

# Pacific Journal of Mathematics

## **APPROXIMATION PROPERTIES OF POLYNOMIALS WITH BOUNDED INTEGER COEFFICIENTS**

VLADIMIR DROBOT AND S. McDONALD

## APPROXIMATION PROPERTIES OF POLYNOMIALS WITH BOUNDED INTEGER COEFFICIENTS

V. DROBOT AND S. McDONALD

For every fixed positive integer  $N$ , let  $\mathcal{P}_N$  denote the set of all polynomials  $p(x) = \sum a_i x^i$  where  $a_i$  is an integer,  $|a_i| \leq N$ . For a fixed real number  $t$  set  $\mathcal{P}_N(t) = \{p(t) : p \in \mathcal{P}_N\}$ .

**THEOREM 1.** Suppose  $1 < t < N + 1$  and  $t$  is not a root of any of the polynomials from  $\mathcal{P}_N$ . Then  $\mathcal{P}_N(t)$  is dense in  $\mathbf{R}$ .

**THEOREM 2.** If  $t$  is an  $S$ -number then  $\mathcal{P}_N(t)$  is discrete for every  $N$ .

1. For every fixed positive integer  $N$ , let  $\mathcal{P}_N$  denote the set of all polynomials  $p(x)$  with integer coefficients,  $p = \sum a_i x^i$ , such that  $|a_i| \leq N$ . For a fixed real number  $t$  set

$$\mathcal{P}_N(t) = \{p(t) : p \in \mathcal{P}_N\}.$$

It was shown in [1] that if  $N = 1$ ,  $t$  is a number such that  $1 < t < 2$  and  $t$  is not a root of any of the polynomials from  $\mathcal{P}_1$  then the set  $\mathcal{P}_1(t)$  is dense in the real line. (It is fairly easy to see that if  $t \notin (1, N + 1)$ ,  $t > 0$  then  $\mathcal{P}_N(t)$  cannot be dense in  $\mathbf{R}$ ). At the same time it was shown that  $\mathcal{P}_1(1/2(1 + \sqrt{5}))$  is discrete. As far as we know this is the only known example of  $N$  and  $t \in (1, N + 1)$  such that  $\mathcal{P}_N(t)$  is discrete. In this paper we prove two extensions of these results. The first is a straightforward generalization of [1]:

**THEOREM 1.** Suppose  $1 < t < N + 1$  and  $t$  is not a root of any of the polynomials from  $\mathcal{P}_N$ . Then  $\mathcal{P}_N(t)$  is dense in  $\mathbf{R}$ .

The second result is more intriguing and has a curious connection with what is known as  $P - V$  numbers or  $S$ -numbers ( $P - V$  numbers for Pisot-Vijayaragharan, see [2] for details).

**DEFINITION.** A number  $t > 1$  is called a  $P - V$  number if it is an algebraic integer and all of its conjugates have absolute value strictly less than 1.

**THEOREM 2.** If  $t$  is a  $P - V$  number then  $\mathcal{P}_N(t)$  is discrete for every  $N$ .

It follows, for instance, that  $\mathcal{P}_N(1/2(1 + \sqrt{5}))$  is discrete for all  $N$ , not just  $N = 1$ .

Let  $\|s\|$  denote the distance from  $s$  to the nearest integer. A number  $\theta$  is said to have property (P) if for some  $\lambda > 1$ ,  $\|\lambda\theta^n\| \rightarrow 0$ . It is known that every  $P - V$  number has property (P). A conjecture is raised in [2] as to whether the converse is true: Is every number with property (P) a  $P - V$  number? It is known that every algebraic number with property (P) is a  $P - V$  number. Thus the conjecture would be settled if one could show that for every number  $t$  having property (P), the set  $\mathcal{P}_N(t)$  is discrete.

The proof of Theorem 1 is essentially no different from the proof given in [1] for  $N = 1$ . We proceed with the proof of Theorem 2 now.

