Pacific Journal of Mathematics

THE PROJECTIVITY OF Ext(T, A) AS A MODULE OVER $E(T)$

JOSEPH NICHOLAS FADYN

Vol. 85, No. 2 October 1979

PACIFIC JOURNAL OF MATHEMATICS
Vol. 85, No. 2, 1979

THE PROJECTIVITY OF EXT (T, A) AS A MODULE OVER $E(T)$

JOSEPH N. FADYN

Let A and T be abelian groups. Then Ext (T, A) can be considered as a right module over $E(T)$, the ring of endomorphisms of T. In this paper necessary and sufficient conditions are developed for Ext (T, A) to be $E(T)$-projective whenever T is reduced torsion and A is reduced.

In this paper A and T will be abelian groups and Ext (T, A) will be considered as a right $E(T)$-module. (See [5].) We consider the question of when Ext (T, A) is a projective $E(T)$-module. Theorems 1 and 2 provide necessary and sufficient conditions for Ext (T, A) to be $E(T)$-projective whenever T is reduced torsion and A is reduced. It is interesting to note (Theorem 3) that if B is any reduced group, a necessary condition for Ext (B, A) to be $E(B)$-projective is that Ext $(B, A) \simeq$ Ext $(T(B), A)$. Hence if Ext (B, A) is $E(B)$-projective, Ext $(B, A) \simeq$ Ext $(T(B), A)$ and Ext $(T(B), A)$ may be considered as an $E(T(B))$-module, where $T(B)$ is, of course, reduced torsion.

We shall employ the following notations and conventions: The word "group" will always mean "abelian group." We reserve the letter T for a torsion group, and in this case, T_p will be the p-primary component of T. For an arbitrary group A, $T_p(A)$ is the p-primary component of the torsion part of A. For a ring R and a left R-module M, $r_R(M)$ will refer to the rank of M as defined in [4], $hd_R(M)$ and $id_R(M)$ will refer, respectively, to the homological and injective dimensions of M as defined in [6]. An isomorphism of R-modules M and N will be denoted by: $M \overset{R}{\simeq} N$. Other notations will follow [2]. Importantly, whenever we speak of Ext (T, A) as a right $E(T)$-module we may assume without loss of generality that A is reduced as a group. Finally, if $A \overset{Z}{\simeq} (v) \oplus A'$, and if $a \in A$, we will write, conveniently, when defining an endomorphism α of A: $\alpha(v) = a$, $\alpha = 0$ otherwise. We mean, more precisely, that: $\alpha(v) = a$, $\alpha|_{A'} = 0$. We now state our main theorems:

THEOREM 1. *Let T be a reduced p-primary group and let A be a reduced group. Then* Ext (T, A) *is a projective right* $E(T)$-*module if and only if either* Ext $(T, A) = 0$, *or all of the following conditions hold*:

(i) *T is bounded, with minimal annihilator p^k, say.*

(ii) *$A[p^k]$ is either zero or is a direct sum of cyclic groups of order p^k.*

(iii) *If D is a divisible hull of $T(A)$ and E is a divisible hull of $A/T(A)$, and if*

$$\max\left\{r_p\left(\frac{D}{T(A)}\right),\quad r_p\left(\frac{E}{A/T(A)}\right)\right\} = m\ ,$$

m an infinite cardinal, then T is either finite, or in a decomposition of T into cyclic groups, there are at least m summands isomorphic to $Z(p^k)$.

THEOREM 2. *Let T be a reduced torsion group and let A be a reduced group. Then* $\operatorname{Ext}(T, A)$ *is a projective $E(T)$-module if and only if for every p,* $\operatorname{Ext}(T_p, A)$ *is a projective $E(T_p)$-module.*

THEOREM 3. *Let A and B be reduced groups. Then a necessary condition for* $\operatorname{Ext}(B, A)$ *to be $E(B)$-projective is that* $\operatorname{Ext}(B, A) \overset{Z}{\simeq} \operatorname{Ext}(T(B), A)$.

