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Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

For all results obtained, attention is restricted to algebraically closed fields of characteristic zero. An affine algebraic group is said to have property (*) if the intersection of its center and its radical is unipotent. Given a Lie algebra L, a characterization is obtained of those affine algebraic groups G having property (*) for which an injection $L \to \mathscr{L}(G)$ exists whose image is algebraically dense. This is applied to obtain a result concerning the embedding of Lie algebras into algebraic Lie algebras, and to questions about the Hopf algebra of representative functions of a Lie algebra L in the case where L is algebraic.

1. Introduction. Let L be a finite-dimensional Lie algebra over a field F of characteristic zero. Let $\mathscr{U}(L)$ denote the universal enveloping algebra of L. If $\mathscr{U}(L)$ is given a topology wherein the two-sided ideals of finite codimension constitute a fundamental system of neighborhoods of 0, then the continuous dual $\mathscr{H}(L)$ of $\mathscr{U}(L)$ is the Hopf algebra of representative functions on $\mathscr{U}(L)$. $\mathscr{H}(L)$ may be viewed as a two-sided $\mathscr{U}(L)$-module as follows: for $u \in \mathscr{U}(L)$ and $f \in \mathscr{H}(L)$, $u \cdot f$ and $f \cdot u$ are defined by $(u \cdot f)(x) = f(xu)$ and $(f \cdot u)(x) = f(ux)$ for all $x \in \mathscr{U}(L)$.

An element $f \in \mathscr{H}(L)$ is termed a *semisimple element* of $\mathscr{H}(L)$ provided f is associated with a semisimple representation of L. That is the case if and only if the left $\mathscr{U}(L)$-module $\mathscr{U}(L) \cdot f$, or equivalently the right $\mathscr{U}(L)$-module $f \cdot \mathscr{U}(L)$, is semisimple. The subalgebra T of the *trigonometric elements* of $\mathscr{H}(L)$ consists of the semisimple elements of $\mathscr{H}(L)$ which are associated with representations that are trivial on the commutator ideal $[L, L]$. The following result is known from [1] and [2]. There exists a left $\mathscr{U}(L)$-stable (or equivalently, left stable under the comultiplication of $\mathscr{H}(L)$) subalgebra B of $\mathscr{H}(L)$ satisfying the following:

(1) B is finitely-generated as an F-algebra;

(2) $\mathscr{H}(L) = T \otimes B$;

(3) the subalgebra of the semisimple elements of B coincides with the portion of $\mathscr{H}(L)$ annihilated by the radical of L by left translation.

Any such subalgebra of $\mathscr{H}(L)$ is termed a *normal basic subalgebra*. Since B is finitely-generated as an F-algebra, so is the smallest Hopf subalgebra of $\mathscr{H}(L)$ containing B. This Hopf subalgebra, which will be labeled B^*, is uniquely determined [5, pp. 173–174].

We assume from now on that F is algebraically closed. The affine algebraic group D, consisting of the F-algebra homomorphisms $B^* \to F$ will be termed the *basic group* of L. The map obtained by composition of the natural injection $L \to \mathscr{L}(\mathscr{H}(L))$ and the restriction Lie algebra homomorphism $\mathscr{L}(\mathscr{H}(L)) \to \mathscr{L}(B^*)$, which by [1, Thm. 6] is an injection, will be called τ. A Lie subalgebra H of the Lie algebra $\mathscr{L}(G)$ of an affine algebraic subgroup G is said to be *algebraically dense* in $\mathscr{L}(G)$ if the smallest algebraic subgroup of G whose Lie algebra contains H has for its Lie algebra $\mathscr{L}(G)$. Then the image of L by τ is algebraically dense in $\mathscr{L}(D)$.

If F is algebraically closed, the subalgebra T of $\mathscr{H}(L)$ is generated by its group-like elements. If A is any Hopf algebra and Δ is the comultiplication of A, the *group-like* elements of A are those nonzero elements $q \in A$ such that $\Delta q = q \otimes q$. The set of group-like elements of $\mathscr{H}(L)$ will be labeled Q.

When speaking of affine algebraic groups or Lie algebras, the following notation will be used throughout. For an affine algebraic group G, G_0 will denote the center of G, G_{rad} the radical of G, G_u the unipotent radical of G, and G_1 the connected component of the identity element of G. Similarly, for a Lie algebra L, L_0 will denote the center of L and L_{rad} the radical of L. An affine algebraic group G will be said to have *property* (*) if $G_0 \cap G_{\mathrm{rad}}$ is unipotent. For any Lie algebra L, the basic group D of L has property (*), [5, Thm. 3.2].

Standard results and terminology used herein concerning affine algebraic groups may be found in [4]. The author wishes to thank Dr. G. Hochschild for reading over this paper and offering many helpful suggestions.

2. The basic group. Let (G, C) be the structure of a connected affine algebraic group having property (*). Let L be a finite-dimensional Lie algebra and assume there exists an injection $\sigma: L \to \mathscr{L}(G)$ whose image is algebraically dense in $\mathscr{L}(G)$. Then from [5, Thm. 3.2], there is a surjective rational homomorphism $\rho: D \to G$ whose differential ϕ coincides with the identity on L if L is identified with its image by τ and by σ. Consequently, $\sigma = \phi \circ \tau$ and the polynomial algebra C of G may be identified with a subalgebra of B^*.

Let H denote the kernel of ρ, then $\mathscr{L}(H)$ is the kernel of ϕ. By [4, Prop. 13.1], if the Lie algebra N is algebraically dense in the Lie algebra $L(G)$ of an affine algebraic group G, then $[N, N] = [\mathscr{L}(G), \mathscr{L}(G)]$. Observe that $\phi([\tau(L), \tau(L)]) = \phi([\mathscr{L}(D), \mathscr{L}(D)]) = [\mathscr{L}(G), \mathscr{L}(G)] = [\sigma(L), \sigma(L)]$, hence the restriction of ϕ to $[\mathscr{L}(D), \mathscr{L}(D)]$ has a trivial kernel. Thus $\mathscr{L}(H) \cap [\mathscr{L}(D), \mathscr{L}(D)] = (0)$; since

$\mathscr{L}(H)$ is an ideal, we also obtain $[\mathscr{L}(H), \mathscr{L}(D)] = (0)$.

In general, an abelian Lie subalgebra Z of a Lie algebra L is a direct summand of L if and only if $Z \subset L_0$ and $Z \cap [L, L] = (0)$. Indeed, if Z satisfies these conditions, a subspace (hence ideal) R of L exists such that $R \supset [L, L]$, $R \cap Z = (0)$, and $R + Z = L$. This gives the direct sum decomposition $L = R \oplus Z$; Z will be called an *abelian direct summand* (ADS) of L, and R a direct sum complement of Z in L.

