ON UNIVERSAL EXTENSIONS OF DIFFERENTIAL FIELDS

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Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

The main result of this paper is the following:

**THEOREM:** Let \( \mathcal{U} \) be a universal extension of the differential field \( \mathcal{F} \) of characteristic zero and let \( \mathcal{F}' \) be a strongly normal extension of \( \mathcal{F} \) in \( \mathcal{U} \). Then \( \mathcal{U} \) is a universal extension of \( \mathcal{F} \).

**Introduction.** We deal with differential fields, always of characteristic zero, relative to a nonempty finite set of commuting derivation operators. By an extension of a differential field, we always mean a differential field extension. An extension \( \mathcal{F}' \) of a differential field \( \mathcal{F} \) is said to be finitely generated if \( \mathcal{F}' \) has a finite subset \( \Phi \) such that \( \mathcal{F}' = \mathcal{F} \langle \Phi \rangle \) is the smallest extension of \( \mathcal{F} \) in \( \mathcal{F}' \) that contains \( \Phi \).

Let \( \mathcal{F} \) be a differential field. Recall that an extension \( \mathcal{U} \) of \( \mathcal{F} \) is called universal if, for any finitely generated extension \( \mathcal{F}_1 \) of \( \mathcal{F} \) in \( \mathcal{U} \) and any finitely generated extension \( \mathcal{C} \) of \( \mathcal{F}_1 \) not necessarily in \( \mathcal{U} \), \( \mathcal{C} \) can be embedded in \( \mathcal{U} \) over \( \mathcal{F}_1 \), i.e., there exists an extension of \( \mathcal{F}_1 \) in \( \mathcal{U} \) that is isomorphic (in the sense of differential fields) to \( \mathcal{C} \) over \( \mathcal{F}_1 \). Such a universal extension of \( \mathcal{F} \) always exists ([2] p. 132, Th. 2). It is not unique, but if \( \mathcal{U} \) and \( \mathcal{V} \) are two universal extensions of \( \mathcal{F} \), then there exist universal extensions \( \mathcal{U}' \) and \( \mathcal{V}' \) of \( \mathcal{F} \) lying in \( \mathcal{U} \) and \( \mathcal{V} \), respectively, such that \( \mathcal{U}' \) is isomorphic to \( \mathcal{V}' \) over \( \mathcal{F} \) ([2] p. 135, Exerc. 7).

Let \( \mathcal{U} \) be a universal extension of the differential field \( \mathcal{F} \) and let \( \mathcal{C} \) be an extension of \( \mathcal{F} \) in \( \mathcal{U} \). Under favorable conditions, \( \mathcal{U} \) is then a universal extension of \( \mathcal{C} \), too. For example, this is the case when \( \mathcal{C} \) is finitely generated over \( \mathcal{F} \) ([2] p. 133, Prop. 4), and also when \( \mathcal{C} \) is algebraic over \( \mathcal{F} \) ([2] p. 134, Exerc. 1). The main purpose of the present note is to point out another such favorable condition. We shall show (§1) that when \( \mathcal{C} \) is a strongly normal extension of \( \mathcal{F} \), in the general sense of Kovacic [4] (i.e., not necessarily finitely generated), then \( \mathcal{U} \) is universal over \( \mathcal{C} \). This result shows that, in the study of strongly normal extensions, it is not necessary to replace \( \mathcal{U} \) by a larger universal extension of \( \mathcal{F} \) (see Kovacic [4] p. 518).

Every strongly normal extension of \( \mathcal{F} \) in \( \mathcal{U} \) is embeddable over \( \mathcal{F} \) in a constrained closure of \( \mathcal{F} \) in \( \mathcal{U} \) ([3] p. 162, Th. 3 or Blum [1] p. 42 (15)) and hence, in particular, is constrained over \( \mathcal{F} \)
It is tempting to conjecture that the above result generalizes to constrained extensions of \( F \) in \( U \). We shall show (§2) by a counterexample that \( U \) can fail to be universal over a constrained closure of \( F \) in \( U \).

