RECURSION FORMULAS FOR THE HOMOLOGY OF $\Omega(X \vee Y)$

Giora Dula and Elyahu Katz
A recursion formula for \( H(\Omega(X \lor Y)) \), the homology of the loop space of the wedge of the spaces \( X \) and \( Y \) is established when \( \Omega X \) and \( \Omega Y \) are connected, and have finite dimensional homology. The recursion formula is expressed in terms of \( H(\Omega X) \) and \( H(\Omega Y) \), and applies to dimensions higher than a fixed integer which depends on the dimension of the highest nonvanishing homologies of \( \Omega X \) and \( \Omega Y \). A similar but much simpler recursion formula for \( H(\Omega X) \Pi H(\Omega Y) \), the co-product of the two algebras \( H(\Omega X) \) and \( H(\Omega Y) \) is also formulated. If \( G_1 \) and \( G_2 \) are topological groups and \( G_1 * G_2 \) is their co-product in the category, then our results definitely hold for \( H(G_1 * G_2) \) by replacing \( \Omega X \) by \( G_1 \), \( \Omega Y \) by \( G_2 \), and \( \Omega(X \lor Y) \) by \( G_1 * G_2 \).

1. Introduction. Over a field \( H(\Omega(X \lor Y)) \) equals \( H(\Omega X) \Pi H(\Omega Y) \) [1] [2], a fact which substantially simplifies the problem of computing the homology of \( \Omega(X \lor Y) \). Over a Dedekind domain a torsion factor is added [5] [3] which significantly complicates the situation. Taking a principal ideal domain as the coefficient ring, \( H(\Omega(X \lor Y)) \) was computed in [3]. However, even if \( \Omega X \) and \( \Omega Y \) are finite dimensional, those computations call for an increasing number of manipulations as the dimension of the homology to be computed gets higher. If \( n_1 \) and \( n_2 \) are the highest dimensions of non vanishing homologies of \( \Omega X \), \( \Omega Y \), then for any \( k > 3(n_1 + n_2) + 4 \) we introduce a recursion formula which expresses \( H_k(\Omega(X \lor Y)) \) in terms of \( H_i(\Omega(X \lor Y)) \) \( i < k \). The number of computations does not increase with \( k \). Of course \( H_i(\Omega(X \lor Y)) \) with \( i \leq 3(n_1 + n_2) + 4 \) has to be computed independently, for example by the method of [5].

In §2 we state the result of [5] in a generalized form which will be used here. We also present in this section most of the relevant notation of this paper. Recursion formulas in general are introduced in §3. The recursion formula for the free component of \( H(\Omega(X \lor Y)) \) is presented in §4. In §5 we derive a recursion formula for \( H(\Omega X) \Pi H(\Omega Y) \). The main result which is a recursion formula for the torsion component of \( H(\Omega(X \lor Y)) \) is proved in §6. We close with an application by computing \( H(\text{SO}_3 * \text{SO}_3) \).

The ring \( R \) will always be a principal ideal domain. The notation and terminology are those of [5].

2. The holomogy of \( \Omega(X \lor Y) \) in dimension \( k \). Let \( L^j \) be
free resolutions of the modules $A^j, j = 1, 2, \cdots, n$. Define
\[ \text{mult}_i^\ast (A^1, \cdots, A^n) = H_i(L^1 \otimes \cdots \otimes L^n). \]
We have [3]:
\[
\tilde{H}_k(\Omega(X \lor Y)) = \sum_{n=1}^{\infty} \sum_{r_1+\cdots+r_n=k} \text{mult}_i^\ast (\tilde{H}_{r_1}(\Omega X_1), \cdots, \tilde{H}_{r_n}(\Omega X_n))
\]
where $r = (r_1, \cdots, r_n)$ is a sequence of nonnegative integers, $j = (j_1, \cdots, j_n)$ is a sequence alternating on 1, 2, and $\Omega X_2 = \Omega Y$. Thus the next step is to express explicitly the elements in the above summation. However, we first introduce some extra notation:

(i) $\text{mult}_i^\ast (j, r) = \text{mult}_i^\ast (\tilde{H}_{r_1}(\Omega X_{j_1}), \cdots, \tilde{H}_{r_n}(\Omega X_{j_n})).$