For a connected affine algebraic group G, given an algebraic subgroup K of G such that $\mathscr{L}(K)$ is an ADS of $\mathscr{L}(G)$, and a subgroup M of G such that for all $m \in M$ and $g \in G$, $mgm^{-1}g^{-1} \in K$, we show $M \subset G_0$. For $m \in M$, consider the map $\psi_m: G \to G$ given by $\psi_m(g) = mgm^{-1}g^{-1}$. Since G is connected and ψ_m is a polynomial map, the Zariski closure of the image $\psi_m(G)$ is connected. On the other hand, by assumption, $\psi_m(G) \subset [G, G] \cap K$ and since $\mathscr{L}(K)$ is an ADS, $[G, G] \cap K$ is a finite subgroup of G. Thus, necessarily, $\psi_m(G) = 1_G$, which gives the desired result. If K is normal, setting $M = K$ gives that K is central. In particular, H, the kernel of ρ, is a central subgroup of D.

We now establish some properties of ADS's. Let L be a finite-dimensional Lie algebra. Let W be a subspace of L_0 such that $L_0 = W \oplus (L_0 \cap [L, L])$; W is seen to be a maximal ADS of L. Let S be a direct sum complement of W in L, so $L = W \oplus S$. Since $S \supset [L, L]$, $S \cap L_0 = S_0 \supset [L, L] \cap L_0$ On the other hand, $L_0 = W \oplus (S \cap L_0) = W \oplus S_0$, so necessarily $S_0 = L_0 \cap [L, L] \subset [L, L] = [S, S]$. Conversely, if $L = Z \oplus R$ with Z any ADS of L, and $R_0 \subset]R, R]$, then Z is a maximal ADS, since $L_0 = Z \oplus (R \cap L_0) = Z \oplus R_0$, and $R_0 \subset [L, L]$.

Now let Z and Z' be ADS's of L of equal dimension and let W and W' be maximal ADS's of L containing Z and Z' respectively. If S and S' are direct sum complements of W and W' in L, we have $L = S \oplus W = S' \oplus W'$. Observe that $W \cap S' \subset W \cap S'_0 \subset W \cap [L, L] = (0)$. From the above paragraph, W and W' have the same dimension, so $L = W \oplus S'$ and thus $S \approx S' \approx L/W$. Hence an automorphism α of L may be chosen so that $\alpha(W) = W'$ and $\alpha(S) = S'$. Since we may say $W = Z \oplus Y$ and $W' = Z' \oplus Y'$, and Y and Y' have the same dimension, we may further choose α so that $\alpha(Z) = Z'$, and, if $Z = Z'$, that α is the identity map on Z. We state results from the above in Lemma 2.1.

LEMMA 2.1. *Let Z be an* ADS *of a finite-dimensional Lie algebra L and let R be a direct sum complement of Z in L. Then*: (a) *Z is maximal if and only if $R_0 \subset [R, R] = [L, L]$; and* (b) *if Z' is an* ADS *of L of dimension equal to that of Z, then $L/Z \approx L/Z'$.*

LEMMA 2.2. *Let G be a connected affine algebraic group having property (*) and let Z be an* ADS *of $\mathscr{L}(G)$. Then there is a direct product decomposition $G = J \times G_Z$ where G_Z is the smallest algebraic subgroup of G such that $\mathscr{L}(G_Z)$ contains Z. Moreover, $\mathscr{L}(G_Z) = Z$ and G_Z is an algebraic vector group.*

Proof. Let $G = M \cdot G_u$ be a standard decomposition of G, where M is a maximal reductive subgroup of G. Since G has property (*), $Z \subset \mathscr{L}(G_u)$. Since $Z \cap [\mathscr{L}(G), \mathscr{L}(G)] = (0)$, we may choose an ideal N of $L(G_u)$ such that $\mathscr{L}(G_u) = Z \oplus N$ and $N \supset [\mathscr{L}(G), \mathscr{L}(G)]_{\text{rad}} = [\mathscr{L}(G), \mathscr{L}(G_u)]$. N is also an ideal of $\mathscr{L}(G)$, hence the unipotent algebraic subgroup G_N of G, whose Lie algebra is N, is normal both in G_u and in G; moreover, we have $G_u = G_Z \times G_N$. If we then define $J = MG_N$, by the normality of G_N in G, J is an algebraic subgroup, and since $N \supset [\mathscr{L}(G), \mathscr{L}(G)]_{\text{rad}}$, $\mathscr{L}(J) \supset [\mathscr{L}(G), \mathscr{L}(G)]$ and J is normal in G. Clearly $JG_Z = G$ and $J \cap G_Z = G_N \cap G_Z = 1_G$, so $G = J \times G_Z$ and Lemma 2.2 is established.

We now establish group automorphism results analogous to those obtained for Lie algebras. Let G be a connected affine algebraic group having property (*), let Z and Z' be ADS's of $\mathscr{L}(G)$ of equal dimension and let W and W' be maximal ADS's of $\mathscr{L}(G)$ containing Z and Z'. By Lemma 2.2, $G = J \times G_W = J' \times G_{W'}$; by Lemma 2.1, $\mathscr{L}(J) \cap W' = (0)$, so necessarily $\mathscr{L}(J) \oplus W' = \mathscr{L}(G)$. Make the observation that $\mathscr{L}(J) \cap \mathscr{L}(G_u) = \mathscr{L}(J \cap G_u)$. Since $W' \subset \mathscr{L}(G_u)$, $\mathscr{L}(J \cap G_u) \oplus W' = \mathscr{L}(G_u)$. It follows that $(J \cap G_u) \times G_{W'} = G_u$, and from the construction in the proof of Lemma 2.2, it is seen that $G = JG_{W'} = J \times G_{W'}$. Hence $J \approx G/G_{W'} \approx J'$ and we may define a rational automorphism β of G such that $\beta(J) = J'$ and $\beta(G_W) = G_{W'}$. Moreover, if we write $W = Z \oplus Y$ and $W' = Z' \oplus Y'$, for the algebraic vector groups G_W and $G_{W'}$, we have $G_W = G_Z \times G_Y$ and $G_{W'} = G_{Z'} \times G_{Y'}$ so we may additionally specify that $\beta(G_Z) = G_{Z'}$ and $\beta(G_Y) = G_{Y'}$. If $Z = Z'$, we may take β as the identity map on G_Z.