Proofs of the theorems. The proof of Theorem 1 will require numerous preliminary results. We postpone its proof. Theorem 2 follows easily from Lemmas 1 and 2 below. We now prove Theorem 3:

Proof of Theorem 3. Since B is reduced, it is easily verified that $E(B)$ is reduced. Now, from the Z-exact sequence: $0 \rightarrow T(B) \xrightarrow{i} B \rightarrow B/T(B) \rightarrow 0$, we obtain the Z-exact sequence: $0 \rightarrow \operatorname{Ker} i^* \rightarrow \operatorname{Ext}(B, A) \xrightarrow{i^*} \operatorname{Ext}(T(B), A) \rightarrow 0$. Since $\operatorname{Ker}\ i^*$ is a subgroup of $\operatorname{Ext}(B/T(B), A)$, and since $B/T(B)$ is torsionfree, $\operatorname{Ker} i^*$ is divisible. (See [2].) Since $\operatorname{Ext}(T(B), A)$ is reduced (see [2]), it follows that $\operatorname{Ker} i^*$ is the maximal divisible subgroup of $\operatorname{Ext}(B, A)$. Now, since $E(B)$ is reduced as a group, any free $E(B)$-module is reduced as a group. So if $\operatorname{Ext}(B, A)$ is to be $E(B)$-projective, we must have $\operatorname{Ker} i^* = 0$.

We will now aim at proving Theorem 1.

LEMMA 1. *Let $M = \prod_{i \in I} M_i$ where each M_i is an R_i-module, $R = \prod_{i \in I} R_i$, and M is an R-module via the coordinatewise action of $\prod_{i \in I} R_i$. Then M is R-projective* (resp. *injective*) *if and only if M_i is R_i-projective* (resp. *injective*) *for all $i \in I$.*

Proof. The proof is easy and is omitted.

LEMMA 2. *Let F: $Ab \times Ab \to Ab$ be either of the functors* Hom *or* Ext. *Then*:

(i) *If $A = \bigoplus_{i \in I} A_i$ where the A_i are fully invariant subgroups of A, then $F(A, B) \overset{E(A)}{\simeq} \prod_{i \in I} F(A_i, B)$.*

(ii) *If $B = \prod_{i \in I} B_i$ where the B_i are fully invariant subgroups of B, then $F(A, B) \overset{E(B)}{\simeq} \prod_{i \in I} F(A, B_i)$.*

Proof. The isomorphism in (i) is given by: $F(A,B) \overset{\psi}{\simeq} \prod_{i \in I} F(A_i,B)$ where, for $f \in F(A, B)$, $\psi(f) = [f\alpha_i]_{i \in I}$ where $\alpha_i \in E(A)$ is defined by: $\alpha_i|_{A_i} = 1_{A_i}$, $\alpha_i = 0$ otherwise. It is easily verified that ψ is an $E(A)$-homomorphism.

The isomorphism for (ii) is similar.

Lemma 3 computes the injective dimension over $E(T)$ of $\operatorname{Ext}(T, A)$ when T is torsion and A is torsionfree:

LEMMA 3. *Let T be torsion and let A be torsionfree. Suppose S is the set of primes for which A is p-divisible. Then*:

(i) $\operatorname{id}_{E(T)}(\operatorname{Ext}(T, A)) = 0$ *if and only if for every prime $p \notin S$, T_p is either bounded or has an unbounded basic subgroup.*

(ii) *Otherwise*, $\operatorname{id}_{E(T)}(\operatorname{Ext}(T, A)) = 1$.

Proof. If D is a divisible hull of A, then D/A is torsion and $\operatorname{Hom}(T, D/A) \overset{E(T)}{\simeq} \operatorname{Ext}(T, A)$. By Lemma 2, it suffices to prove the result in the case in which T is a p-group, and we may assume $(D/A)_p \neq 0$, since otherwise $\operatorname{Ext}(T, A) = 0$. Assuming this, we note that by [8, Lemma 2], $\operatorname{Hom}(T, D/A)$ is $E(T)$-injective if and only if T is $E(T)$-flat. From [9] we know that this holds if and only if the condition (i) of the lemma holds (where T is a p-group, and $p \notin S$.) Otherwise, from [1], we know T has dimension one as an $E(T)$-module, and if we take a projective resolution of T and dualize it, applying [8, Lemma 2] again, we obtain an injective resolution for $\operatorname{Hom}(T, D/A)$, establishing part (ii) of the lemma.