LEMMA 1. *Suppose  $t > 1$  and  $0$  is an accumulation point of  $\mathcal{P}_N(t)$ . Let  $k, m$  be any positive integers. There exists polynomial  $p$  of the form  $p(x) = x^{m_1}f(x)$ ,  $f \in \mathcal{P}_N$ ,  $m_1 > m$  such that*

$$t^{-k-1} \leq p(t) < t^{-k}.$$

*Proof.* Let  $r(x)$  be a polynomial in  $\mathcal{P}_N$  such that

$$0 < r(t) < t^{-k-m_1}.$$

Let  $m_1$  be the smallest integer such that

$$t^{-k-m_1-1} < r(t).$$

Then  $m_1 > m$  and  $r(t) < t^{-k-m_1}$ . Thus

$$t^{-k-1} < t^{m_1}r(t) \leq t^{-k}.$$

LEMMA 2. *Suppose  $t > 1$  and  $0$  is an accumulation point of  $\mathcal{P}_N(t)$ . Then  $\mathcal{P}_N(t)$  is dense in  $\mathbf{R}$ .*

*Proof.* Let  $u > 0$  and  $\eta > 0$  be fixed. Let  $k$  be so large that the interval  $[t^{-k-1}, t^{-k}]$  has length less than  $\eta$ . There is a sequence of polynomials  $p_1, p_2, \dots$ , having no common terms  $\alpha_j x^j$  such that

$$t^{-k-1} < p_n(t) \leq t^{-k}.$$

This follows by applying Lemma 1 with fixed  $k$  and making  $m_1$  larger and larger. If

$$q_m(t) = p_1(t) + \dots + p_m(t)$$

then  $q_m(t) > mt^{-k}$ , so  $q_m(t) \rightarrow \infty$ . Hence for some  $m$ ,  $q_m(t)$  will be inside the interval  $[u - \eta, u + \eta]$ . Since  $u$  and  $\eta$  were arbitrary, the result follows.

*Proof of Theorem 2.* It is enough to show that  $\mathcal{P}_N(t)$  is not dense for any  $N = 1, 2, \dots$ . Indeed, suppose this is done and assume that  $\mathcal{P}_{N_0}(t)$  is not discrete. Then clearly  $\mathcal{P}_{2N_0}(t)$  has 0 as an accumulation point and by Lemma 2 is dense. To show  $\mathcal{P}_N(t)$  is not dense for any  $N$  we argue as follows. Let

$$t = t_1, t_2, \dots, t_p$$

be all the roots of the irreducible monic polynomial of  $t$ , and let

$$\sigma = \max\{|t_2|, |t_3|, \dots, |t_p|\}$$

so that  $0 < \sigma < 1$ . For any  $k$

$$t^k + t_2^k + \dots + t_p^k$$

is an integer, hence

$$|t^k\text{-integer}| \leq |t_2|^k + \dots + |t_p|^k \leq (p-1)\sigma^k.$$

Let  $p(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_j x^j$  be a polynomial in  $\mathcal{P}_N$ . Then

$$t^k p(t) = \sum_{n=0}^j \alpha_n t^{k+n}$$

so

$$\begin{aligned} |t^k p(t) - \text{integer}| &\leq \sum_{n=0}^j |\alpha_n| (p-1)\sigma^{k+n} \\ &\leq N(p-1) \frac{\sigma^k}{1-\sigma}. \end{aligned}$$

Choose  $k$  so large that the right hand side is less than  $1/3$ . Then

$$\left| p(t) - \frac{\text{integer}}{t^k} \right| < \frac{1}{3} \frac{1}{t^k}$$

or, if the integer is odd

$$\left| p(t) - \frac{1}{2} \frac{\text{integer}}{t^k} \right| \geq \frac{1}{6} \frac{1}{t^k}$$

for any  $p \in \mathcal{P}_N$ .

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