LEMMA 2.3. *Let G be an affine algebraic group and let $\sigma: L \to \mathscr{L}(G)$ be an injection of the Lie algebra L into $\mathscr{L}(G)$ whose image is algebraically dense. Then if Y is central or an* ADS *of L, $\sigma(Y)$ is central or an* ADS *respectively of $\mathscr{L}(G)$.*

Proof. Recall that if V is a finite-dimensional polynomial representation space for G and V_1 and V_2 are F-subspaces of V such that $V_2 \subset V_1$, then if L' is a Lie subalgebra of $\mathscr{L}(G)$ which by the induced representation of $\mathscr{L}(G)$ on V sends V_1 into V_2, $[L']$, the algebraic hull of L', also sends V_1 into V_2. Consider the adjoint representation of G on $\mathscr{L}(G)$. In the above terms, we will have

$V = \mathscr{L}(G)$, $V_1 = \sigma(Y)$ and $V_2 = (0)$. Assume first that Y is central in L. Then, since $ad_{\sigma(L)}\sigma(Y)$ is trivial, it follows that $ad_{[\sigma(L)]}\sigma(Y)$, or $ad_{\mathscr{L}(G)}\sigma(Y)$ is trivial, hence $\sigma(Y)$ is central in $\mathscr{L}(G)$. Further, by the algebraic density of $\sigma(L)$ in $\mathscr{L}(G)$, and since σ is an injection, the preimage of $[\mathscr{L}(G), \mathscr{L}(G)]$ is exactly $[L, L]$. Whence, if Y is an ADS of L, $\sigma(Y)$ is an ADS of $\mathscr{L}(G)$.

We now give our first principal result. L and D shall have the same meaning as in the introduction and w and x shall signify the dimensions of the maximal ADS's of $\mathscr{L}(D)$ and L respectively.

PROPOSITION 2.4. *Let G be a connected affine algebraic group having property* (*) *such that an injection $\sigma: L \to \mathscr{L}(G)$ exists whose image is algebraically dense. Let ρ be the canonical surjective rational homomorphism $D \to G$ of* [5, Thm. 3.2]. *Let H be the kernel of ρ, so $G \approx D/H$. Then H is in the center of D, H_1 is an algebraic vector group, and there is a direct sum decomposition $D = J \times H_1$ such that the restriction of ρ to J is a covering of G,* i.e., *a surjective rational homomorphism with a finite kernel. The* ADS's *of $\mathscr{L}(G)$, of $\mathscr{L}(J)$, and those of $\mathscr{L}(D)$ which contain $\mathscr{L}(H)$ are in bijective correspondence. These, in turn, are in bijective correspondence with the abelian direct factors of G, of J, and those of D which contain H_1. The injective images of the* ADS's of L *in $\mathscr{L}(G)$, $\mathscr{L}(J)$, and $\mathscr{L}(D)$ are* ADS's *of each respectively. Lastly,* $\dim(\mathscr{L}(H)) \leqq w - x$.

Proof. $\mathscr{L}(H)$ is an ADS of $\mathscr{L}(D)$ and D property (*), hence the direct product decomposition follows from Lemma 2.2. $J \cap H$ is finite, so the restriction of ρ to J is a covering of G. The differential of the covering $J \to G$ is an isomorphism $\mathscr{L}(J) \to \mathscr{L}(G)$, hence gives a bijective correspondence between the ADS's of $\mathscr{L}(J)$ and those of $\mathscr{L}(G)$. There also exists a clear bijective correspondence between the ADS's of $\mathscr{L}(J)$ and those of $\mathscr{L}(D)$ containing $\mathscr{L}(H)$. Since G, J, and D, all are connected and have property (*), it follows from Lemma 2.2 that each ADS of $\mathscr{L}(G)$, $\mathscr{L}(J)$, or $\mathscr{L}(D)$ determines a corresponding abelian direct factor of G, J, or D. These correspondences are all bijective, and the ADS's of $\mathscr{L}(D)$ containing $\mathscr{L}(H)$ correspond to the abelian direct factors of D containing H_1. From Lemma 2.3, it is seen that the ADS's of L have for their injective images by the given injections of L into $\mathscr{L}(G)$, $\mathscr{L}(J)$, and $\mathscr{L}(D)$, ADS's of each of these three respectively. Hence w must exceed x by an amount not less than $\dim(\mathscr{L}(H))$. This completes the proof of Proposition 2.4.

Lemma 2.5 is preliminary to Proposition 2.6, which is a partial converse to [5, Thm. 2.3] and Proposition 2.4.

LEMMA 2.5. *Let G be an arbitrary connected affine algebraic group and let K be a normal algebraic subgroup of G such that $\mathscr{L}(K)$ is an* ADS *of $\mathscr{L}(G)$. (Hence K_1 is central.) Then $(G/K_1)_0 \cap (G/K_1)_{\mathrm{rad}} = (G_0 \cap G_{\mathrm{rad}})/K_1$. If G has property* (*), *then G/K has property* (*).

Proof. Say $(G/K_1)_0 = M/K_1$. Then M is the subgroup of G consisting of all elements m such that for all $g \in G$, $mgm^{-1}g^{-1} \in K_1$. From the argument of pg. 289, $M \subset G_0$, thus $M = G_0$. Since $K_1 \subset G_{\mathrm{rad}}$, $G_{\mathrm{rad}}/K_1 = (G/K_1)_{\mathrm{rad}}$ is immediate from the corresponding fact for Lie algebras. Thus $(G/K_1)_0 \cap (G/K_1)_{\mathrm{rad}} = G_0/K_1 \cap G_{\mathrm{rad}}/K_1 = (G_0 \cap G_{\mathrm{rad}})/K_1$, and if G has property (*), so does G/K_1.

$G/K \approx (G/K_1)/(K/K_1)$ and K/K_1 is a finite, normal algebraic subgroup of G/K_1, hence a central reductive subgroup. We show in general that if H is a connected affine algebraic group having property (*), and T is a finite, central subgroup of H, H/T has property (*). It follows from the argument of pg. 289 that $(H/T)_0 = H_0/T$. Since T is a central reductive subgroup of H, $T \cap H_{\mathrm{rad}}$ is trivial. Hence $TH_{\mathrm{rad}}/T = (T \times H_{\mathrm{rad}})/T \approx H_{\mathrm{rad}}$ is a connected algebraic subgroup of T/H clearly equal to $(T/H)_{\mathrm{rad}}$. Thus $(H/T)_0 \cap (H/T)_{\mathrm{rad}} = (H_0 \cap (T \times H_{\mathrm{rad}}))/T = (T \times (H_0 \cap H_{\mathrm{rad}}))/T$ since $T \subset H_0$. Thus $(H/T)_0 \cap (H/T)_{\mathrm{rad}} \approx H_0 \cap H_{\mathrm{rad}}$ which gives the desired result.

