A NOTE ON TAMELY RAMIFIED POLYNOMIALS

JOE PETER BUHLER
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J. P. Buhler

Let \( f(x) \) be a monic polynomial with coefficients in a Dedekind ring \( A \). If \( P \) is a prime ideal and \( A_P \) denotes the completion of \( A \) at \( P \) then \( f(x) \) is said to be integrally closed at \( P \) if \( A_P[X]/(f(X)) \) is isomorphic to a product of discrete valuation rings. The purpose of this note is to show that if \( f(x) \) appears to be tamely ramified and integrally closed at \( P \) (in terms of its discriminant and factorization mod \( P \)) then in fact it is.

If \( f(\alpha) = 0 \), where \( f(x) \) is a monic irreducible polynomial with coefficients in \( Z \), then the ring \( Z[\alpha] \) is of finite index in the ring \( R \) of algebraic integers in \( Q(\alpha) \). The full ring of integers can be obtained by applying a very general algorithm due to Zassenhaus ([6]). There are well known cases where this is unnecessary. If, for instance, \( f(x) \) is an Eisenstein polynomial at \( p \), or if \( p^2 \) does not divide the discriminant of \( f(x) \), then the polynomial \( f(x) \) is integrally closed at \( p \) (which is equivalent to saying that \( p \) does not divide the index \( [R: Z[\alpha]] \)). The theorem below asserts that if the power of \( p \) that divides the discriminant of \( f(x) \) is consistent with the factorization of \( f(x) \) mod \( P \) and the hypothesis that \( R \) is tamely ramified at \( p \), then \( f(x) \) is integrally closed at \( p \).

If \( P \) is a prime ideal in the Dedekind ring \( A \) let \( v_P: A \to Z \cup \{\infty\} \) denote the corresponding normalized valuation. Let \( d(g) \) and \( \text{Disc}(g) \) denote the degree and discriminant of a polynomial \( g(x) \).

**Theorem.** Suppose that \( f(x) \in A[x] \) is a monic polynomial that satisfies

(a) \( f(x) \equiv \Pi g_i(x)^{e_i} \mod P \)

(b) \( v_P(\text{Disc}(f)) = \Sigma_i (e_i - 1)d(g_i) \)

where the \( g_i(x) \in (A/P)[x] \) are distinct monic, irreducible and separable polynomials. Then \( f(x) \) is integrally closed at \( P \). Moreover, \( p \nmid e_i \), and \( A_P[x]/(f(x)) \) is isomorphic to a product of discrete valuation rings that are tamely ramified over \( A_P \).

The proof given in the third section is an easy consequence of a purely local result given in the second section. The first section recalls some basic formulas concerning resultants.

**Remarks.** (1) It is a standard fact that if \( f(x) \) is integrally closed and tamely ramified at \( P \) then conditions (a) and (b) must
The characteristic of $A/P$ is larger than $n = d(f)$ then the ramification has to be tame. Thus the test above usually determines the power of $P$ in the discriminant of the root field in the case in which $\text{char}(A/P) > n$: it can fail only if $v_P(\text{Disc}(f)) \geq 4$.

(2) The condition that $f(x)$ be integrally closed at $P$ is equivalent to saying that every ideal in $A[x]/(f(x))$ lying over $P$ is invertible, or to saying that the index (in the sense of [2], p. 10) of $A[x]/(f(x))$ in the maximal order in $K[x]/(f(x))$ is prime to $P$ (where $K$ is the fraction field of $A$).

1. Resultants. Let $f(x)$ and $g(x)$ be polynomials with coefficients in any ring and let $R(f, g)$ denote their resultant (which could be defined, for instance, as the determinant of the “Sylvester matrix” formed from the coefficients). Let $L(g)$ denote the leading coefficient of the polynomial $g(x)$.

The following properties of the resultant $R(f, g)$ are standard and will be used freely below. Proofs can be found in [1] and [5].

