

# Pacific Journal of Mathematics

**A NOTE ON TAMELY RAMIFIED POLYNOMIALS**

JOE PETER BUHLER

## A NOTE ON TAMELY RAMIFIED POLYNOMIALS

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Let  $f(x)$  be a monic polynomial with coefficients in a Dedekind ring  $A$ . If  $P$  is a prime ideal and  $A_P$  denotes the completion of  $A$  at  $P$  then  $f(x)$  is said to be integrally closed at  $P$  if  $A_P[X]/(f(X))$  is isomorphic to a product of discrete valuation rings. The purpose of this note is to show that if  $f(x)$  appears to be tamely ramified and integrally closed at  $P$  (in terms of its discriminant and factorization mod  $P$ ) then in fact it is.

If  $f(\alpha) = 0$ , where  $f(x)$  is a monic irreducible polynomial with coefficients in  $\mathbb{Z}$ , then the ring  $\mathbb{Z}[\alpha]$  is of finite index in the ring  $R$  of algebraic integers in  $\mathbb{Q}(\alpha)$ . The full ring of integers can be obtained by applying a very general algorithm due to Zassenhaus ([6]). There are well known cases where this is unnecessary. If, for instance,  $f(x)$  is an Eisenstein polynomial at  $p$ , or if  $p^2$  does not divide the discriminant of  $f(x)$ , then the polynomial  $f(x)$  is integrally closed at  $p$  (which is equivalent to saying that  $p$  does not divide the index  $[R:\mathbb{Z}[\alpha]]$ ). The theorem below asserts that if the power of  $p$  that divides the discriminant of  $f(x)$  is consistent with the factorization of  $f(x)$  mod  $P$  and the hypothesis that  $R$  is tamely ramified at  $p$ , then  $f(x)$  is integrally closed at  $p$ .

If  $P$  is a prime ideal in the Dedekind ring  $A$  let  $v_P: A \rightarrow \mathbb{Z} \cup \{\infty\}$  denote the corresponding normalized valuation. Let  $d(g)$  and  $\text{Disc}(g)$  denote the degree and discriminant of a polynomial  $g(x)$ .

**THEOREM.** *Suppose that  $f(x) \in A[x]$  is a monic polynomial that satisfies*

- (a)  $f(x) \equiv \prod g_i(x)^{e_i} \pmod{P}$
- (b)  $v_P(\text{Disc}(f)) = \sum_i (e_i - 1)d(g_i)$

where the  $g_i(x) \in (A/P)[x]$  are distinct monic, irreducible and separable polynomials. Then  $f(x)$  is integrally closed at  $P$ . Moreover,  $p \nmid e_i$  and  $A_P[x]/(f(x))$  is isomorphic to a product of discrete valuation rings that are tamely ramified over  $A_P$ .

The proof given in the third section is an easy consequence of a purely local result given in the second section. The first section recalls some basic formulas concerning resultants.

**REMARKS.** (1) It is a standard fact that if  $f(x)$  is integrally closed and tamely ramified at  $P$  then conditions (a) and (b) must

hold. If the characteristic of  $A/P$  is larger than  $n = d(f)$  then the ramification has to be tame. Thus the test above usually determines the power of  $P$  in the discriminant of the root field in the case in which  $\text{char}(A/P) > n$ : it can fail only if  $v_P(\text{Disc}(f)) \geq 4$ .

(2) The condition that  $f(x)$  be integrally closed at  $P$  is equivalent to saying that every ideal in  $A[x]/(f(x))$  lying over  $P$  is invertible, or to saying that the index (in the sense of [2], p. 10) of  $A[x]/(f(x))$  in the maximal order in  $K[x]/(f(x))$  is prime to  $P$  (where  $K$  is the fraction field of  $A$ ).

1. **Resultants.** Let  $f(x)$  and  $g(x)$  be polynomials with coefficients in any ring and let  $R(f, g)$  denote their resultant (which could be defined, for instance, as the determinant of the ‘‘Sylvester matrix’’ formed from the coefficients). Let  $L(g)$  denote the leading coefficient of the polynomial  $g(x)$ .