PROPOSITION 2.6. *Let D be the basic group of a finite-dimensional Lie algebra L. Let H be a central subgroup of D such that $\mathscr{L}(H)$ is an* ADS *and $\dim(\mathscr{L}(H)) \leqq w - x$. Let ϕ be the differential of the quotient morphism $D \to D/H$. Then D/H has property* (*). *Then also there exists a rational automorphism β of D having differential α such that $\phi \circ \alpha \circ \tau$ is an injection of L into $\mathscr{L}(D/H)$ whose image is algebraically dense.*

Proof. D/H has property (*) from Lemma 2.5.

Let Z be a maximal ADS of L. By Lemma 2.3, $\tau(Z)$ is an ADS of $\mathscr{L}(D)$. Let W be a maximal ADS of $\mathscr{L}(D)$ containing $\tau(Z)$ and let V be a subspace of W such that $W = V \oplus \tau(Z)$. Then $\tau(L) \cap W$ is an ADS of $\tau(L)$ and $\tau(L) \cap W$ contains $\tau(Z)$, so $\tau(L) \cap W = \tau(Z)$ and hence $\tau(L) \cap V = (0)$. Let W' be a maximal ADS of $\mathscr{L}(D)$ containing $\mathscr{L}(H)$. By hypothesis, W' contains a subspace Y of dimension x such that $Y \cap \mathscr{L}(H) = (0)$. Thus we write $W' = U \oplus Y$ where $U \supset \mathscr{L}(H)$.

As pointed out on pg. 290, a rational automorphism β of D exists such that $\beta(D_W) = D_{W'}$ and we may further specify that $\beta(D_{\tau(Z)}) = D_Y$ and $\beta(D_V) = D_U$, so $\mathscr{L}(H) \subset \alpha(V)$ and it follows that $(\alpha \circ \tau)(L) \cap \mathscr{L}(H) = (0)$. Therefore $\phi \circ \alpha \circ \tau$ is an injection. To show that the image of $\alpha \circ \tau$ is algebraically dense in $\mathscr{L}(D)$, for a Lie subalgebra L' of $\mathscr{L}(D)$, let $[L']$ denote the algebraic Lie algebra hull of L'. Then $\tau(L) \subset \alpha^{-1}([(\alpha \circ \tau)(L)])$ and since α^{-1} preserves algebraicity, the latter is an algebraic Lie algebra. Hence $[\tau(L)] = \mathscr{L}(D) \subset \alpha^{-1}([(\alpha \circ \tau)(L)])$ and thus $\mathscr{L}(D) = [(\alpha \circ \tau)(L)]$. It is thus immediate that the image of $\phi \circ \alpha \circ \tau$ is algebraically dense in $\mathscr{L}(D/H)$. This completes the proof of Proposition 2.6.

For an arbitrary Hopf algebra A with comultiplication Δ, an element $p \in A$ is called a *primitive* element of A if $\Delta p = 1 \otimes p + p \otimes 1$. The exponential map of $\mathscr{H}(L)$, which is a group isomorphism from the vector F-group P of the primitive elements of $\mathscr{H}(L)$ onto the multiplicative group Q of the group-like elements of $\mathscr{H}(L)$ will be called ρ. An element $\delta \in \mathscr{L}(\mathscr{H}(L))$ is in the image of the natural injection $L \to \mathscr{L}(\mathscr{H}(L))$ if and only if $\delta(p) = \delta(\rho(p))$ for all $p \in P$. Henceforth we shall label as P_0 the F-subspace span $(\rho^{-1}(B^* \cap Q))$ of P, and as P_1 the F-subspace $P^{\mathscr{L}(D)_0}$ of P.

PROPOSITION 2.7. $w - x = \dim(P_0) - \dim(P_0 \cap P_1)$.

Proof. Identify L with its image in $\mathscr{L}(D)$ by τ. By Lemma 2.3, and since D has property (*), $L_0 \subset \mathscr{L}(D)_0 \subset \mathscr{L}(D_u)$, so $L_0 = L \cap \mathscr{L}(D)_0$. If $\delta \in \mathscr{L}(D_u)$, $\delta(B^* \cap Q) = 0$ since D_u fixes the semisimple part of B^* by right and left translation; thus from the defining property of L in $\mathscr{L}(D)$ it is seen that $\delta \in L_0$ if and only if $\delta \in \mathscr{L}(D)_0$ and $\delta(\rho^{-1}(B^* \cap Q)) = 0$. Let K be the fixer in D of $\rho^{-1}(B^* \cap Q)$; then K fixes P_0 and $\mathscr{L}(K)$ consists of those elements δ of $\mathscr{L}(D)$ such that $\delta(P_0) = 0$. Evidently, $L_0 = \mathscr{L}(D)_0 \cap \mathscr{L}(K)$.

We signify by "$'$" the restriction of a differentiation on B^* to $F[P]$; then $L_0' = (\mathscr{L}(D)_0 \cap \mathscr{L}(K))'$. Consider $\delta \in \mathscr{L}(D)_0' \cap \mathscr{L}(K)'$; necessarily, there exists $\delta_0 \in \mathscr{L}(D)_0$ such that $\delta_0' = \delta \in \mathscr{L}(K)'$, hence $\delta_0(P_0) = 0$ and $\delta_0 \in \mathscr{L}(D)_0 \cap \mathscr{L}(K)$, thus $\delta \in (\mathscr{L}(D)_0 \cap \mathscr{L}(K))'$. Since clearly $(\mathscr{L}(D)_0 \cap \mathscr{L}(K))' \subset \mathscr{L}(D)_0' \cap \mathscr{L}(K)'$, we obtain that $L_0' = \mathscr{L}(D)_0' \cap \mathscr{L}(K)'$. The space of differentiations of $F[P]$ is now identified with $\mathrm{Hom}_F(P, F)$. For any subspace N of P, we label by N^0 those differentiations on $F[P]$ which annihilate N. Then $(P^{L_0})^0 = L_0'$, $(P_1)^0 = \mathscr{L}(D)_0'$, and $(P_0)^0 = \mathscr{L}(K)'$, hence $(P^{L_0})^0 = (P_0)^0 \cap (P_1)^0$. In view of the natural homomorphism between P and $\mathrm{Hom}_F(\mathrm{Hom}_F(P, F), F)$, we obtain from the latter equality $P^{L_0} = P_1 + P_0$. Consequently, $\dim(P^{L_0}) = \dim(P_0) + \dim(P_1) - \dim(P_0 \cap P_1)$.