(ii) $R(M) = \text{the number of } R \text{ direct summands in the module } M.$

$R_p(M) = \text{the number of } R_p \text{ direct summands in } M \text{ where } p \text{ is a prime in } R \text{ and } h \text{ is a nonnegative integer.}$

(iii) $a_i = R(\tilde{H}_i(\Omega X)), b_i = R(H_i(\Omega Y)),$

\[
c_i = \sum_{h' > h} R_{p^{h'}}(\tilde{H}_i(\Omega X)), \quad \tilde{c}_i = \sum_{h' \geq h} R_{p^{h'}}(\tilde{H}_i(\Omega X)),
\]

\[
d_i = \sum_{h' > h} R_{p^{h'}}(\tilde{H}_i(\Omega Y)), \quad \tilde{d}_i = \sum_{h' \geq h} R_{p^{h'}}(\tilde{H}_i(\Omega Y)).
\]

(iv) $m_i(l) = \text{the number of times that } H_i(\Omega X_t) \text{ appears in } \text{mult}_i^\ast (j, r), t = 1, 2, l = 1, 2, \cdots, k.$

(v) $\phi(s_1, \cdots, s_k, t_1, \cdots, t_k) = \prod_{i=1}^{k} c_i^{s_i}d_i^{t_i} - \prod_{i=1}^{k} c_i^{t_i}d_i^{s_i},$

\[
\psi(s_1, \cdots, s_k, t_1, \cdots, t_k) = \prod_{i=1}^{k} (t_i^{m_i(l)} - s_i^{m_i(l)}) \cdot \prod_{i=1}^{k} (s_i^{m_i(l)} - t_i^{m_i(l)}),
\]

where $0 \leq s_i \leq m_i(l), 0 \leq t_i \leq m_i(l)$ and

\[
\begin{pmatrix}
 p \\
 q
\end{pmatrix} = \begin{cases} 
 0 & q > p \text{ or } q < 0 \\
 1 & q = p \text{ or } 0 = q < p
\end{cases} \quad \left( p! \over (p - q)!q! \right) \quad \text{otherwise}.
\]

With this notation we have [3]:

**Theorem 1.** $\text{R(mult}_0^\ast (j, r)) = \prod_{i=1}^{k} a_i^{m_i(l)} \cdot b_i^{m_i(l)}$,

\[
R_{p^h}(\text{mult}_i^\ast (j, r)) = \sum_{0 \leq s_1 \leq m_i(l)} \sum_{0 \leq t_1 \leq m_i(l)} \sum_{0 \leq s_2 \leq m_i(l)} \sum_{0 \leq t_2 \leq m_i(l)} \psi(s_1, \cdots, s_k, t_1, \cdots, t_k) \cdot \phi(s_1, \cdots, t_k) \left( \sum_{i=1}^{k} (s_i + t_i) - 1 \right).
\]

We close this section with some further notation:

\[
\tilde{H}(\Omega(X \lor Y)) = \text{mult}_0^\ast (\Omega X, \Omega Y) \oplus \text{mult}_1^\ast (\Omega X, \Omega Y)
\]

where $\text{mult}_0^\ast (\Omega X, \Omega Y) = \sum_{n, j, r} \text{mult}_n^\ast (j, r)$

\[
\text{mult}_1^\ast (\Omega X, \Omega Y) = \sum_{n=1}^{\infty} \sum_{j, r} \text{mult}_n^\ast (j, r).
\]

Note that $\text{mult}_0^\ast (\Omega X, \Omega Y)$ is exactly $H(\Omega X) \amalg H(\Omega Y)$. 
3. Recursion formulas. In this section we will make the general preparation for setting up the recursion formulas mentioned in the introduction.

Let \( \{c_r\}_{r=1}^{\infty} \) be a sequence of numbers, and \( q(x) = 1 - u_1x - u_2x^2 - \cdots - u_rx^r \) a polynomial. We define a new sequence \( \{c'_r\}_{r=1}^{\infty} \) as follows:

\[
c'_r = q_r(c_r) = c_r - u_1c_{r-1} - \cdots - u_rc_{r-1}.
\]

The sequence \( \{c_r\} \) satisfies the recursion formula corresponding to the polynomial \( q(x) \) at \( t \) if \( c'_r = q_r(c_r) = 0 \).