1. Strongly normal extensions. Recall ([2] p. 393), for a finitely generated extension \( G \) of \( F \) in a given universal extension \( U \) of \( F \), that \( G \) is called strongly normal over \( F \) if every isomorphism \( \sigma \) over \( F \) of \( G \) onto an extension of \( F \) in \( U \) is strong, i.e., has the property that \( \sigma c = c \) for every constant \( c \) in \( G \) and \( G \cdot K = \sigma G \cdot K \), where \( K \) denotes the field of constants of \( F \).

This definition is apparently a relative one, depending on the universal extension \( U \) of \( F \) in which \( G \) is embedded. It is easy to see, however, that if \( G \) is strongly normal over \( F \) relative to one \( U \), then \( G \) is strongly normal over \( F \) relative to every \( U \), so that the notion of strongly normal finitely generated extension is an absolute one. When \( G \) is not necessarily finitely generated over \( F \), \( G \) is said, following Kovacic [4] p. 518, to be strongly normal over \( F \) if \( G \) is the union of strongly normal finitely generated extensions. Hence, also this more general notion is absolute.

It follows from [2] pp. 402–403, Th. 5, and the definition that if \( G \) is any strongly normal extension of \( F \) and \( \mathcal{C} \) is any extension of \( F \), both contained in an extension of \( F \) having the same field of constants as \( F \), then \( G \cdot \mathcal{C} \) is a strongly normal extension of \( \mathcal{C} \), and \( G \) and \( \mathcal{C} \) are linearly disjoint over \( G \cap \mathcal{C} \).

We now prove the main theorem of this paper which was stated in the opening paragraph.

Proof. (a) We must show that if \( G_1 \) is a finitely generated extension of \( G \) in \( U \) and \( H \) is any finitely generated extension of \( G_1 \) not necessarily in \( U \), then there exists an embedding \( H \to U \) over \( G_1 \). As before, denote the field of constants of \( U \) by \( K \), and put \( C = F \cap K, G_1 = G_1 \cap K \). Then \( C = G \cap K \) ([2] p. 393, Prop. 9), \( C_1 \) is a finitely generated field extension of \( C \) ([2] p. 113, Cor. 1 to Prop. 14), \( U \) is a universal extension of \( FC_1 \), and \( FC_1 \) is a strongly normal extension of \( FC_1 \) ([2] p. 396, Th. 2). Thus, we may replace \((F, \mathcal{C}, G, H) \) by \((FC_1, FC_1, G_1, H) \), i.e., we may suppose that \( F, G, G_1 \) have the same field of constants \( C \).

(b) That being the case, fix a finite family \( \beta \) of generators of \( G_1 \) over \( G \). Then \( U \) is a universal extension of \( F \langle \beta \rangle \) and \( G_1 = FC_1 \langle \beta \rangle \) is a strongly normal extension of \( \langle F \langle \beta \rangle \rangle \). Thus, we may replace \((F, G, G_1, H) \) by \((F \langle \beta \rangle, G_1, G_1, H) \), i.e., we may suppose that \( G_1 = G \).
(c) That being the case, let \( \mathcal{D} \) denote the field of constants of \( \mathcal{A} \). Then \( \mathcal{D} \) is a finitely generated field extension of \( \mathcal{C} \), so that there exists an isomorphism \( \mathcal{A} \cong \mathcal{D}' \) over \( \mathcal{C} \) with \( \mathcal{D}' \) a field extension of \( \mathcal{C} \) in \( \mathcal{A} \). Because \( \mathcal{C} \) and \( \mathcal{D} \) are linearly disjoint over \( \mathcal{C} \), so that \( \mathcal{C}' \) and \( \mathcal{D}' \) are linearly disjoint over \( \mathcal{C} \), and likewise \( \mathcal{C} \) and \( \mathcal{D} \), this can be extended to an isomorphism \( \mathcal{A} \cong \mathcal{D}' \) over \( \mathcal{C} \). This can in turn be extended to an isomorphism \( \mathcal{A} \cong \mathcal{D}' \) over \( \mathcal{C} \). Thus, we may replace \( (\mathcal{F}, \mathcal{D}, \mathcal{H}) \) by \( (\mathcal{F}', \mathcal{D}', \mathcal{H}') \), i.e., we may suppose that the field of constants of \( \mathcal{A} \) is \( \mathcal{C} \).