Let Z and W be maximal ADS's of L and $\mathscr{L}(D)$ respectively. Then $P^Z = P^{Z+(L_0 \cap [L,L])} = P^{L_0}$, and similarly $P^W = P_1$, so we write for the above:

$$(1) \qquad \dim(P^Z) = \dim(P^W) + \dim(P_0) - \dim(P_0 \cap P_1) .$$

For any ADS Y of $\mathscr{L}(D)$, we obtain from Lemma 2.2 that $D = J \times D_Y$, where D_Y is an algebraic vector group. Thus $(B^*)^J$, which may be regarded as the polynomial algebra of D_Y, is $F[P^J]$, hence $\dim(Y) = \dim(P^J)$. Since $P^{G_Y} = P^Y$, it follows from the F-space decomposition $P = P^Y \oplus P^J$ that $\dim(Y) = \dim(P) - \dim(P^Y)$. In particular, $w = \dim(P) - \dim(P^W)$ and $x = \dim(P) - \dim(P^Z)$. These two equalities, combined with equation (1), give $w - x = \dim(P_0) - \dim(P_0 \cap P_1)$. This completes the proof of Proposition 2.7.

Propositions 2.4, 2.6, and 2.7 are now summarized in Theorem 2.8. Let L and D be as in the introduction, and let P, P_0, and P_1 be as for Proposition 2.7.

THEOREM 2.8. *G is a connected affine algebraic group having property (*) for which an injection $\sigma: L \to \mathscr{L}(G)$ exists whose image is algebraically dense if and only if $G \approx D/H$ where H is a central algebraic subgroup of D, $\mathscr{L}(H)$ is an* ADS *of $\mathscr{L}(D)$ and* $\dim(\mathscr{L}(H)) \leqq \dim(P_0) - \dim(P_0 \cap P_1)$.

3. Lie algebra embeddings.

Our main result will be a characterization of the Lie algebra of an affine algebraic group of smallest dimension (equivalently, algebraic Lie algebra of smallest dimension) into which a finite-dimensional Lie algebra L may be embedded. As expected from Theorem 2.8, such a Lie algebra is a quotient algebra of the Lie algebra of the basic group of L by an ADS.

The next lemma is a result already known from [6]. This alternate proof is given because it is of interest that the characterization of algebraic Lie algebras of Theorem 4.1 is independent of Goto's characterization of algebraic Lie algebras.

LEMMA 3.1. *If L is a finite-dimensional algebraic Lie algebra, there exists a connected affine algebraic group with a unipotent center whose Lie algebra is L.*

Proof. Let G be some connected affine algebraic group whose Lie algebra is L. By the standard decomposition of G_0 into unipotent and reductive components, we have $G_0 = (G_0)_u \times (G_0)_r$. If we label $\mathscr{L}((G_0)_u)$ as U and $\mathscr{L}((G_0)_r)$ as Z then $L_0 = U \oplus Z$. Now consider $Y = L_0 \cap [L, L]$; Y is central in $[L, L]$, so $Y \subset [L, L]_{\mathrm{rad}}$, and in the Lie algebra of any affine algebraic group $[L, L]_{\mathrm{rad}} \subset \mathscr{L}(G_u)$.

Hence $Y \cap Z = (0)$, and a maximal ADS, W, of L may be chosen so that $W \supset Z$; then $L_0 = Y \oplus W$ and $W = (U \cap W) \oplus Z$. Thus $G_W = G_{U \cap W} \times ((G_0)_r)_1$ and G_W is a central algebraic subgroup of G.

Label G/G_W by G', then the Lie algebra of G' is L/W which we label M. By Lemma 2.1, $M_0 \subset [M, M]$, so as above, $M_0 \subset [M, M]_{\text{rad}} \subset \mathscr{L}(G'_u)$. Hence the reductive component T of G'_0 is a finite group. Therefore G'/T, which we label G'', has Lie algebra M and a unipotent center. Now let V be an algebraic vector group whose dimension is that of W. Then the affine algebraic group $H = G'' \times V$ has a unipotent center and $\mathscr{L}(H) = \mathscr{L}(G'') \oplus \mathscr{L}(V) \approx (L/W) \oplus W \approx L$; the latter isomorphism holds because W is an ADS of L. This completes the proof of Lemma 3.1.

Let L and D be as in the introduction and let Z be an ADS of $\mathscr{L}(D)$ of dimension equal to $\dim(P_0) - \dim(P_0 \cap P_1)$. We now label the Lie algebra $\mathscr{L}(D)/Z$ as L^*. By Lemma 2.1, L^* is uniquely determined up to isomorphism class.

Let G be an affine algebraic group and L be a Lie algebra and suppose $\sigma: L \to \mathscr{L}(G)$ is a Lie algebra injection. We adopt the notation $(G, \sigma)_L$ for this circumstance. For any finite-dimensional Lie algebra, L, if G is the basic group of L, σ exists so that $(G, \sigma)_L$ holds. In case $\mathscr{L}(G)$ is of minimal dimension, we have the following.

THEOREM 3.2. *If L is a finite-dimensional Lie algebra, given $(G, \sigma)_L$ such that $\mathscr{L}(G)$ is of minimal dimension, $\mathscr{L}(G) \approx L^*$.*

Proof. Since L is fixed for the duration of the argument, we supress the subscript L from the notation $(G, \sigma)_L$. Given any pair (G, σ), we first show a pair (G', σ') exists such that (1) $\dim(\mathscr{L}(G')) \leqq \dim(\mathscr{L}(G))$, (2) G' has property (*), and (3) the image of σ' is algebraically dense in $\mathscr{L}(G')$. Hence from Theorem 2.8 it will follow that $\dim(\mathscr{L}(G)) \geqq \dim(\mathscr{L}(G')) \geqq \dim(\mathscr{L}(D)) - \dim(P_0) + \dim(P_0 \cap P_1) = \dim(L^*)$.