LEMMA 4. *Let A be a reduced group with $T_p(A)$ unbounded. Then if M is a right $E(A)$-module with* $\operatorname{Hom}_Z(A, Z(p^\infty)) \subseteq M$, *then M is not $E(A)$-projective.*

Proof. We will show that there is no $E(A)$-monic map ψ:

$$0 \longrightarrow \operatorname{Hom}(A, Z(p^\infty)) \xrightarrow{\psi} \bigoplus_{b \in B} E(A)_b$$

for any indexing set B. This will complete the proof. Consider

$I = \{1, 2, 3, \cdots\}$. Since $T_p(A)$ is reduced and unbounded, for each $i \in I$, we may choose $\nu_i \in T_p(A)$ with the property that (ν_i) is a cyclic summand of $T_p(A)$ and such that $O(\nu_i) < O(\nu_{i+1})$, $i = 1, 2, 3, \cdots$. Say $(\nu_i) = Z(p^{n_i})$ for $i = 1, 2, 3, \cdots$. Now, let $h_i \in \mathrm{Hom}\,(A, Z(p^\infty))$ be defined by:

$$h_i(\nu_i) = \frac{1}{p^{n_i}}, \quad h_i = 0 \text{ otherwise} .$$

Let:

$$\psi(h_i) = \alpha_{b_{1i}} + \alpha_{b_{2i}} + \cdots + \alpha_{b_{k_i i}}$$

where $\alpha_{b_{ji}} \in E(A)_{b_{ji}}$ for all $j = 1, 2, \cdots, k_i$. Define $\beta_i \in E(A)$ by:

$$\beta_i(\nu_i) = 0, \quad \beta_i = 1 \text{ otherwise} .$$

Then the computation:

$$0 = \psi(0) = \psi(h_i\beta_i) = \alpha_{b_{1i}}\beta_i + \alpha_{b_{2i}}\beta_i + \cdots + \alpha_{b_{k_i i}}\beta_i$$

shows that $\alpha_{b_{ji}}\beta_i = 0$ for all $j = 1, 2, \cdots, k_i$, and hence that $\alpha_{b_{ji}} = 0$, except possibly on ν_i, for all $j = 1, 2, \cdots, k_i$ and for all $i = 1, 2, 3, \cdots$. Suppose $\alpha_{b_{ji}}(\nu_i) = t_{ji}$. Then $t_{ji} \in T_p(A)$, and not all t_{ji} are zero for a fixed i, where $j = 1, 2, \cdots, k_i$. By defining $\delta_i \in E(A)$ by:

$$\delta_i(\nu_{i-1}) = p^{n_i - n_{i-1}}\nu_i$$
$$\delta_i = 0 \text{ otherwise}$$

for $i = 2, 3, 4, \cdots$, the computation:

$$\begin{aligned}\psi(h_i\delta_i) &= \psi(h_{i-1}) = \alpha_{b_{1i-1}} + \alpha_{b_{2i-1}} + \cdots + \alpha_{b_{k^{i-1}i-1}} \\ &= \psi(h_i)\delta_i = \alpha b_{1i}\delta_i + \alpha_{b_{2i}}\delta_i + \cdots + \alpha_{b_{k_i i}}\delta_i\end{aligned}$$

shows that we may assume $k_1 = k_2 = \cdots = k$, say, and that:

$$\alpha_{b_{ji-1}} = \alpha_{b_{ji}}\delta_i \text{ for all } j = 1, 2, \cdots, k .$$

Now, since not all t_{ji} are zero for a fixed i, where $j = 1, 2, \cdots, k$, assume that:

$$\alpha_{b_{s1}}(\nu_1) = t_{s1} \neq 0 \text{ where } s \in \{1, 2, \cdots, k\} .$$

From this, we easily obtain the relations:

$$\begin{aligned} t_{s1} &= p^{n_2 - n_1}t_{s2} \\ t_{s2} &= p^{n_3 - n_2}t_{s3} \\ t_{s_3} &= p^{n_4 - n_3}t_{s4} \\ &\ \ \vdots \end{aligned}$$

However, this is a contradiction, since the subgroup of A generated by the t_{si}, $i = 1, 2, 3, \cdots$ is isomorphic to $Z(p^\infty)$, and A was assumed to be a reduced group.