The following results will be very useful for the sequel:

**Lemma 1.** Let \( p(x), q(x) \) be polynomials and \( \{c_r\}_{r=1}^{\infty} \) a sequence of numbers. Then:

\[
q_t\{p_s\{c_r\}\} = (pq)_t\{c_r\},
\]

where \( (pq)(x) = p(x) \cdot q(x) \), the product of the two polynomials.

**Proof.** For \( p(x) = \sum_{i=0}^{l} u_i x^i \) and \( q(x) = \sum_{j=0}^{l'} v_j x^j \) we have:

\[
q_t\{p_s\{c_r\}\} = \sum_{j=0}^{l} v_j p_{t-j}\{c_r\} = \sum_{j=0}^{l} v_j \sum_{i=0}^{k} u_i c_{t-j-i} = \sum_{h=0}^{l} v_h \sum_{i=0}^{k} u_i c_{t-h} = (pq)_t\{c_r\},
\]

which completes the proof.

**Lemma 2.** Let \( \{c_r\}_{r=1}^{\infty} \) satisfy the polynomial \( p(x) = \sum_{i=0}^{k} u_i x^i \) at \( t, t-1, \ldots, t-l \), and \( \{d_r\}_{r=1}^{\infty} \) satisfy the polynomial \( q(x) = \sum_{j=0}^{l'} v_j x^j \) at \( t, t-1, \ldots, t-k \). Then the sequence \( \{c_r + d_r\}_{r=1}^{\infty} \) satisfies the polynomial \( q(x) \cdot p(x) \) at \( t \).

**Proof.**

\[
(qp)_t\{c_r + d_r\} = (qp)_t\{c_r\} + (pq)_t\{d_r\}
\]

\[
= q_t\{p_s\{c_r\}\} + p_t\{q_s\{d_r\}\}
\]

\[
= \sum_{j=0}^{l} v_j p_{t-j}\{c_r\} + \sum_{i=0}^{k} u_i q_{t-i}\{d_r\} = 0.
\]

We are now ready for the construction of the recursion formulas.

4. A recursion formula for the free part of \( H(\Omega(X \lor Y)) \). Our interest in this section is focused on the sequence \( \{\alpha_k\} \) where

\[
\alpha_k = R(H_k(\Omega(X \lor Y))).
\]
Since \( \bar{H}(Q(X \lor Y)) = \bar{H}(QX) \amalg \bar{H}(QY) \oplus \text{mult}^1(QX, QY) \) and
\( R(\text{mult}^1(QX, QY)) = 0 \)
we actually have \( \alpha_k = R(\bar{H}(QX) \amalg \bar{H}(QY)) \), \( k \geq 1 \).

**Theorem 2.** Let
\[
R(H_i(QX)) = \begin{cases} 
0 & i > n_1 \\
n_i & i \leq n_1 
\end{cases}
\]
\[
R(H_i(QY)) = \begin{cases} 
0 & i > n_2 \\
n_i & i \leq n_2 
\end{cases}
\]
Then the sequence \( \{\alpha_k\}_{k=0}^\infty \) satisfies the recursion formula:
\[
q_1(x) = 1 - \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \alpha_i b_j x^{i+j}, \quad \text{for any } k > n_1 + n_2.
\]
The proof of this theorem derives from the following:

