

# Pacific Journal of Mathematics

## **A COHOMOLOGICAL INTERPRETATION OF BRAUER GROUPS OF RINGS**

RAYMOND TAYLOR HOOBLER

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*Dedicated to Gerhard Hochschild on the occasion of his 65th birthday*

Quillen's proof of the Serre conjecture introduced a new tool for passing from local to global results on affine schemes. We use this to prove the theorem below characterizing the image of the injection  $i: Br(X) \rightarrow H^2(X_{et}, G_m)$  when  $X = \text{Spec } A$ , is a regular scheme. A result of M. Artin then allows us to conclude that  $Br(X) \cong H^2(X_{et}, G_m)$  if  $X = \text{Spec } A$  is a smooth, affine scheme over a field. For such rings, this proves the Auslander Goldman conjecture [2],  $Br(A) = \bigcap Br(A_\nu)$ ,  $\mathfrak{p} \in P(A)$ , the set of height one primes of  $A$ .

We begin with following theorem.

**THEOREM.** *Let  $X = \text{Spec } A$  be a regular scheme. If  $c \in H^2(X_{et}, G_m)$  and  $c_y = i([A_y])$  in  $H^2(\text{Spec}(A_{m_y})_{et}, G_m)$  for all closed points  $y \in X$ , then  $c = i([A])$ .*

*Proof.* If  $f \in A$ , let  $c_f$  denote the restriction of  $c$  to

$$H^2(\text{Spec}(A_f)_{et}, G_m).$$

Let  $S = \{f \in A \mid c_f = i([A]) \text{ for some Azumaya algebra } A \text{ over } A_f\}$ . We will show that  $S$  is an ideal. Then  $S = A$  since the hypothesis on  $c$  prevents  $S$  from being contained in any maximal ideal of  $A$ .

Suppose  $f_1, f_2 \in S$  and  $f \in Af_1 + Af_2$ . Then  $\text{Spec}(A_f) = D_{f_1} \cup D_{f_2}$  where  $D_{f_i} = \text{Spec}(A_{f_i})$ . Hence we may assume  $A_f = A$  and  $\text{Spec}(A)$  is covered by  $D_{f_1} \cup D_{f_2}$ . Let  $A_1, A_2$  be Azumaya algebras over  $A_{f_1}, A_{f_2}$  with  $i([A_1]) = c_{f_1}$  and  $i([A_2]) = c_{f_2}$ . Since  $i$  is injective,  $[A_{1f_2}] = [A_{2f_1}]$ ; that is, there are locally free coherent  $A_{f_1f_2}$  modules  $P_1, P_2$  such that  $A_{1f_2} \otimes \text{End}(P_1) \cong A_{2f_2} \otimes \text{End}(P_2)$ . Since  $K^0(A_{f_i}) \rightarrow K^0(A_{f_1f_2})$  is onto ( $A$  is regular) [3] and we may assume the rank of  $P_i$  is large, there are locally free coherent  $A_{f_i}$  modules  $Q_i$  such that  $Q_{if_i} \cong P_i$  [3, Chapter IX, 4.1]. Replacing  $A_i$  by  $A_i \otimes \text{End}(Q_i)$ , we may assume that  $A_{1f_2} \cong A_{2f_1}$ . Using this patching isomorphism we produce an algebra  $A$  with  $A_{f_1} \cong A_1, A_{f_2} \cong A_2$ . Since  $H^2(X_{et}, G_m) \rightarrow H^2(D_{f_{et}}, G_m)$  is a monomorphism,  $c = i([A])$ .

**COROLLARY 1.** *Let  $X = \text{Spec}(A)$  be a smooth  $k$  scheme where  $k$  is a field. Then  $Br(X) \cong H^2(X_{et}, G_m)$ .*

*Proof.* Since  $X$  is regular,  $H^2(X_{et}, G_m)$  is torsion [4]. If  $c \in$

$H^2(X_{et}, G_m)$  has order  $n$  and  $n$  is relatively prime to the characteristic of  $k$ , then the Kummer sequence

$$0 \longrightarrow \mu_n \longrightarrow G_m \longrightarrow G_m \longrightarrow 0$$

shows that  $c$  is in the image of  $H^2(X_{et}, \mu_n)$ . Now the existence of good neighborhoods [1] on  $\bar{X} = X \otimes_k \bar{k}$ ,  $\bar{k}$  = algebraic closure of  $k$ , shows that elements of  $H^2(\bar{X}_{et}, \mu_n)$  are locally isotrivial, i.e., there is a Zariski covering  $\{U_i\}$  of  $\bar{X}$  and a finite, etale covering space  $V_i \rightarrow U_i$  which splits elements of  $H^2(X_{et}, \mu_n)$ . Consequently  $X$  has a Zariski open covering  $\{U_\alpha\}$  and finite, flat coverings  $\pi_\alpha: W_\alpha \rightarrow U_\alpha$  such that  $\pi_\alpha^*(c|_{U_\alpha}) = 0$ . Hence by the criterion in [6],  $c|_{U_\alpha}$  is in the image of  $Br(U_\alpha)$  and so by the theorem  $c$  is in the image of  $Br(X)$ . If  $c$  has order  $p^n$ ,  $p = \text{char } k$ , then we know that  $F_X^{p^n}(c) = 0$  where  $F_X$  is the Frobenius map. Since it defines a finite flat covering of  $X$ , the same criterion shows that  $c$  is in the image of  $Br(X)$  (see [7] where this argument is given in more detail.).