The following properties of the resultant  $R(f, g)$  are standard and will be used freely below. Proofs can be found in [1] and [5].

$$\text{R1. } R(f, g) = L(g)^{d(f)} \prod_{i=1}^{d(g)} f(\alpha_i) \quad \text{if } \alpha_1, \dots, \alpha_{d(g)} \text{ are the roots of } g(x) \\ = L(g)^{d(f)} \quad \text{if } d(g) = 0$$

$$\text{R2. } R(g, f) = (-1)^{d(f)d(g)} R(f, g)$$

$$\text{R3. } R(fg, h) = R(f, h)R(g, h)$$

$$\text{R4. } R(f, g) = L(g)^{d(f)-d(r)} R(r, g) \quad \text{if } f = qg + r$$

$$\text{R5. there exist polynomials } a(x), b(x) \text{ such that } R(f, g) = af + bg$$

$$\text{R6. } \text{Disc}(f) = (-1)^{d(f)(d(f)-1)/2} R(f, f')$$

$$\text{R7. } \text{Disc}(fg) = \text{Disc}(f) \text{Disc}(g) R(f, g)^2.$$

REMARK. The resultant  $R(f, g)$  can be efficiently computed by forming a ‘‘polynomial remainder sequence’’ ([3])  $f_1 = f, f_2 = g, f_3, \dots$  with

$$c_i f_i = d_i f_{i+1} + f_{i+2}, \quad \deg(f_{i+2}) < \deg(f_{i+1}).$$

The relationship R4 then can be used to express  $R(f_i, f_{i+1})$  in terms of  $R(f_{i+1}, f_{i+2})$ . It is easy to check that this algorithm can be used to compute the discriminant of a polynomial of degree  $n$  in  $O(n^2)$  steps, as opposed to the usual algorithms (e.g., taking the determinant of the Sylvester matrix or of the power sum matrix) which take  $O(n^3)$  steps.

2. **A local result.** Throughout this section  $A$  will be a discrete valuation ring with valuation  $v: A \rightarrow \mathbf{Z} \cup \{\infty\}$ , uniformizing parameter  $\pi$ , and residue field  $k$  of characteristic  $p$ . Moreover let  $f(x)$  be a monic polynomial with coefficients in  $A$  that satisfies

$$(a)' \quad f(x) \equiv g(x)^e \pmod{\pi}, \quad \text{where } \bar{g}(x) \in k[x] \text{ is irreducible and}$$

separable

$$(b)' \quad v(\text{Disc}(f)) = d(f) - d(g) = (e - 1)d(g).$$

Let  $B_f$  denote the ring  $A[x]/(f(x))$ . It is easy to show ([4], Lemma 4 of Chapter I, § 6) that  $B_f$  is a local ring with unique maximal ideal  $(\pi, g(x))$  and residue field  $k[x]/(\bar{g}(x))$ . The goal of this section is to show that (a)' and (b)' imply that  $B_f$  is a discrete valuation ring.

We follow the pattern of [4] and use the fact that a local noetherian ring is a discrete valuation ring if its maximal ideal is principal and is generated by a nonnilpotent element ([4], Prop. 2 of Chapter I, § 2). In fact we will show that  $\pi$  is in the ideal generated by  $g(x)$  so that the maximal ideal is  $(\pi, g(x)) = (g(x))$  and the ring must be a discrete valuation ring as claimed.

Use (a)' to define a polynomial  $h(x)$  by

$$f(x) = g(x)^e + \pi h(x).$$

LEMMA.  $v(R(g, h)) = 0$ .

Assume this lemma for the moment. By the definition of  $h(x)$ , R4, and R3 it follows that  $v(R(f, h)) = 0$ . By R5 it follows that there exist  $a(x), b(x) \in A[x]$  such that

$$1 = af + bh.$$

Now work in the ring  $B_f = A[x]/(f(x))$ . We have

$$1 = b(x)h(x) \quad g(x)^e = -\pi h(x)$$

so that  $\pi = -b(x)g(x)^e$ . Hence the maximal ideal in  $B_f$  is generated by  $g(x)$ . This reduces the proof of the assertion that  $B_f$  is a discrete valuation ring to the proof of the lemma.

*Proof of the lemma.* Put  $n = d(f)$ ,  $m = d(g)$ . By (b)' together with R6

$$v(R(f, f')) = v(R(g^e + \pi h, eg'g^{e-1} + \pi h')) = n - m.$$

Note that it is clear from this formula that  $e$  is prime to  $p$ . Indeed, if  $p$  divides  $e$  then the second term above is divisible by  $\pi$  so that by R1 and R3 the valuation would be at least  $n$ .

