every prime $p \notin S$, $T_p$ is either bounded or has an abelian group summand isomorphic to $\mathbb{Z}(p^\infty)$.

**Proof.** Let $D$ be a divisible hull of $A$. Then: $D/A \cong \bigoplus_{p \in P'} D_p$, where $D_p$ is a divisible torsion group of finite rank, and where $P' = P - S$. Then:

$$\text{Ext}(T, A) \overset{E(T)}{\cong} \text{Hom}\left(T, \frac{D}{A}\right) \overset{E(T)}{\cong} \prod_{p \in P'} \text{Hom}(T_p, D_p).$$
The proof is completed by Lemma 1 and Corollary 2, recalling also that a direct sum of projective modules is projective.

**Lemma 5.** Let $T$ be a bounded $p$-primary group with minimal annihilator $p^k$ and let $n$ be any cardinal. Further assume that in a decomposition of $T$ into cyclic groups, there are at least $n$ summands isomorphic to $\mathbb{Z}(p^k)$. Then for any indexing set $I$ with $|I| \leq n$, $\text{Hom}(T, \bigoplus_{i \in I} \mathbb{Z}(p^k))$ is a cyclic projective $E(T)$-module.

**Proof.** There is a set $\{v_j\}_{j \in J}$ where $v_j$ generates a cyclic abelian group summand of $T$ of order $p^k$, and where $|J| = |I|$. Then $T \cong \bigoplus_{j \in J} (v_j) \oplus T'$, where the isomorphism is one of abelian groups. Thus, $\text{Hom}(T, \bigoplus_{j \in J} (v_j)) \cong \text{Hom}(T, \bigoplus_{i \in I} \mathbb{Z}(p^k))$ is seen to be an $E(T)$-direct summand in $E(T)_{E(T)}$.

**Lemma 6.** Let $V$ be a vector space of infinite dimension over a field $k$, and $E = \text{End}(V)$. Let $H = \text{Hom}(V, \bigoplus_{i \in I} V)$. Then $H$ is not projective as an $E$-module if $|I| > \dim(V)$.

**Proof.** We first note that if $F$ is a countable subset of $H$, then $F$ is contained in a cyclic submodule of $H$. To see this, let $W$ be a subspace of $\bigoplus_{i \in I} V$ containing $f(V)$ for all $f \in F$, and such that $\dim(W) = \dim(V)$. We may regard $\text{Hom}(V, W)$ as an $E$-submodule of $H$ and this submodule certainly contains $F$. Since $W \cong V$, $\text{Hom}(V, W) \cong E$.

We next note that any module with the above property cannot have an infinite direct sum decomposition (clearly). Now if $H$ were projective, it would be a direct sum of countably generated submodules (by Kaplansky’s theorem in [3]). Since $H$ is clearly not countably generated, this would mean that it had an infinite direct sum decomposition, which, as we have just seen, it does not.

**Corollary 4.** Let $T$ be a bounded $p$-group, of exponent $p^k$, and such that in a direct sum decomposition of $T$ there are $n$ summands of order $p^k$ where $n$ is an infinite cardinal (i.e., $n = \dim(T/T[p^{k-1}])$, where $T/T[p^{k-1}]$ is viewed as a vector-space over $\mathbb{Z}/p\mathbb{Z}$). Let $H = \text{Hom}(T, U)$, regarded as an $E(T)$-module, where $U$ is a direct sum of $m$ copies of $\mathbb{Z}(p^\infty)$, for some $m, m > n$. Then $H$ is not a projective $E(T)$-module.

**Proof.** Let $E = \text{End}(T/T[p^{k-1}])$. There is a natural map $\text{End}(T) \to E$ (since $T[p^{k-1}]$ is a fully invariant submodule of $T$) which is clearly surjective. If $I$ is the kernel of this map of rings,
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(\(I = \{ f \in \text{End}(T) : f(T) \leq T[p^{k-1}] \}\), then one easily identifies \(H/HI\) with \(\text{Hom}(T/T[p^{k-1}], U[p^k]/U[p^{k-1}])\). If \(H\) is projective as an \(E(T)\)-module, then \(H/HI\) must be projective as an \(E\)-module, which, according to the previous lemma, it is not.

**Lemma 7.** Let \(T\) be a bounded \(p\)-group which is infinite, but such that its highest nonzero Ulm invariant is finite. Let \(U\) be the direct sum of a countable number of copies of \(Z(p^\infty)\), and let \(E = \text{End}(T)\). Then \(H = \text{Hom}(T, U)\) is not a projective \(E\)-module.