Given (G_k, σ_k), $\alpha: (G_k, \sigma_k) \to (G_{k+1}, \sigma_{k+1})$, termed construction α, will mean that G_{k+1} is the smallest algebraic subgroup of G_k whose Lie algebra contains $\sigma_k(L)$ and $\sigma_{k+1} = \sigma_k$. Given (G_j, σ_j), $\beta: (G_j, \sigma_j) \to (G_{j+1}, \sigma_{j+1})$, construction β, will mean that G_{j+1} has property (*), $\mathscr{L}(G_j) \approx \mathscr{L}(G_{j+1})$, and for some Lie algebra isomorphism $\lambda: \mathscr{L}(G_j) \to \mathscr{L}(G_{j+1})$, $\sigma_{j+1} = \lambda \circ \sigma_j$. By Lemma 3.1, construction β can always be performed. Let (G, σ) be an arbitrary pair and consider a sequence of pairs in which the first pair is (G, σ) and the $n + 1$st pair is obtained by applying constructions α and β in succession to the nth pair. A sequence of affine algebraic groups is obtained in which affine algebraic group of the sequence has dimension greater than or equal to its successor. Hence some affine algebraic group in the sequence after the first, say G' of the pair (G', σ'), has the same dimension as its successor. Then it is easily verified that the pair (G', σ') satisfies (1), (2), and (3).

It remains to be shown that a pair (G, σ) exists such that $\mathscr{L}(G) \approx L^*$, and that if (G_1, σ_1) is another pair such that $\dim(\mathscr{L}(G_1)) = \dim(L^*)$, then $\mathscr{L}(G_1) \approx L^*$. Recalling Proposition 2.7, it is clear that $\mathscr{L}(D)$ has an ADS Z of dimension $\dim(P_0) - \dim(P_0 \cap P_1)$. If D_Z is the central algebraic subgroup of D whose Lie algebra is Z, then by Lemma 2.1, D/D_Z has Lie algebra isomorphic to L^*. Applying Theorem 2.8 with $H = D_Z$, we see a pair $(D/D_Z, \sigma)$ exists. Let (G_1, σ_1) be another pair such that $\dim(G_1) = \dim(L^*)$. Apply the construction $\beta\colon (G_1, \sigma_1) \to (G_2, \sigma_2)$. The image of σ_2 is necessarily algebraically dense in $\mathscr{L}(G_2)$; if not $\alpha\colon (G_2, \sigma_2) \to (G_3, \sigma_3)$ would yield a pair such that $\dim(\mathscr{L}(G_3)) < \dim(L^*)$ which, as shown above, is impossible. Hence, applying Theorem 2.8 again, we find $\mathscr{L}(G_1) \approx \mathscr{L}(G_2) \approx \mathscr{L}(D)/\mathscr{L}(H)$, where $\mathscr{L}(H)$ is an ADS of $\mathscr{L}(D)$. Thus by Lemma 2.1, $\mathscr{L}(G_1) \approx L^*$. This completes the proof of Theorem 3.2.

4. The algebra of representative functions of an algebraic Lie algebra. Let L, D, Q, and B^* have their meanings from the introduction, and P_0 and P_1 their meanings from Proposition 2.7. Let $\operatorname{alg\,dim}(B^* \cap Q)$ signify the degree of transcendence of $B^* \cap Q$ over F. From [1, Thm. 5] and [2, Thm. 1], it follows that $\dim(\mathscr{L}(D_u)) = \dim(L_{\mathrm{rad}})$. Adding $\dim(\mathscr{L}(B_s)) = \dim(L/L_{\mathrm{rad}})$ to both sides, we obtain the F-dimension of the space of differentiations on B, i.e., $\dim(\mathscr{L}(D)) - \operatorname{alg\,dim}(B^* \cap Q)$ on the left side, and $\dim(L)$ on the right. Hence $\dim(\mathscr{L}(D)) - \dim(L) = \operatorname{alg\,dim}(B^* \cap Q)$.

THEOREM 4.1. *L is an algebraic Lie algebra if and only if $P_0 \cap P_1 = (0)$ and $\dim(P_0) = \operatorname{alg\,dim}(B^* \cap Q)$.*

Proof. Assume L is algebraic. By Lemma 3.1, a connected affine algebraic group G having property (*) exists whose Lie algebra is L. By Theorem 2.8, a central algebraic subgroup H of D exists such that $G \approx D/H$ and from Theorem 3.2 and the above, $\dim(\mathscr{L}(H)) = \dim(P_0) - \dim(P_0 \cap P_1) = \operatorname{alg\,dim}(B^* \cap Q)$.

Let p_i, $i = 1, \cdots, n$ be an F-basis of P_0. We claim $\rho(p_i) = q_i$, $i = 1, \cdots, n$ are algebraically independent, and hence $\dim(P_0) \leqq \operatorname{alg\,dim}(B^* \cap Q)$. Assume that

$$\sum f_{(e_1, \cdots, e_n)} q_1^{e_1} \cdots q_n^{e_n} = 0 ,$$

with each e_i a positive integer. Then $q_1^{e_1} \cdots q_n^{e_n} = \rho(e_1 p_1 + \cdots + e_n p_n)$ is a distinct group-like element of $\mathscr{H}(L)$ for each distinct n-tuple

$(e_1, \cdots, e_n)$ since ρ is bijective. By [8, pg. 55], any collection of distinct group-like elements of a Hopf algebra is linearly independent, so necessarily $f_{(e_1,\cdots,e_n)} = 0$ for all n-tuples. Consequently, $\operatorname{alg}\dim(B^* \cap Q) = \dim(P_0)$ and $P_0 \cap P_1 = (0)$.

For the converse, assume that $\dim(P_0) = \operatorname{alg}\dim(B^* \cap Q)$ and $P_0 \cap P_1 = (0)$. Then, by Theorem 3.2, there is an affine algebraic group G of dimension equal to that of L such that a Lie algebra injection $\sigma: L \to \mathscr{L}(G)$ exists, so necessarily $L \approx \mathscr{L}(G)$. This completes the proof of Theorem 4.1.