Without loss of generality we can assume that  $A$  is complete. Since

$$f' \equiv eg'g^{e-1} \pmod{\pi}$$

and since  $eg'$  is relatively prime to  $g^{e-1}$  ( $\bar{g}$  is irreducible and separable) it follows from Hensel's lemma that we can find polynomials  $a(x)$  and  $b(x)$  such that

$$f' = (eg' + \pi a)(g^{e-1} + \pi b)$$

with  $d(b) < d(g^{e-1})$ . Substituting in \* yields

$$** \quad n - m = v(R(g^e + \pi h, eg' + \pi a)) + v(R(g^e + \pi h, g^{e-1} + \pi b)).$$

Now apply the obvious fact that if the coefficients of two pairs of monic polynomials are congruent mod  $\pi$  then their resultants are congruent mod  $\pi$ . This shows that the first term on the right of \*\* is zero since

$$v(R(g, eg')) = 0.$$

In the second term rearrange to take advantage of R4:

$$\begin{aligned} v(R(g^e + \pi h, g^{e-1} + \pi b)) &= v(R(g(g^{e-1} + \pi b) + \pi(h - bg), g^{e-1} + \pi b)) \\ &= v(R(\pi, g^{e-1} + \pi b)) + v(R(h - bg, g^{e-1} + \pi b)) \\ &= m(e - 1) + v(R(h - bg, g^{e-1} + \pi b)). \end{aligned}$$

Since  $R(h - bg, g^{e-1} + \pi b) \equiv R(h - bg, g)^{e-1} \equiv R(h, g)^{e-1} \pmod{\pi}$  we are forced to conclude that  $v(R(h, g)) = 0$  which finishes the proof of the lemma.

The above results can be summarized as follows:

**PROPOSITION.** *Suppose that  $f(x)$  is a monic polynomial with coefficients in a discrete valuation ring and that  $f(x)$  satisfies*

(a)'  $f(x) \equiv g(x)^e \pmod{\pi}$ , where  $g(x)$  is irreducible and separable mod  $\pi$ ,

(b)'  $v(\text{Disc}(f)) = (e - 1)d(g)$ .

Then  $p \nmid e$  and  $B_f = A[x]/(f(x))$  is a discrete valuation ring with residue field  $k[x]/(\bar{g}(x))$  and maximal ideal  $(g(x))$ .

**COROLLARY.** *With the above notation,  $f(x)$  is irreducible,  $B_f$  is integrally closed, and  $B_f$  is tamely ramified over  $A$ .*

*Proof.* As in Chapter I, § 6, corollary to Proposition 15 of [4].

**REMARKS.** (1) It can be shown that the irreducibility criterion above reduces to the Eisenstein irreducibility criterion if  $e = 1$  and  $d(f)$  is prime to  $p$ .

(2) It is clear from the proof of the lemma that the valuation of the discriminant given in (b)' is in fact a lower bound on the discriminant of a polynomial that factors mod  $\pi$  as in (a)'.

**3. Proof of the theorem.** Now let the notation be as in the statement of the theorem:  $A$  is a Dedekind ring with prime ideal

$P$ ,  $v_P$  is the corresponding valuation,  $A_P$  is the completion of  $A$  at  $P$ , and  $f(x)$  is a monic irreducible polynomial satisfying (a) and (b).

By Hensel's lemma we can find polynomials  $G_i(x) \in A_P[x]$  such that

$$\begin{aligned} G_i(x) &\equiv g_i(x)^{e_i} \pmod{P} \\ f(x) &= \prod G_i(x). \end{aligned}$$

By remark (2) above

$$v_P(\text{Disc}(G_i)) \geq (e_i - 1)d(g_i).$$

The iteration of R7 shows that the discriminant of a product is divisible by the product of the discriminants so that

$$v_P(\text{Disc}(f)) \geq \sum v_P(\text{Disc}(G_i)) \geq \sum (e_i - 1)d(g_i) = v_P(\text{Disc}(f))$$

(using the hypothesis (b)). Therefore we must have equality throughout and  $v_P(\text{Disc}(G_i)) = (e_i - 1)d(g_i)$ . The proposition of the preceding section applies to the polynomial  $G_i(x)$  and we conclude that

$$A_P[x]/(f(x)) \simeq \prod A_P[x]/(G_i(x))$$

is a product of discrete valuation rings and that  $f(x)$  is integrally closed at  $P$ . Also the  $e_i$ 's are prime to  $p$  and  $f(x)$  is tamely ramified at  $P$ . This finishes the proof of the theorem.

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