(d) That being the case, fix a finite family \( \alpha \) of generators of the extension \( \mathcal{A}' \) of \( \mathcal{A} \), and put \( \mathcal{E} = \mathcal{F}(\alpha) \). Then \( \mathcal{C} \cap \mathcal{E} \) is a finitely generated extension of \( \mathcal{F} \) ([2] p. 112, Prop. 14), so that \( \mathcal{U} \) is universal over \( \mathcal{C} \cap \mathcal{E} \). Thus, we may replace \( (\mathcal{F}, \mathcal{C}, \mathcal{H}, \mathcal{E}) \) by \( (\mathcal{F}, \mathcal{C}, \mathcal{H}, \mathcal{E}) \), i.e., we may suppose that \( \mathcal{C} \cap \mathcal{E} = \mathcal{F} \). Since \( \mathcal{C} \) is strongly normal over \( \mathcal{F} \), then the differential field \( \mathcal{H}' = \mathcal{F} \mathcal{E} \) is strongly normal over \( \mathcal{C} \) and \( \mathcal{C} \) and \( \mathcal{E} \) are linearly disjoint over \( \mathcal{F} \).

(e) Because \( \mathcal{U} \) is universal over \( \mathcal{F} \), there exists an isomorphism \( \mathcal{E} \cong \mathcal{E}_0 \) over \( \mathcal{F} \) with \( \mathcal{E}_0 \) an extension of \( \mathcal{F} \) in \( \mathcal{U} \), and this isomorphism can be extended to an isomorphism \( \sigma: \mathcal{H} \cong \mathcal{H}_0 \), where \( \mathcal{H}_0 \) is an extension of \( \mathcal{F} \) (and of \( \mathcal{E}_0 \) not necessarily in \( \mathcal{U} \)). Put \( \mathcal{C}_0 = \sigma \mathcal{C} \). Then \( \mathcal{H}_0 = \mathcal{C}_0 \mathcal{E}_0 \), this differential field is a strongly normal extension of \( \mathcal{E}_0 \), and \( \mathcal{E}_0 \) and \( \mathcal{E}_0 \) are linearly disjoint over \( \mathcal{F} \). Evidently \( \mathcal{U} \) is universal over \( \mathcal{E}_0 \) (because \( \mathcal{E}_0 \) is finitely generated over \( \mathcal{F} \)), and hence the strongly normal extension \( \mathcal{E}_0 \mathcal{E}_0 \) of \( \mathcal{E}_0 \) can be embedded in \( \mathcal{U} \) over \( \mathcal{E}_0 \), i.e., there exists an isomorphism \( \sigma_0: \mathcal{E}_0 \mathcal{E}_0 \cong \mathcal{C}_2 \mathcal{E}_0 \) over \( \mathcal{E}_0 \) with \( \sigma_0 \mathcal{C}_0 = \mathcal{C}_2 \subset \mathcal{U} \). The field of constants of \( \mathcal{C}_2 \mathcal{E}_0 \), like those of \( \mathcal{H}_0 = \mathcal{C}_0 \mathcal{E}_0 \) and \( \mathcal{H} = \mathcal{F} \mathcal{E} \), is \( \mathcal{C} \), and hence \( \mathcal{C}_2 \mathcal{E}_0 \) and \( \mathcal{H} \) are linearly disjoint over \( \mathcal{C} \). Therefore \( \mathcal{C}_2 \mathcal{E}_0 \) and \( \mathcal{C}_2 \mathcal{H} \) are linearly disjoint over \( \mathcal{C}_2 \). But by (d), \( \mathcal{C} \) and \( \mathcal{E} \) are linearly disjoint over \( \mathcal{F} \), so that \( \mathcal{C} \) and \( \mathcal{C}_0 \) are, too, and hence also \( \mathcal{E}_0 \) and \( \mathcal{C}_2 \). Therefore \( \mathcal{E}_0 \) and \( \mathcal{C}_2 \mathcal{H} \) are linearly disjoint over \( \mathcal{F} \). But \( \mathcal{G} \) is strongly normal over \( \mathcal{F} \), so that \( \mathcal{G} \subset \sigma_0 \sigma \mathcal{C} \cdot \mathcal{H} = \mathcal{C}_2 \mathcal{H} \). Hence \( \mathcal{G} \) and \( \mathcal{G} \) are linearly disjoint over \( \mathcal{F} \). Therefore, \( \mathrm{id}_{\mathcal{E}_0} \) and the isomorphism \( \mathcal{C}_2 \cong \mathcal{C} \) (restriction of \( (\sigma_0 \circ \sigma)^{-1} \)) extend to an isomorphism \( \tau: \mathcal{C}_2 \mathcal{E}_0 \cong \mathcal{C} \mathcal{E}_0 \). The composite isomorphism \( \tau \circ \sigma_0 \circ \sigma \) is an embedding of \( \mathcal{H}' \) into \( \mathcal{U} \) over \( \mathcal{C} \).
2. A counterexample for constrained extensions. Recall that an extension $F$ of a differential field is said to be constrained ([3] p. 144) if every finite family of elements of $F$ is constrained over $F$ in the sense of [2] p. 142, that a differential field is said to be constrainedly closed ([3] p. 145) if it has no constrained extension other than itself, and that $F$ is said to be a constrained closure of $F$ ([3] p. 147) if $F$ is constrainedly closed and is embeddable over closed $F$ in every constrainedly extension of $F$. A constrained closure of $F$ always exists, and it is a constrained extension of $F$.