COROLLARY 1. *Let T be a reduced torsion group. Then the following statements are equivalent:*

(i) *T is a projective left $E(T)$-module.*

(ii) Hom $(T, Q/Z)$ *is a projective right $E(T)$-module.*

(iii) *Every p-primary component of T is bounded.*

Proof. In [7] it is shown that a torsion group T is a projective left $E(T)$-module $\langle = \rangle T_p$ is bounded for all p.

Now, to prove the equivalence of (i) and (ii), we note first that by Lemma 1 we may assume that T is p-primary. Let T be bounded with minimal annihilator p^k and let ν generate a cyclic summand of T of order p^k. Then $T \overset{Z}{\simeq} (\nu) \oplus T'$, the isomorphism being one of abelian groups. Hence:

$$\mathrm{Hom}\,(T, (\nu)) \overset{E(T)}{\simeq} \mathrm{Hom}\,(T, Z(p^\infty)) ,$$

is seen to be an $E(T)$ direct summand in $E(T)_{E(T)}$. If T is not bounded, then Lemma 4 completes the proof.

COROLLARY 2. *Let T be a torsion group. Then* Hom $(T, Q/Z)$ *is a projective right $E(T)$-module if and only if for every prime p, T_p is either bounded or has an abelian group summand isomorphic to $Z(p^\infty)$.*

Proof. We may assume that T is p-primary. If T is reduced, the result follows from Corollary 1. If T is not reduced, then $T = Z(p^\infty) \oplus T'$ for some group T', and it is clear that Hom $(T, Z(p^\infty))$ is an $E(T)$-direct summand in $E(T)_{E(T)}$.

COROLLARY 3. *Let A be a torsionfree group of finite rank, and let T be a torsion group. Further, let S be the set of primes for which A is p-divisible. Then* Ext (T, A) *is a projective right $E(T)$-module if and only if for every prime $p \notin S$, T_p is either bounded or has an abelian group summand isomorphic to $Z(p^\infty)$.*

Proof. Let D be a divisible hull of A. Then: $D/A \simeq \bigoplus_{p \in P'} D_p$, where D_p is a divisible torsion group of finite rank, and where $P' = P - S$. Then:

$$\mathrm{Ext}\,(T, A) \overset{E(T)}{\simeq} \mathrm{Hom}\left(T, \frac{D}{A}\right) \overset{E(T)}{\simeq} \prod_{p \in P'} \mathrm{Hom}\,(T_p, D_p) .$$

The proof is completed by Lemma 1 and Corollary 2, recalling also that a direct sum of projective modules is projective.

LEMMA 5. *Let T be a bounded p-primary group with minimal annihilator p^k and let n be any cardinal. Further assume that in a decomposition of T into cyclic groups, there are at least n summands isomorphic to $Z(p^k)$. Then for any indexing set I with $|I| \leqq n$, $\mathrm{Hom}\,(T, \bigoplus_{i \in I} Z(p^\infty)_i)$ is a cyclic projective $E(T)$-module.*

Proof. There is a set $\{v_j\}_{j \in J}$ where v_j generates a cyclic abelian group summand of T of order p^k, and where $|J| = |I|$. Then $T \overset{Z}{\simeq} \bigoplus_{j \in J} (v_j) \oplus T'$, where the isomorphism is one of abelian groups. Thus, $\mathrm{Hom}\,(T, \bigoplus_{j \in J} (v_j)) \overset{E(T)}{\simeq} \mathrm{Hom}\,(T, \bigoplus_{i \in I} Z(p^\infty)_i)$ is seen to be an $E(T)$-direct summand in $E(T)_{E(T)}$.

LEMMA 6. *Let V be a vector space of infinite dimension over a field k, and $E = \mathrm{End}\,(V)$. Let $H = \mathrm{Hom}\,(V, \bigoplus_{i \in I} V)$. Then H is not projective as an E-module if $|I| > \dim\,(V)$.*

Proof. We first note that if F is a countable subset of H, then F is contained in a cyclic submodule of H. To see this, let W be a subspace of $\bigoplus_{i \in I} V$ containing $f(V)$ for all $f \in F$, and such that $\dim\,(W) = \dim\,(V)$. We may regard $\mathrm{Hom}\,(V, W)$ as an E-submodule of H and this submodule certainly contains F. Since $W \simeq V$, $\mathrm{Hom}\,(V, W) \simeq E$.