Now say A is a Hopf algebra satisfying the conditions of [7, Thm. 2.1], hence A is isomorphic to the algebra of representative functions $\mathscr{H}(L)$ of some Lie algebra L. We use ρ to signify a group isomorphism from the additive F-group P_A of the primitive elements of A onto the multiplicative group Q_A of the group-like elements of A. If L is the Lie algebra determined by A and ρ such that $A \approx \mathscr{H}(L)$, we write L as $L(A, \rho)$. The canonical image of $L(A, \rho)$ in $\mathscr{L}(A)$ is those differentiations $\delta \in \mathscr{L}(A)$ such that $\delta(p) = \delta(\rho(p))$ for all $p \in P_A$. If Φ is the canonical Hopf algebra isomorphism $A \to \mathscr{H}(L)$ and exp is the exponential map of $\mathscr{H}(L)$, then $\exp = \Phi \circ \rho \circ \Phi^{-1}$ [7, Prop. 2.4]. To obtain a complementary result, let A_1 and A_2 be two Hopf algebras that satisfy the conditions of [7, Thm. 2.1], and let ρ_1 and ρ_2 be group isomorphisms from the primitive elements onto the group-like elements for A_1 and A_2 respectively. Say further that there is a Hopf algebra isomorphism $\lambda: A_1 \to A_2$ such that $\lambda \circ \rho_1 = \rho_2 \circ \lambda$. It is then easily verified that for the Lie algebra isomorphism $\mathscr{L}(A_2) \to \mathscr{L}(A_1)$ induced by λ, which is given by $\delta \to \delta \circ \lambda$ for $\delta \in \mathscr{L}(A_2)$, the canonical image of $L(A_2, \rho_2)$ in $\mathscr{L}(A_2)$ is mapped isomorphically onto the canonical image of $L(A_1, \rho)$ in $\mathscr{L}(A_1)$; thus $L(A_1, \rho_1) \approx L(\lambda(A_1), \lambda \circ \rho_1 \circ \lambda^{-1}) \approx L(A_2, \rho_2)$. Recall that for any Lie algebra L, $\mathscr{H}(L)$ is known to satisfy the conditions of [7, Thm. 2.1]. If exp is the exponential map of $\mathscr{H}(L)$ then surely $L(\mathscr{H}(L), \exp) = L$. Therefore, let A be as above, let L be any Lie algebra such that $A \approx \mathscr{H}(L)$, and let λ be a Hopf algebra isomorphism $A \to \mathscr{H}(L)$. It then follows that $L(A, \lambda^{-1} \circ \exp \circ \lambda) \approx L(\mathscr{H}(L), \exp) = L$. It cannot be asserted that λ is the canonical isomorphism $A \to \mathscr{H}(L)$.

By hypothesis, A has a left-stable, finitely-generated subalgebra E such that $A = E \otimes F[Q_A]$ and the semisimple elements of E are a fully stable subalgebra of E with no proper affine unramified extension. We call such a subalgebra *regular*. From [7, Prop. 2.3], the image of a regular subalgebra by the canonical isomorphism Φ is a normal basic subalgebra of $\mathscr{H}(L)$. If E^* is the smallest Hopf subalgebra of A containing E, it follows that the image of E^*

by Φ is the Hopf algebra B^* of $\mathscr{H}(L)$. Thus E^* is a uniquely-determined, finitely-generated subalgebra of A which contains all regular subalgebras of A. If $D_A = \mathscr{G}(E^*)$, the restriction Hopf algebra isomorphism $\Phi: E^* \to B^*$ induces a rational isomorphism $D \to D_A$, so D_A is isomorphic to the basic group of $L(A, \rho)$ for any ρ. Clearly the image by Φ of $E^* \cap Q_A$ is $B^* \cap Q$, and it is easily seen further that the image by Φ of $P_A^{\mathscr{G}(D_A)_0}$ is P_1. Hence we identify E^* with B^* and identify P_A, Q_A, and D_A with their counterparts in relation to $\mathscr{H}(L)$, dropping the subscript. These identifications are possible it is emphasized, because these features of A are all independent of ρ.

THEOREM 4.2. *Let A be a Hopf algebra satisfying the conditions of* [7, Thm. 2.1]. *Then $A \approx \mathscr{H}(L)$ for some algebraic Lie algebra L if and only if* $\dim(P) - \dim(P_1) \geqq \operatorname{alg\,dim}(B^* \cap Q)$.

Proof. Assume that $A \approx \mathscr{H}(L)$ for an algebraic Lie algebra L. Let exp be the exponential map of $\mathscr{H}(L)$ and let P_0 be the subspace $\operatorname{span}(\exp^{-1}(B^* \cap Q))$ of the primitive elements of $\mathscr{H}(L)$. Then by Theorem 4.1, $\dim(P_0) = \operatorname{alg\,dim}(B^* \cap Q)$ and $P_0 \cap P_1 = (0)$. Thus $\dim(P) \geqq \dim(P_0 + P_1) = \operatorname{alg\,dim}(B^* \cap Q) + \dim(P_1)$. From the previous discussions, it is clear that the numbers $\dim(P)$, $\dim(P_1)$, and $\operatorname{alg\,dim}(B^* \cap Q)$ are the same for A as for any $\mathscr{H}(L)$ such that $L = L(A, \rho)$ for some ρ, and moreover, that given an L such that $A \approx \mathscr{H}(L)$, a ρ_0 exists such that $L = L(A, \rho_0)$. Hence our conclusion for $\mathscr{H}(L)$ gives the same result for A.

Conversely, let A be as hypothesized, and let B^* be the smallest Hopf subalgebra of A containing a regular subalgebra of A. $F[B^* \cap Q]$ is a fully stable subalgebra of B^* hence is finitely-generated. Thus $\mathscr{G}(F[B^* \cap Q])$ is a toroid and we may write $F[B^* \cap Q]$ as $F[q_1, q_1^{-1}, \cdots, q_n, q_n^{-1}]$. Then $q_1, \cdots, q_n$ is a maximal algebraically independent subset of $B^* \cap Q$. By hypothesis, a subspace P' of the primitive elements of A exists such that $\dim(P') = n = \operatorname{alg\,dim}(B^* \cap Q)$ and $P' \cap P_1 = (0)$. Let $p_1, \cdots p_n$ be a basis of P'. Label $\rho^{-1}(q_i)$ by m_i, $i = 1, \cdots, n$ and let F_0 be the rational subfield of F. We now demonstrate the existence of a group homomorphism $\rho': P \to Q$ such that $\rho'(p_i) = q_i$ for $i = 1, \cdots, n$. Assume $\sum_{i=1}^n r_i m_i = 0$, with $r_i \in F_0$ and some $r_i \neq 0$. Multiplying by a suitable integer, we may assume the r_i's are integers. Then we have

$$\rho\Big(\sum_{i=1}^{n} r_i m_i\Big) = \prod_{i=1}^{n} q_i^{r_i} = \rho(0) = 1\,,$$

so $\prod_{i=1}^n q_i^{r_i} - 1 = 0$, contrary to the algebraic independence of the q_i's. Thus the m_i's are linearly independent over F_0 so an F_0-linear automorphism θ of P exists such that $\theta(p_i) = m_i$ for $i = 1, \cdots, n$; we choose $\rho' = \rho \circ \theta$.