We are going to exhibit an ordinary differential field $F$, a universal extension $U$ of $F$, and an extension $V$ of $F$ in $U$ such that $V$ is a constrained closure of $F$ and $U$ is not universal over $F$.

Let $C'$ be any denumerable field of characteristic zero and put $F = C(x) = \text{the field of rational fractions over } C$ in an indeterminate $x$; $F$ has a unique structure of ordinary differential field with field of constants $C$ in which the derivative of $x$ is 1. By [3] p. 149, Prop. 4, we may fix a denumerable universal extension $V$ of $F$. By [3] p. 146, Cor. 1 to Prop. 3, $U$ is constrainedly closed.

The set of solutions in $U$ different from 0 and 1 of the differential equation

$$y' = y^3 - y^2$$

is denumerable and hence can be arranged in a sequence

$$\eta_0, \eta_1, \eta_2, \ldots$$

By [3] §8, this set is infinite and is an independent set of conjugates over $F$, and $F(\eta_0, \eta_1, \eta_2, \ldots)$ is constrained over $F$ (see [3] p. 144, Prop. 1). Because $U$ is constrainedly closed, $F(\eta_0, \eta_1, \eta_2, \ldots)$ has a constrained closure $C$ in $U$. The differential ideal $[y' - y^3 + y^2]$ of the differential polynomial algebra $F[y]$ is evidently prime and does not have a generic zero in $U$ (because all its zeros in $U$ are in $C$). Therefore, $U$ is not universal over $C$. (The same argument shows that $U$ is even not universal over $F(\eta_0, \eta_1, \eta_2, \ldots)$.) We are going to show that $C$ is a constrained closure of $F$.

By [3] p. 144, Prop. 2(a), $C$ is constrained over $F$. Let $U$ be any denumerable constrained closure of $F$ (e.g., any constrained closure of $F$ in $U$). The set of solutions in $U$ of the above differential equation can be arranged in a sequence

$$\xi_0, \xi_1, \xi_2, \ldots$$

As before, this set is infinite and is an independent set of conjugates over $F$. Therefore, there exists an isomorphism
\[ \mathcal{P}: \mathcal{F}\langle \eta_0, \eta_1, \eta_2, \cdots \rangle \approx \mathcal{F}\langle \zeta_0, \zeta_1, \zeta_2, \cdots \rangle. \]

Now, \( \mathcal{F}\langle \zeta_0, \zeta_1, \zeta_2, \cdots \rangle \) is normal over \( \mathcal{F} \) in \( \mathcal{H} \) (see [3] §6 p. 153). Hence, by [3] p. 159, Cor. 1 to Th. 2, \( \mathcal{H} \) is a constrained closure of \( \mathcal{F}\langle \zeta_0, \zeta_1, \zeta_2, \cdots \rangle \). Therefore, by [3] p. 158, Th. 2(b), \( \mathcal{P} \) can be extended to an isomorphism \( \mathcal{G} \approx \mathcal{H} \), so that \( \mathcal{G} \) is a constrained closure of \( \mathcal{F} \).

**REFERENCES**


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COLUMBIA UNIVERSITY

NEW YORK, NY 10027