We next note that any module with the above property cannot have an infinite direct sum decomposition (clearly). Now if H were projective, it would be a direct sum of countably generated submodules (by Kaplansky's theorem in [3]). Since H is clearly not countably generated, this would mean that it had an infinite direct sum decomposition, which, as we have just seen, it does not.

COROLLARY 4. *Let T be a bounded p-group, of exponent p^k, and such that in a direct sum decomposition of T there are n summands of order p^k where n is an infinite cardinal* (i.e., $n = \dim(T/T[p^{k-1}])$, *where $T/T[p^{k-1}]$ is viewed as a vector-space over Z/pZ). Let $H = \mathrm{Hom}\,(T, U)$, regarded as an $E(T)$-module, where U is a direct sum of m copies of $Z(p^\infty)$, for some m, $m > n$. Then H is not a projective $E(T)$-module.*

Proof. Let $E = \mathrm{End}\,(T/T[p^{k-1}])$. There is a natural map $\mathrm{End}\,(T) \to E$ (since $T[p^{k-i}]$ is a fully invariant submodule of T) which is clearly surjective. If I is the kernel of this map of rings, (so $I=\{f \in \mathrm{End}(T): f(T) \leqq T[p^{k-1}]\}$), then one easily identifies H/HI with $\mathrm{Hom}(T/T[p^{k-1}], U[p^k]/U[p^{k-1}])$. If H is projective as an End (T)-module, then H/HI must be projective as an E-module, which, according to the previous lemma, it is not.

LEMMA 7. *Let T be a bounded p-group which is infinite, but such that it's highest nonzero Ulm invariant is finite. Let U be the direct sum of a countable number of copies of $Z(p^\infty)$, and let $E = \mathrm{End}(T)$. Then $H = \mathrm{Hom}(T, U)$ is not a projective E-module.*

Proof. If there is a split mono $\mu: H \to \bigoplus_{i \in I} E$, then it induces a split mono $H/H[p^{k-1}] \to \bigoplus_{i \in I} E/E[p^{k-1}]$, where we choose k such that $p^k T = 0$, $p^{k-1}T \neq 0$ (i.e., p^k is the exponent of T). We note that $H/H[p^{k-1}]$ is infinite dimensional and all of the terms on the right above are finite dimensional over Z/pZ. We now let $f: T/pT \to U[p]$ be a surjective homomorphism. If $g: T \to U[p]$ is any homomorphism with $T[p^{k-1}]$ in its kernel, then there is an endomorphism $\varepsilon_g: T \to T$ such that $g = f\varepsilon_g$. It follows that if $h \in H$, then for some endomorphism ϕ of E, $p^{k-1}h = f\phi$. Now if π_i is the projection onto the ith summand in the above free E-module, then $\pi_i\mu(f) \neq 0$ for only a finite number of indicies i. Let this finite subset of I be J. It follows that $p^{k-1}\pi_i\mu(h) = 0$ unless $i \in J$, for all $h \in H$. Hence the image of the induced map

$$H/H[p^{k-1}] \longrightarrow \bigoplus_{i \in I} E/E[p^{k-1}]$$

is actually in the submodule $\bigoplus_{i \in J} E/E[p^{k-1}]$. This is a contradiction, since this is finite dimensional, and $H/H[p^{k-1}]$ is not.

COROLLARY 5. *Let A be torsionfree, and let D be a divisible hull of A. Let S be the set of primes for which A is p-divisible, and let T be a reduced torsion group. Then* $\mathrm{Ext}(T, A)$ *is a projective $E(T)$-module if and only if for every $p \notin S$ the following two conditions hold:*

(i) *Whenever $r_p(D/A)$ is finite, T_p is bounded.*

(ii) *Whenever $r_p(D/A) = m$, m being an infinite cardinal, T_p is either finite, or T_p is bounded of exponent p^k and in a decomposition of T_p into cyclic groups, there are at least m summands isomorphic to $Z(p^k)$.*

Proof. We note first that for any finite group T, and any index set I, and groups $A_i (i \in I)$, there is a natural isomorphism: $\mathrm{Hom}(T, \bigoplus_{i \in I} A_i) \simeq \bigoplus_{i \in I} \mathrm{Hom}(T, A_i)$. Hence if T is finite and p-primary, $\mathrm{Hom}(T, \bigoplus_{i \in I} Z(p^\infty)_i)$ is a projective $E(T)$-module. The proof follows from this fact and from Lemma 7 and Corollary 4.