We have therefore span $(\rho'^{-1}(B^* \cap Q)) = P'$. If Φ is the canonical isomorphism $A \to \mathscr{H}(L)$, where $L = L(A, \rho')$, the exponential map exp of $\mathscr{H}(L)$ is given by $\Phi \circ \rho' \circ \Phi^{-1}$, thus $\exp^{-1} \circ \Phi = \Phi \circ \rho'^{-1}$. It follows that span $(\exp^{-1}(B^* \cap Q)) = \Phi(P')$, so by Theorem 4.1, $L(A, \rho')$ is algebraic. This completes the proof of Theorem 4.2.

THEOREM 4.3. *Let A be a Hopf algebra satisfying the conditions of* [7, Thm. 2.1]. *Let ρ_1 and ρ_2 be two group isomorphisms from the primitive elements of A onto the group-like elements of A such that $L_1 = L(A, \rho_1)$ and $L_2 = L(A, \rho_2)$ are both algebraic. Then $L_1 \approx L_2$.*

Proof. If B^* is the smallest Hopf subalgebra of A containing a regular subalgebra of A, then $\mathscr{G}(B^*) = D$ is the basic group for both L_1 and L_2. By Theorems 3.2 and 4.1, $L_1 \approx \mathscr{L}(D)/Z_1$ where Z_1 is an ADS of $\mathscr{L}(D)$ and $\dim(Z_1) = \operatorname{alg}\dim(B^* \cap Q)$; similarly $L_2 \approx \mathscr{L}(D)/Z_2$, Z_2 is an ADS of $\mathscr{L}(D)$ and $\dim(Z_2) = \dim(Z_1)$. Hence from Lemma 2.1, $L_1 \approx L_2$.

COROLLARY 4.4. *Let L_1 and L_2 be two algebraic Lie algebras such that $\mathscr{H}(L_1) \approx \mathscr{H}(L_2)$. Then $L_1 \approx L_2$.*

Proof. Let λ be a Hopf algebra isomorphism $\mathscr{H}(L_1) \to \mathscr{H}(L_2)$ and let ρ_1 and ρ_2 be the exponential maps of $\mathscr{H}(L_1)$ and $\mathscr{H}(L_2)$ respectively. Then $L_1 = L(\mathscr{H}(L_1), \rho_1)$ and $L_2 = (\mathscr{H}(L_2), \rho_2) \approx L(\mathscr{H}(L_1), \lambda^{-1} \circ \rho_2 \circ \lambda)$. By Theorem 4.3, $L(\mathscr{H}(L_1), \lambda^{-1} \circ \rho_2 \circ \lambda) \approx L(\mathscr{H}(L_1), \rho_1)$ since both are algebraic, so $L_1 \approx L_2$.

A stronger result than Theorem 4.3 is available for the case of complex analytic groups. From [3, Thm. 3.2] we see that if G is a complex affine algebraic group and G' is an arbitrary complex analytic group such that $\mathscr{H}(G) \approx \mathscr{H}(G')$, then $G \approx G'$ as complex analytic groups. The following example shows that our Theorem 4.3 cannot be strengthened to a similar extent.

Consider a four-dimensional algebraic Lie algebra L determined as follows. Let V be a two-dimensional abelian Lie algebra generated by elements λ_1 and λ_2. Let λ_3 be a linear endomorphism on V that sends λ_1 to λ_2 and λ_2 to 0. Then considering $F\lambda_3$ to be a one-dimensional Lie algebra, we label as N the three-dimensional nilpotent Lie algebra $V + F\lambda_3$ where the sum is the natural semidirect Lie algebra sum. Since N is nilpotent, N is the Lie algebra of a unipotent affine algebraic group U; U may be identified with the group of triples of elements of F with group composition

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2 + c_1 b_2, b_1 + b_2, c_1 + c_2) .$$

Let λ_4 be a linear endomorphism on N that is the identity map on V and trivial on λ_3 and label by L the Lie algebra $N + F\lambda_4$. Correspondingly, let F^* act on U by $t \cdot (a, b, c) = (ta, tb, c)$ for $t \in F^*$; then L is the Lie algebra of the affine algebraic group obtained by the semidirect preduct $F^* \cdot U$.

For L, we have $[\lambda_3, \lambda_1] = [\lambda_4, \lambda_2] = \lambda_2$, $[\lambda_4, \lambda_1] = \lambda_1$ and $[\lambda_1, \lambda_2] = [\lambda_2, \lambda_3] = [\lambda_3, \lambda_4] = 0$. A normal basic subalgebra B of $\mathscr{H}(L)$ is generated by elements b_1, b_2, b_3, and b_4 with b_3 and b_4 the generators of the F-space of the primitive elements of $\mathscr{H}(L)$. If Δ is the comultiplication of $\mathscr{H}(L)$, we have, for the non-primitive generators of $\mathscr{H}(L)$, $\Delta(b_1) = b_1 \otimes \exp(-b_4) + 1 \otimes b_1$ and $\Delta(b_2) = 1 \otimes b_2 + b_3 \otimes b_1 + b_2 \otimes \exp(-b_4)$. $\mathscr{H}(L)$ is generated by b_1, b_2, b_3, b_4, and the elements $\exp(rb_3 + sb_4)$ with $r, s \in F$, and $B^* = F[b_1, b_2, b_3, b_4, \exp(\pm b_4)]$. The elements of L are given on B by $\lambda_i(b_j) = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The center of $\mathscr{G}(B^*)$ is those elements g which satisfy $g(b_1) = g(b_2) = g(b_3) = 0$, and $g(\exp(b_4)) = 1$, hence $P_1 = \mathrm{span}(b_3)$.

Let exp′ be a group isomorphism from the primitive elements of $\mathscr{H}(L)$ onto the group-like elements of $\mathscr{H}(L)$ defined such that $\exp'(b_3) = \exp(b_4)$. Then if generators of $L' = L(\mathscr{H}(L), \exp')$ are defined on B by $\lambda'_i(b_j) = \delta_{ij}$, the Lie algebra bracket of L' is given by $[\lambda'_3, \lambda'_1] = \lambda'_1 + \lambda'_2$, $[\lambda'_3, \lambda'_2] = \lambda'_2$, and $[\lambda'_1, \lambda'_2] = [\lambda'_1, \lambda'_4] = [\lambda'_2, \lambda'_4] = [\lambda'_3, \lambda'_4] = 0$. Since $\exp'^{-1}(B^* \cap Q) \cap P_1$ is nontrivial, L' is not algebraic, so $L' \not\approx L$, but $\mathscr{H}(L) \approx \mathscr{H}(L')$.

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