R1. $R(f, g) = L(g)^{d(f)} \prod_{i=1}^{d(g)} f(\alpha_i)$ if $\alpha_1, \ldots, \alpha_{d(g)}$ are the roots of $g(x)$

R2. $R(g, f) = (-1)^{d(f)d(g)}R(f, g)$

R3. $R(fg, h) = R(f, h)R(g, h)$

R4. $R(f, g) = L(g)^{d(f) - d(r)}R(r, g)$ if $f = qg + r$

R5. there exist polynomials $a(x), b(x)$ such that $R(f, g) = af + bg$

R6. $\text{Disc}(f') = (-1)^{d(f)(d(f)-1)/2}R(f, f')$

R7. $\text{Disc}(fg) = \text{Disc}(f)\text{Disc}(g)R(f, g)^2$.

REMARK. The resultant $R(f, g)$ can be efficiently computed by forming a “polynomial remainder sequence” ([3]) $f_1 = f, f_2 = g, f_3, \ldots$ with

$$c_i f_i = d_i f_{i+1} + f_{i+2}, \deg(f_{i+2}) < \deg(f_{i+1}).$$

The relationship R4 then can be used to express $R(f_i, f_{i+1})$ in terms of $R(f_{i+1}, f_{i+2})$. It is easy to check that this algorithm can be used to compute the discriminant of a polynomial of degree $n$ in $O(n^2)$ steps, as opposed to the usual algorithms (e.g., taking the determinant of the Sylvester matrix or of the power sum matrix) which take $O(n^3)$ steps.

2. A local result. Throughout this section $A$ will be a discrete valuation ring with valuation $v: A \to \mathbb{Z} \cup \{\infty\}$, uniformizing parameter $\pi$, and residue field $k$ of characteristic $p$. Moreover let $f(x)$ be a monic polynomial with coefficients in $A$ that satisfies

$(a') f(x) \equiv g(x) \mod \pi$, where $g(x) \in k[x]$ is irreducible and
Let $B_f$ denote the ring $A[x]/(f(x))$. It is easy to show ([4], Lemma 4 of Chapter I, §6) that $B_f$ is a local ring with unique maximal ideal $(\pi, g(x))$ and residue field $k[x]/(g(x))$. The goal of this section is to show that (a)' and (b)' imply that $B_f$ is a discrete valuation ring.

We follow the pattern of [4] and use the fact that a local noetherian ring is a discrete valuation ring if its maximal ideal is principal and is generated by a nonnilpotent element ([4], Prop. 2 of Chapter I, §2). In fact we will show that $\pi$ is in the ideal generated by $g(x)$ so that the maximal ideal is $(\pi, g(x)) = (g(x))$ and the ring must be a discrete valuation ring as claimed.

Use (a)' to define a polynomial $h(x)$ by

$$f(x) = g(x)^e + \pi h(x).$$

**Lemma.** $v(R(g, h)) = 0$.

Assume this lemma for the moment. By the definition of $h(x)$, R4, and R3 it follows that $v(R(f, h)) = 0$. By R5 it follows that there exist $a(x), b(x) \in A[x]$ such that

$$1 = af + bh.$$

Now work in the ring $B_f = A[x]/(f(x))$. We have

$$1 = b(x)h(x), \quad g(x)^e = -\pi h(x)$$

so that $\pi = -b(x)g(x)^e$. Hence the maximal ideal in $B_f$ is generated by $g(x)$. This reduces the proof of the assertion that $B_f$ is a discrete valuation ring to the proof of the lemma.

**Proof of the lemma.** Put $n = d(f), m = d(g)$. By (b)' together with R6

$$v(R(f, f')) = v(R(g^e + \pi h, eg'g^{e-1} + \pi h')) = n - m.$$

Note that it is clear from this formula that $e$ is prime to $p$. Indeed, if $p$ divides $e$ then the second term above is divisible by $\pi$ so that by R1 and R3 the valuation would be at least $n$.