**COROLLARY 2.** *Let  $A$  be an algebra of finite type over a field  $k$  such that  $A \otimes \bar{k}$  is regular, i.e.,  $\text{Spec } A$  is smooth over  $k$ . Then*

$$Br(A) = \bigcap Br(A_\mathfrak{p}), \mathfrak{p} \in P(A) = \{\mathfrak{p} / \text{height } \mathfrak{p} = 1\} .$$

*Proof.* We will use induction on  $n = \dim A$ . If  $n = 0, 1$ , or  $2$ , the result was proven in [2, 4]. Since  $A$  is a regular ring,  $Br(A) \subset \bigcap Br(A_\mathfrak{p}), \mathfrak{p} \in P(A)$ . Hence the argument of the theorem shows that

$$S = \{f \in A / Br(A_\mathfrak{p}) = \bigcap Br(A_\mathfrak{p}), \mathfrak{p} \in P(A_\mathfrak{p})\}$$

is an ideal in  $A$  and so is either  $A$  or is contained in a maximal ideal of  $A$ . Hence we may assume  $A$  is a local ring of dimension greater than 2.

Let  $c \in \bigcap Br(A_\mathfrak{p}), \mathfrak{p} \in P(A)$ , and  $X = \text{Spec } A$  and  $U$  be the punctured spectrum. Since  $Br(A_\mathfrak{p}) = H^2(\text{Spec}(A_\mathfrak{p})_{et}, G_m)$ , there is a cohomology class  $c_1 \in \bigcap H^2(\text{Spec}(A_\mathfrak{p})_{et}, G_m) \subseteq H^2(\text{Spec}(K)_{et}, G_m), \mathfrak{p} \in P(A)$ , with  $c_1 = i(c)$  where  $K$  is the quotient field of  $A$ . Now the Mayer-Vietoris sequence, which may be viewed as the Cech spectral sequence for the covering  $\{U_1, U_2\}$  of  $U_1 \cup U_2$ , and the induction hypothesis show that there is a  $c' \in H^2(U_{et}, G_m)$  whose restriction to  $H^2(\text{Spec}(A_{et}, G_m)$  is  $i(c)$ . Suppose  $c'$  is of order  $n$  where  $(n, \text{char } k) = 1$ . Then the Kummer sequence shows that there is a cohomology class in  $H^2(U_{et}, \mu_n)$  whose image in  $H^2(U_{et}, G_m)$  is  $c'$ . But  $H^2(X_{et}, \mu_n) = H^2(U_{et}, \mu_n)$  by relative cohomological purity [1, Expose XVI] and so there is a cohomology class  $c''$  in  $H^2(X_{et}, G_m)$  whose restriction to  $U$  is  $c'$ . By the first corollary  $c''$  is in the image of  $Br(X)$  as desired.

If  $n = p^m, p = \text{char}(k)$ , the same argument will work if we can

show that  $c'$  is in the image of  $H^2(X_{et}, G_m)$ . We will then be done since  $c' = c'_1 + c'_2$  where the order of  $c'_1 = p^m$  and the order of  $c'_2$  is prime to  $p$ . The obstruction to  $c'$  being in the image of  $H^2(X_{et}, G_m)$  lies in the local cohomology group  $H^2_P(X_{et}, G_m)$  where  $P$  is the closed point of  $X$ . Moreover since  $F_{X^*}^m(c') = p^m c' = 0$  where  $F_X: X \rightarrow X$  is the purely inseparable Galois covering defined by the Frobenius map, the obstruction lies in the kernel of  $F_X^{m*}: H^2_P(X_{et}, G_m) \rightarrow H^2_P(X_{et}, G_m)$ .

We have an exact sequence of sheaves on  $X_{et}$  [7]

$$0 \longrightarrow G_m \xrightarrow{j} F_{X^*} G_m \longrightarrow \mathcal{Z}_X^1 \longrightarrow \Omega_X^1 \longrightarrow 0$$

where  $\mathcal{Z}_X^1$  and  $\Omega_X^1$  are free  $A$ -modules ( $A$  is smooth and local) whose definition is unimportant. If  $C$  denotes the cokernel of  $j$ , then  $H^2_P(X_{et}, C)$  is trapped between  $H^2_P(X_{et}, \Omega_X^1)$  and  $H^2_P(X_{et}, \mathcal{Z}_X^1)$ . Since  $\Omega_X^1$  and  $\mathcal{Z}_X^1$  are coherent sheaves, their local cohomology in the etale and Zariski topology coincide and hence vanish because  $H^i_{(m)}(\text{Spec}(A), A) = 0$  if  $A$  is regular and  $i < \dim A$ . Since  $\dim A > 2$ , we conclude that

$$F_X^*: H^2_P(X_{et}, G_m) \longrightarrow H^2_P(X_{et}, F_{X^*} G_m) = H^2_P(X_{et}, G_m)$$

is injective and so  $c'$  is in the image of  $H^2(X_{et}, G_m)$ .

**COROLLARY 3.** *Let  $\pi: X \rightarrow Y$  be a proper, smooth morphism of fibre dimension one where  $X$  is a smooth scheme over a field. Then  $R^2 \pi_* G_m = 0$ .*

*Proof.* We may assume that  $Y$  is a strictly local  $k$ -scheme,  $k$  a field, and we must show that  $H^2(X_{et}, G_m) = 0$ . Since  $\pi$  has fibre dimension one and is proper,  $X$  is a union of two affine schemes which are limits of smooth  $k$ -schemes. Consequently  $H^2(X_{et}, G_m) = Br(X)$ . But  $Br(\pi^{-1}(y)) = 0$  by Tsen's theorem. Thus Artin approximation may be used as in [5] to lift a trivialization of an Azumaya algebra on the fibre to a trivialization of the algebra on  $X$ .

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