**Proposition 1.** \( \alpha_k = a_k + b_k + 2 \sum_{i+j=k} a_i b_j + \sum_{i+j<k} a_i b_j \alpha_{k-i-j} \).

**Proof.** According to the definition of \( \alpha_k \) we have
\[
\alpha_k = \sum_{n, \sum s_i = k} \text{mult}^n(j, r).
\]
We can split up this sum into, \( \alpha_k = A + B + C \), where:
\[
A = \sum_{n, \sum s_i = k} R(\text{mult}^n(j, r))
\]
\[
B = \sum_{n, \sum s_i = k} R(\text{mult}^n(j, r))
\]
\[
C = \sum_{n, \sum s_i = k} R(\text{mult}^n(j, r)).
\]
Next we compute each term separately:
\[
A = R(\text{mult}^1(\bar{H}(QX))) + R(\text{mult}^1(\bar{H}(QY))) = a_k + b_k
\]
\[
B = \sum_{r_1 + r_2 = k} R(\text{mult}^1((1, 2), r)) + \sum_{r_1 + r_2 = k} R(\text{mult}^1((2, 1), r))
= \sum_{r_1 + r_2 = k} a_r \cdot b_r + \sum_{r_1 + r_2 = k} b_r \cdot a_r = 2 \sum_{i+j=k} a_i b_j.
\]
The computation of \( C \) is somewhat more complicated. Let \( \text{mult}^n(j, r) \) be a direct summand of \( (\text{mult}^n(QX, QY))_k \), with \( n \geq 3 \). We denote \( \tilde{j} = (j_1, \ldots, j_{n-2}) \) and \( \tilde{r} = (r_1, \ldots, r_{n-2}) \). Then it is not difficult to see that:
\[
R(\text{mult}^n(j, r)) = R(QX_{j_{n-1}}) \cdot R(QX_{j_n}) \cdot R(\text{mult}^n(\tilde{j}, \tilde{r})).
\]
Summing up the last equality on the proper possibilities of \( j \) and \( r \) we get the desired equality for \( A \).

**Proof of Theorem 2.** If \( k > n_1 + n_2 \) then each one of \( a_k, b_k \) and \( a_ib_j \) with \( i + j = k \), equals zero. Thus for \( k > n_1 + n_2 \) the equation of Proposition 2 reduces to:

\[
\alpha_k = \sum_{i+j<k} a_ib_j\alpha_{k-i-j} = \sum_{\substack{i,j \leq n_1 \\ j \leq n_2}} a_ib_j\alpha_{k-i-j},
\]

which is exactly the result of Theorem 2.

5. A recursion formula for the torsion component of \( H(\Omega X) \| H(\Omega Y) \). In this section we want to find a convenient way of expressing the \( k \) dimensional part of \( H(\Omega X) \| H(\Omega Y) \). We do it by forming a recursion formula for the number of \( R_{p^k} \) direct summands in each dimension, for each \( R_{p^k} \), which is a direct summand of either \( \tilde{H}(\Omega X) \) or \( \tilde{H}(\Omega Y) \). If \( R_{p^k} \) is one of these modules, we denote:

\[
\beta_k = R_{p^k}(\tilde{H}(\Omega X) \| \tilde{H}(\Omega Y)).
\]

**Theorem 3.** Let \( n_3 \) and \( n_4 \) be integers such that: \( a_k + \bar{c}_k > 0 \) implies that \( k \leq n_3 \) and \( b_k + \bar{d}_k > 0 \) implies that \( k \leq n_4 \). Consider the polynomials:

\[
q_2(x) = 1 - \sum_{i=1}^{n_3} \sum_{j=1}^{n_4} (a_i + \bar{c}_i)(b_j + \bar{d}_j)x^{i+j}
\]

\[
q_3(x) = 1 - \sum_{i=1}^{n_3} \sum_{j=1}^{n_4} (a_i + c_i)(b_j + d_j)x^{i+j}.
\]

Then for \( k > 2(n_3 + n_4) \) the polynomial \( q_2(x) \cdot q_3(x) \) corresponds to the recursion formula for \( \{\beta_k\}_{k=1}^{\infty} \).

For the proof we need some intermediate result as well as some auxiliary functions. The following functions are similar to functions introduced in §2.

(i) \( \phi_j(s_1, \ldots, s_k, t_1, \ldots, t_k) = c_1^{s_1} \cdots c_k^{s_k} \cdot d_1^{t_1} \cdots d_k^{t_k} \)

(ii) \( R_{p^k}(\text{mult}_a^b(j, r)) = \sum_{0 \leq i \leq b^k(l) \atop 0 \leq j \leq b^k(l) \atop 1 \leq l \leq k} \psi(s_1, \ldots, t_k) \cdot \phi_j(s_1, \ldots, t_k) \left( \sum_{i=1}^{k} s_i + t_i - 1 \right) \).