LEMMA 8. *Let T be a reduced primary group. Then* Hom $(T, Z(p^\infty))$ *is an indecomposable $E(T)$-module.*

Proof. Suppose Hom $(T, Z(p^\infty)) \overset{E(T)}{\simeq} M_1 \oplus M_2$ where $M_1 \neq 0$ and $M_2 \neq 0$. Now let ν and ω generate cyclic summands of T, where $o(\nu) \leqq o(\omega)$, say, and where ν and w need not be distinct. Suppose there exists $h_1 \in M_1$, $h_2 \in M_2$ with $h_1 \neq 0$, $h_2 \neq 0$, and having:

$$h_1(\nu) = \frac{r}{p^s},\ h_2(\omega) = \frac{u}{p^z}$$

where

$$(r, p) = (u, p) = 1 .$$

We consider the case where $s \leqq z$, $z - s = d \geqq 0$. The case $s > z$ is similar. Define $\alpha, \beta \in E(T)$ by:

$$\alpha(\omega) = x\nu \qquad \beta(\omega) = p^d y\omega$$

$$\alpha = 0 \text{ otherwise } \beta = 0 \text{ otherwise}$$

where x and y are nonzero solutions of the linear congruence:

$$rx - uy \equiv 0 \pmod{p^s} .$$

Then $h_1\alpha = h_2\beta \neq 0$, a contradiction. Thus we may suppose that for any $h \in M_1$, say, and any generator ν of a cyclic summand of T, that $h(\nu) = 0$. Since, if T is bounded this implies that $h = 0$, the proof is complete in the case of T bounded. For T not bounded, let $h \in M_1$, $h \neq 0$. Say $h(t) \neq 0$, for some $t \in T$, where $o(t) = p^k$. Choose ν to be a generator of a cyclic summand of T of order $p^r \geqq p^s$, and define $\alpha \in E(T)$ by: $\alpha(\nu) = t$, $\alpha = 0$ otherwise. Then $(h\alpha)(\nu) \neq 0 - a$ contradiction.

LEMMA 9. *Let T be a reduced unbounded p-primary group, and let A be a reduced group. Then* Ext (T, A) *is a projective $E(T)$-module if and only if* Ext $(T, A) = 0$.

Proof. We show first that if $k \geqq 1$, and k is finite, Ext $(T, Z(p^k))$ is not $E(T)$-projective. For this, consider an injective resolution of $Z(p^k)$: $0 \to Z(p^k) \xrightarrow{i} Z(p^\infty) \xrightarrow{\beta} Z(p^\infty) \to 0$. This induces: Hom $(T, Z(p^\infty)) \xrightarrow{\beta^*}$ Ext $(T, Z(p^k)) \to 0$. Since T is unbounded, β_* is not an $E(T)$-isomorphism, and hence it follows from Lemma 8 that Ext $(T, Z(p^k))$ is not $E(T)$-projective. Now, if $T_p(A) \neq 0$, A has a cyclic abelian group summand isomorphic to $Z(p^k)$ for some $k \geqq 1$, and the lemma follows. Hence, suppose that $T_p(A) = 0$. Then the sequence of abelian groups: $0 \to T(A) \to A \to A/T(A) \to 0$ yields the $E(T)$-isomorphism: $\mathrm{Ext}\,(T, A) \overset{E(T)}{\simeq} \mathrm{Ext}\,(T, A/T(A))$. Since $A/T(A)$ is torsionfree, Corollary 5 completes the proof.