Without loss of generality we can assume that $A$ is complete. Since

$$f' \equiv eg'g^{e-1} \mod \pi$$

and since $eg'$ is relatively prime to $g^{e-1}(g$ is irreducible and separable) it follows from Hensel's lemma that we can find polynomials $a(x)$ and $b(x)$ such that
with $d(b) < d(g^{*-1})$. Substituting in * yields

$$** \ n - m = \nu(R(g^* + \pi h, eg' + \pi a)) + \nu(R(g^* + \pi h, g^{*-1} + \pi b)) .$$

Now apply the obvious fact that if the coefficients of two pairs of monic polynomials are congruent mod $\pi$ then their resultants are congruent mod $\pi$. This shows that the first term on the right of ** is zero since

$$\nu(R(g, eg')) = 0 .$$

In the second term rearrange to take advantage of $R4$:

$$\nu(R(g^* + \pi h, g^{*-1} + \pi b)) = \nu(R(g^{*-1} + \pi b) + \pi(h - bg), g^{*-1} + \pi b))$$

$$= \nu(R(\pi, g^{*-1} + \pi b)) + \nu(R(h - bg, g^{*-1} + \pi b)))$$

$$= m(e - 1) + \nu(R(h - bg, g^{*-1} + \pi b)) .$$

Since $R(h - bg, g^{*-1} + \pi b) = R(h - bg, g^{*-1}) = R(h, g)^{*-1}$ mod $\pi$ we are forced to conclude that $\nu(R(h, g)) = 0$ which finishes the proof of the lemma.

The above results can be summarized as follows:

**PROPOSITION.** Suppose that $f(x)$ is a monic polynomial with coefficients in a discrete valuation ring and that $f(x)$ satisfies

(a) $f(x) \equiv g(x)^e \mod \pi$, where $g(x)$ is irreducible and separable mod $\pi$,

(b) $\nu(\text{Disc}(f)) = (e - 1)d(g)$.

Then $p \mid e$ and $B_f = A[x]/(f(x))$ is a discrete valuation ring with residue field $k[x]/(\bar{g}(x))$ and maximal ideal $(g(x))$.

**COROLLARY.** With the above notation, $f(x)$ is irreducible, $B_f$ is integrally closed, and $B_f$ is tamely ramified over $A$.

**Proof.** As in Chapter I, § 6, corollary to Proposition 15 of [4].

**REMARKS.** (1) It can be shown that the irreducibility criterion above reduces to the Eisenstein irreducibility criterion if $e = 1$ and $d(f)$ is prime to $p$.

(2) It is clear from the proof of the lemma that the valuation of the discriminant given in (b)' is in fact a lower bound on the discriminant of a polynomial that factors mod $\pi$ as in (a)'.
P, v_\mathfrak{P} is the corresponding valuation, A_\mathfrak{P} is the completion of A at P, and f(x) is a monic irreducible polynomial satisfying (a) and (b).

By Hensel's lemma we can find polynomials G_i(x) \in A_\mathfrak{P}[x] such that

\[ G_i(x) \equiv g_i(x)^{e_i} \mod P \]
\[ f(x) = \Pi G_i(x) . \]

By remark (2) above

\[ v_\mathfrak{P}(\text{Disc } G_i) \geq (e_i - 1)d(g_i) . \]

The iteration of R7 shows that the discriminant of a product is divisible by the product of the discriminants so that

\[ v_\mathfrak{P}(\text{Disc } f) \geq \Sigma v_\mathfrak{P}(\text{Disc } G_i) \geq \Sigma (e_i - 1)d(g_i) = v_\mathfrak{P}(\text{Disc } f) \]

(using the hypothesis (b)). Therefore we must have equality throughout and \[ v_\mathfrak{P}(\text{Disc } G_i) = (e_i - 1)d(g_i) . \] The proposition of the preceding section applies to the polynomial G_i(x) and we conclude that

\[ A_\mathfrak{P}[x]/(f(x)) \simeq \Pi A_\mathfrak{P}[x]/(G_i(x)) \]

is a product of discrete valuation rings and that f(x) is integrally closed at P. Also the e_i's are prime to p and f(x) is tamely ramified at P. This finishes the proof of the theorem.

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REFERENCES


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