(iii) \( \beta_k = \sum_{n,j, r} R_{p^k}(\text{mult}_a^b(j, r)) \).

Note that in the expression \( R_{p^k}(\text{mult}_a^b(j, r)) \) the binomial term \( \left( \sum_{i=1}^{k} s_i + t_i - 1 \right) \) can be omitted. For if \( \sum_{i=1}^{k} s_i + t_i \geq 1 \) the binomial
term equals 1, and if $\sum_{i=1}^k s_i + t_i = 0$ the function $\phi(s_1, \ldots, t_k)$ is zero.

**Proposition 2.**

$$\beta_k = (\bar{c}_k - c_k) + (\bar{d}_k - d_k) + 2 \sum_{i+j=k} [(a_i + \bar{c}_i)(b_j - \bar{d}_j) - (a_i + c_i)(b_j + d_j)] + \sum_{i+j=k} (a_i + \bar{c}_i)(b_j + \bar{d}_j) - (a_i + c_i)(b_j + d_j) \beta_{k-i-j}.$$ 

**Proof.** We split up $\beta_k$ into three, $\beta_k = \sum_{i+j=r} R_p^h(\text{mult}_0^i(j, r)) = A + B + C$, and compute each term separately:

$$A = R_p^h(\text{mult}_0^i((1), (r))) + R_p^h(\text{mult}_0^i((2), (r))) = (\bar{c}_k - c_k) + (\bar{d}_k - d_k),$$

$$B = \sum_{j=k} R_p^h(\text{mult}_0^i(j, r)) = 2 \sum_{i+j=k} R_p^h(\bar{H}_i(\Omega X) \otimes \bar{H}_j(\Omega Y)) = 2 \sum_{i+j=k} [(a_i + \bar{c}_i)(b_j + \bar{d}_j) - (a_i + c_i)(b_j + d_j)].$$

The last term is more complicated, and needs some preliminary computations. For $n \geq 3$ and $j = (j_1, \ldots, j_n)$, $r = (r_1, \ldots, r_n)$ we denote $\hat{j} = (j_1, \ldots, j_{n-2})$, $\hat{r} = (r_1, \ldots, r_{n-2})$. If $j_{n-1} = 1$ and $j_n = 2$ we denote the following:

$$\hat{m}_1(i) = \begin{cases} m_2(i) & i \neq r_{n-1} \\ m_2(i) - 1 & i = r_{n-1} \end{cases}$$

$$\hat{m}_2(i) = \begin{cases} m_2(i) & i \neq r_n \\ m_2(i) - 1 & i = r_n \end{cases}$$

Consider the following:

$$R_p^h(\text{mult}_0^i(j, r)) = \prod_{0 \leq s_1 \leq m_1(l), l \neq r_{n-1}} \prod_{0 \leq t_1 \leq m_2(l)} \left( \left( m_2(l) \right)_{s_1} \right) \alpha_{s_1}^{(m_1(l) - s_1)} \times \prod_{l \neq r_n} \left( m_2(l) \right)_{t_1} \cdot \left( m_2(r_{n-1}) - 1 \right) \left( m_2(r_n) - 1 \right) \cdot \alpha_{s_1}^{(m_1(r_{n-1}) - s_{r_{n-1}})} \times b_{1}^{m_2(r_n) - t_{r_n}} \cdot \phi(s_1, \ldots, t_k) = U_1 + U_2 + U_3 + U_4$$

where each of the $U_i$ equals the previous sum except that $\left( \hat{m}_1(r_{n-1}) - 1 \right) \left( \hat{m}_2(r_n) + 1 \right)$ is replaced by one of the terms
respectively. Now we compare $\text{mult}_0^n(j, r)$ with $\text{mult}_0^{n-2}(\hat{j}, \hat{r})$:

$$U_1 = a_{r_{n-1}} b_{r_n} \sum_{0 \leq l \leq m_1(l)} \prod_{l=1}^{k} \left\{ \left( \frac{m_1(l)}{s_l} \right) a_{l}^{(m_1(l)-s_l)} \right\}.$$