LEMMA 10. *Let T be a reduced torsion group. Then* $\mathrm{Ext}\,(T, Z(p^r))$ *is a projective $E(T)$-module if and only if T_p is bounded with minimal annihilator p^k where $k \leqq r$.*

Proof. By Lemma 9, it is necessary that T_p be bounded in order that $\mathrm{Ext}\,(T, Z(p^r))$ be $E(T)$-projective. Consider the injective resolution of $Z(p^r)$:

$$0 \longrightarrow Z(p^r) \xrightarrow{i} Z(p^\infty) \xrightarrow{\pi} \frac{Z(p^\infty)}{Z(p^r)} \longrightarrow 0 .$$

This induces:

$$\mathrm{Hom}\,(T, Z(p^\infty)) \xrightarrow{\pi_*} \mathrm{Hom}\left(T, \frac{Z(p^\infty)}{Z(p^r)}\right) \xrightarrow{\Delta} \mathrm{Ext}\,(T, Z(p^r)) \longrightarrow 0 .$$

Now if $k > r$, let ν generate a cyclic summand of T of order p^k. Define $h \in \mathrm{Hom}\,(T, Z(p^\infty))$ by: $h(\nu) = 1/p^k$, $h = 0$ otherwise. Then $\pi_* h \neq 0$, and so $\ker \Delta \neq 0$. Lemma 8 completes the proof in this case, since

$$\mathrm{Hom}\left(T, \frac{Z(p^\infty)}{Z(p^r)}\right) \overset{E(T)}{\simeq} \mathrm{Hom}\,(T, Z(p^\infty)) .$$

If $k \leqq r$, we have: $\mathrm{Hom}\,(T, Z(p^\infty)) \overset{E(T)}{\simeq} \mathrm{Ext}\,(T, Z(p^r))$, and Corollary 1 completes the proof of the lemma.

We are now in a position to complete the proof of Theorem 1.

Proof of Theorem 1. If $A[p^k\}$ is homogeneous (i.e., if $A[p^k] \simeq \bigoplus_{i \in I} Z(p^k)_i$ for some indexing set I), and D is a divisible hull for A, then it is clear that $D[p^k] \leqq A$, whence, it is also clear that if $p^k T = 0$, that the map $\mathrm{Hom}\,(T, D) \to \mathrm{Hom}\,(T, D/T(A))$ is the zero map. Since $\mathrm{Ext}\,(T, D/T(A)) = 0$, this means that $\mathrm{Hom}\,(T, D/T(A)) \overset{E(T)}{\simeq} \mathrm{Ext}\,(T, T(A))$. Since T is bounded, an earlier result immediately says

$$\mathrm{Ext}\,(T, T(A)) \oplus \mathrm{Ext}\,(T, A/T(A)) \overset{E(T)}{\simeq} \mathrm{Ext}\,(T, A) .$$

The statement of the theorem for such A follows immediately from Corollaries 4 and 5 and from Lemmas 5 and 7.

If $A[p^k]$ is not homogeneous, it is routine that A has a cyclic summand of order p^r for some $r < k$, and the result follows from Lemma 10.

The author wishes to thank the referee, whose suggestions considerably shortened the work of proving Theorem 1.

References

1. A. J. Douglas and H. K. Farahat, *The homological dimension of an Abelian group as a module over its ring of endomorphisms*, Montashefte Math., **69** (1965).
2. L. Fuchs, "*Infinite Abelian Groups*," Vols. I and II, Pure and Appl. Math., **36**, Academic Press, New York, 1970, 1973.
3. I. Kaplansky, *Projective modules*, Ann. Math., **68** (1958), 372-377.
4. S. A. Khabbaz and E. H. Toubassi, *Ext(A, T) as a module over End(T)*, Proc. Amer. Math. Soc., **47**, Number 2, February 1975.
5. S. MacLane, *Homology*, Springer-Verlag, New York, 1967.
6. D. G. Northcott, *An Introduction to Homological Algebra*, Cambridge, 1960.
7. F. Richman and E. A. Walker, *Homological dimension of Abelian groups over their endomorphism rings*, Proc. Amer. Math. Soc., January, 1976.
8. ———, *Modules over PIDs that are injective over their endomorphism rings, in ring theory*, ed. by R. Gordon, Academic Press, (1972), 363-372.
9. ———, *Primary Abelian groups as modules over their endomorphism rings*, Math. Zeitschr., **89** (1965), 77-81.

Received December 14, 1977 and in revised form August 16, 1978.

PENN STATE
P. O. BOX 1830
WILKES-BARRE, PA 18708

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $84.00 a year (6 Vols., 12 issues). Special rate: $42.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Older back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics
Vol. 85, No. 2 October, 1979