**Proof.** If there is a split mono \(\mu : H \to \bigoplus_{i \in I} E\), then it induces a split mono \(H/H[p^{k-1}] \to \bigoplus_{i \in I} E/E[p^{k-1}]\), where we choose \(k\) such that \(p^kT = 0\), \(p^{k-1}T \neq 0\) (i.e., \(p^k\) is the exponent of \(T\)). We note that \(H/H[p^{k-1}]\) is infinite dimensional and all of the terms on the right above are finite dimensional over \(Z/pZ\). We now let \(f : T/pT \to U[p]\) be a surjective homomorphism. If \(g : T \to U[p]\) is any homomorphism with \(T[p^{k-1}]\) in its kernel, then there is an endomorphism \(\varepsilon_g : T \to T\) such that \(g = f\varepsilon_g\). It follows that if \(h \in H\), then for some endomorphism \(\phi \) of \(E\), \(p^{k-1}h = f\phi\). Now if \(\pi_i\) is the projection onto the \(i\)th summand in the above free \(E\)-module, then \(\pi_i\mu(f) \neq 0\) for only a finite number of indices \(i\). Let this finite subset of \(I\) be \(J\). It follows that \(p^{k-1} \pi_i \mu(h) = 0\) unless \(i \in J\), for all \(h \in H\). Hence the image of the induced map

\[
H/H[p^{k-1}] \to \bigoplus_{i \in I} E/E[p^{k-1}]
\]

is actually in the submodule \(\bigoplus_{i \in J} E/E[p^{k-1}]\). This is a contradiction, since this is finite dimensional, and \(H/H[p^{k-1}]\) is not.

**Corollary 5.** Let \(A\) be torsionfree, and let \(D\) be a divisible hull of \(A\). Let \(S\) be the set of primes for which \(A\) is \(p\)-divisible, and let \(T\) be a reduced torsion group. Then \(\text{Ext}(T, A)\) is a projective \(E(T)\)-module if and only if for every \(p \in S\) the following two conditions hold:

(i) Whenever \(r_p(D/A)\) is finite, \(T_p\) is bounded.

(ii) Whenever \(r_p(D/A) = m\), \(m\) being an infinite cardinal, \(T_p\) is either finite, or \(T_p\) is bounded of exponent \(p^k\) and in a decomposition of \(T_p\) into cyclic groups, there are at least \(m\) summands isomorphic to \(Z(p^k)\).

**Proof.** We note first that for any finite group \(T\), and any index set \(I\), and groups \(A_i (i \in I)\), there is a natural isomorphism:

\[
\text{Hom}(T, \bigoplus_{i \in I} A_i) \simeq \bigoplus_{i \in I} \text{Hom}(T, A_i).
\]

Hence if \(T\) is finite and \(p\)-primary, \(\text{Hom}(T, \bigoplus_{i \in I} Z(p^\infty)_i)\) is a projective \(E(T)\)-module. The
proof follows from this fact and from Lemma 7 and Corollary 4.

**Lemma 8.** Let $T$ be a reduced primary group. Then $\text{Hom}(T, Z(p^\infty))$ is an indecomposable $E(T)$-module.

**Proof.** Suppose $\text{Hom}(T, Z(p^\infty)) \cong M_1 \oplus M_2$ where $M_1 \neq 0$ and $M_2 \neq 0$. Now let $\nu$ and $\omega$ generate cyclic summands of $T$, where $o(\nu) \leq o(\omega)$, say, and where $\nu$ and $w$ need not be distinct. Suppose there exists $h_1 \in M_1, h_2 \in M_2$ with $h_1 \neq 0, h_2 \neq 0$, and having:

$$h_1(\nu) = \frac{r}{p^s}, \ h_2(\omega) = \frac{u}{p^s}$$

where

$$(r, p) = (u, p) = 1.$$

We consider the case where $s \leq z, z - s = d \geq 0$. The case $s > z$ is similar. Define $\alpha, \beta \in E(T)$ by:

$$\alpha(\omega) = x\nu \quad \beta(\omega) = p^s y \omega$$

$$\alpha = 0 \quad \text{otherwise} \quad \beta = 0 \quad \text{otherwise}$$

where $x$ and $y$ are nonzero solutions of the linear congruence:

$$rx - uy \equiv 0 \pmod{p^s}.$$ 

Then $h_1\alpha = h_2\beta \neq 0$, a contradiction. Thus we may suppose that for any $h \in M_1$, say, and any generator $\nu$ of a cyclic summand of $T$, that $h(\nu) = 0$. Since, if $T$ is bounded this implies that $h = 0$, the proof is complete in the case of $T$ bounded. For $T$ not bounded, let $h \in M_1, h \neq 0$. Say $h(t) \neq 0$, for some $t \in T$, where $o(t) = p^k$. Choose $\nu$ to be a generator of a cyclic summand of $T$ of order $p^r \geq p^s$, and define $\alpha \in E(T)$ by: $\alpha(\nu) = t, \alpha = 0$ otherwise. Then $(h\alpha)(\nu) \neq 0 - a$ contradiction.