$$= a_{r_{n-1}} b_{r_n} \sum_{0 \leq l \leq m_1(l)} \prod_{l=1}^{k} \left( \frac{m_2(l)}{s_l} \right) b_{l}^{(m_2(l)-t_l)} \cdot \phi(s_1, \ldots, t_k).$$

$$U_2 = a_{r_{n-1}} d_{r_n} \cdot R_{p^h}(\text{mult}_0^{n-2}(\hat{j}, \hat{r}))$$

$$= a_{r_{n-1}} \cdot \sum_{0 \leq l \leq m_2(l)} \prod_{l=1}^{k} \left( \frac{m_2(l)}{s_l} \right) b_{l}^{(m_2(l)-t_l)} \cdot a_{l}^{(m_1(l)-s_l)} \cdot \phi(s_1, \ldots, t_k)$$

Adding up the last equations we get:

$$R_{p^h}(\text{mult}_0^n(j, r)) - (a_{r_{n-1}} + c_{r_{n-1}})(b_{r_n} + d_{r_n})R_{p^h}(\text{mult}_0^{n-2}(\hat{j}, \hat{r}))$$

$$= [(a_{r_{n-1}} + c_{r_{n-1}})(b_{r_n} + d_{r_n}) - (a_{r_{n-1}} + c_{r_{n-1}})(b_{r_n} + d_{r_n})] \times R_{p^h}(\text{mult}_0^{n-2}(\hat{j}, \hat{r})).$$

We observe that the equation holds also when $j_{n-1} = 2$, $j_n = 1$ and $r = (r_1, \ldots, r_{n-2}, r_n, r_{n-1})$. Summing up the later equation for all $j$ and $r$, we get:

$$C = \sum_{i+j<k} (a_i + \bar{c}_i)(b_j + \bar{d}_j)\beta_{k-i-j} + \sum_{i+j<k} (a_i + \bar{c}_i)(b_j + \bar{d}_j) - (a_i + c_i)(b_j + d_j)\beta_{k-i-j}.$$

This concludes the proof of Proposition 2:
PROPOSITION 3.

\[
\beta_k = (a_k + c_k) + (b_k + d_k) + 2 \sum_{i+j=k} (a_i + c_i)(b_j + d_j) \\
+ \sum_{i+j>k} (a_i + c_i)(b_j + d_j) \beta_{k-i-j}.
\]

Although the proof of this proposition is lengthy, it is similar to the proof of Proposition 2 and will therefore be skipped.

Proof of Theorem 3. Under the conditions of the theorem, the formulas of \( \beta_k \) and \( \beta'_k \) reduce to the following:

\[
\beta_k = \sum_{i=1}^{n_3} \sum_{j=1}^{n_4} (a_i + c_i)(b_j + \bar{d}_j) \beta_{k-i-j} \\
+ \sum_{i=1}^{n_3} \sum_{j=1}^{n_4} [(a_i + \bar{c}_i)(b_j + \bar{d}_j) - (a_i + c_i)(b_j + d_j)] \beta_{k-i-j}.
\]

We observe that:

\[
(q_1[\beta_i])_k = \sum_{i=1}^{n_3} \sum_{j=1}^{n_4} [(a_i + \bar{c}_i)(b_i + \bar{d}_j) - (a_i + c_i)(b_j + d_j)] \beta_{k-i-j} \\
(q_2[\beta'_i])_k = 0.
\]

As a consequence of the results of \( \S 3 \) the proof of the theorem is obtained.

6. A recursion formula for the torsion component of \( H(\Omega(X \vee Y)) \). We denote the number of \( R_{p,h} \) direct summands in \( H_k(\Omega(X \vee Y)) \) by \( \gamma_k \). A recursion formula for \( \{\gamma_k\} \) will be stated next precisely:

THEOREM 4. Let \( n_3, n_4 \) satisfy: \( a_i + \bar{c}_i > 0 \) and \( b_j + \bar{d}_j > 0 \) imply that \( i \leq n_3, j \leq n_4 \).