**Lemma 9.** Let $T$ be a reduced unbounded $p$-primary group, and let $A$ be a reduced group. Then $\text{Ext}(T, A)$ is a projective $E(T)$-module if and only if $\text{Ext}(T, A) = 0$.

**Proof.** We show first that if $k \geq 1$, and $k$ is finite, $\text{Ext}(T, Z(p^k))$ is not $E(T)$-projective. For this, consider an injective resolution of $Z(p^k)$: $0 \to Z(p^k) \to Z(p^\infty) \to Z(p^r)^{\beta} \to 0$. This induces: $\text{Hom}(T, Z(p^\infty))^{\beta} \to \text{Ext}(T, Z(p^r)) \to 0$. Since $T$ is unbounded, $\beta_*$ is not an $E(T)$-isomorphism, and hence it follows from Lemma 8 that $\text{Ext}(T, Z(p^k))$ is not $E(T)$-projective. Now, if $T_\infty(A) \neq 0$, $A$ has a cyclic
abelian group summand isomorphic to \( Z(p^k) \) for some \( k \geq 1 \), and the lemma follows. Hence, suppose that \( T_\pi(A) = 0 \). Then the sequence of abelian groups: \( 0 \to T(A) \to A \to A/T(A) \to 0 \) yields the \( E(T) \)-isomorphism: \( \text{Ext}(T, A) \cong \text{Ext}(T, A/T(A)) \). Since \( A/T(A) \) is torsionfree, Corollary 5 completes the proof.

**Lemma 10.** Let \( T \) be a reduced torsion group. Then \( \text{Ext}(T, Z(p^r)) \) is a projective \( E(T) \)-module if and only if \( T_\pi \) is bounded with minimal annihilator \( p^k \) where \( k \leq r \).

**Proof.** By Lemma 9, it is necessary that \( T_\pi \) be bounded in order that \( \text{Ext}(T, Z(p^r)) \) be \( E(T) \)-projective. Consider the injective resolution of \( Z(p^r) \):

\[
0 \longrightarrow Z(p^r) \xrightarrow{i} Z(p^\infty) \xrightarrow{\pi} \frac{Z(p^\infty)}{Z(p^r)} \longrightarrow 0.
\]

This induces:

\[
\text{Hom}(T, Z(p^\infty)) \xrightarrow{\pi_*} \text{Hom}
\left(T, \frac{Z(p^\infty)}{Z(p^r)}\right) \xrightarrow{\partial} \text{Ext}(T, Z(p^r)) \longrightarrow 0.
\]

Now if \( k > r \), let \( \nu \) generate a cyclic summand of \( T \) of order \( p^k \). Define \( h \in \text{Hom}(T, Z(p^\infty)) \) by: \( h(\nu) = 1/p^k \), \( h = 0 \) otherwise. Then \( \pi_*h \neq 0 \), and so \( \ker \partial \neq 0 \). Lemma 8 completes the proof in this case, since

\[
\text{Hom}
\left(T, \frac{Z(p^\infty)}{Z(p^r)}\right) \xrightarrow{E(T)} \text{Hom}(T, Z(p^r)).
\]

If \( k \leq r \), we have: \( \text{Hom}(T, Z(p^\infty)) \cong \text{Ext}(T, Z(p^r)) \), and Corollary 1 completes the proof of the lemma.

We are now in a position to complete the proof of Theorem 1.

**Proof of Theorem 1.** If \( A[p^k] \) is homogeneous (i.e., if \( A[p^k] \cong \bigoplus_{i \in I} Z(p^k) \), for some indexing set \( I \)), and \( D \) is a divisible hull for \( A \), then it is clear that \( D[p^k] \leq A \), whence, it is also clear that if \( p^kT = 0 \), that the map \( \text{Hom}(T, D) \to \text{Hom}(T, D/T(A)) \) is the zero map. Since \( \text{Ext}(T, D/T(A)) = 0 \), this means that \( \text{Hom}(T, D/T(A)) \cong \text{Ext}(T, T(A)) \). Since \( T \) is bounded, an earlier result immediately says

\[
\text{Ext}(T, T(A)) \oplus \text{Ext}(T, A/T(A)) \cong \text{Ext}(T, A).
\]

The statement of the theorem for such \( A \) follows immediately from Corollaries 4 and 5 and from Lemmas 5 and 7.

If \( A[p^k] \) is not homogeneous, it is routine that \( A \) has a cyclic
summand of order $p^r$ for some $r < k$, and the result follows from Lemma 10.

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