Denote:

\[
q_i(x) = 1 - \sum_{i=1}^{n_3} \sum_{j=1}^{n_4} [a_i b_j x^{i+j} + (a_i \bar{d}_j + b_j \bar{c}_i) x^{i+j}(1 + x) + \bar{c}_j \bar{d}_j x^{i+j}(1 + x)^2] \\
q_j(x) = 1 - \sum_{i=1}^{n_3} \sum_{j=1}^{n_4} [a_i b_j x^{i+j} + (a_i d_j + b_j c_i) x^{i+j}(1 + x) + c_i d_j x^{i+j}(1 + x)^2].
\]

Then \( \{\gamma_k\}_{k=1}^{\infty} \) satisfies the recursion formula corresponding to \( q_4(x) \times q_5(x) \cdot q_i(x) \) at any \( k > 3(n_3 + n_4) + 4 \).

The proof of Theorem 4 is much more complicated than the proofs of the previous theorems. However, in principal it is similar to them. We state the intermediate results and leave the proofs for the reader. We denote:
\[
\gamma_k = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \sum_{r_j+i=k} R_{\nu}^n(\text{mult}_j(n, r))
\]

**Proposition 4.**

\[
\gamma_k = [(\bar{c}_k - c_k) + (\bar{d}_k - d_k)] \\
+ 2 \sum_{i+j=k} \{(a_i + \bar{c}_i)(b_j + \bar{d}_j) - (a_i + c_i)(b_j + d_j)\} \\
+ 2 \sum_{i+j=k} \{\bar{c}_i \bar{d}_j - c_i d_j\} \\
+ \sum_{i+j<k} \{(a_i b_j \gamma_{k-i-j} + a_i \bar{d}_j(\gamma_{k-i-j} + \gamma_{k-i-j-1}) \\
+ b_j \bar{c}_i(\gamma_{k-i-j} + \gamma_{k-i-j-1}) + \bar{c}_i \bar{d}_j(\gamma_{k-i-j} + 2 \gamma_{k-i-j-1} + \gamma_{k-i-j-2}) \\
+ \sum_{i+j<k} \{a_i(\bar{d}_j - d_j)(\gamma_{k-i-j} + \gamma_{k-i-j-1}) + b_j(\bar{c}_i - c_i)(\gamma_{k-i-j} + \gamma_{k-i-j-1}) \\
+ (\bar{c}_i \bar{d}_j - c_i d_j)(\gamma_{k-i-j} + 2 \gamma_{k-i-j-1} + \gamma_{k-i-j-2})\}.
\]

**Proposition 5.**

\[
\gamma_k = c_k + d_k + 2 \sum_{i+j=k} \{(a_i + c_i)(b_j + d_j) - a_i b_j\} \\
+ 2 \sum_{i+j=k} c_i d_j + \sum_{i+j<k} \{(a_i b_j \gamma_{k-i-j} + a_i \bar{d}_j)(\gamma_{k-i-j} + \gamma_{k-i-j-1}) \\
+ b_j c_i(\gamma_{k-i-j} + \gamma_{k-i-j-1}) + c_i d_j(\gamma_{k-i-j} + 2 \gamma_{k-i-j-1} + \gamma_{k-i-j-2}) \\
+ (c_i b_j + d_j a_i + c_i d_j) \alpha_{k-i-j}\}.
\]

7. An example. In this section we apply the recurrence formulas obtained, to compute the homology of the free product of two groups $G_1 \ast G_2$. Our method holds in this case, for $G_1 \ast G_2$ is of the homotopy type of $\Omega(BG_1 \vee BG_2)$ and $\Omega BG_i$ is of the homotopy type of $G_i$, $i = 1, 2$, where $BG_i$ is the classifying space of $G_i$, $i = 1, 2$, $[4], [7]$. We actually demonstrate our method of computation on the free product of the special orthogonal group $SO_3$ with itself. The homology of $SO_3$ is computed $[6]$ and equals:

\[
H_j(SO_3) = \begin{cases} 
Z & j = 0, 3 \\
Z & j = 1 \\
0 & \text{otherwise}
\end{cases}
\]

We are content with this group, because though its homology is simple the homology of $SO_3 \ast SO_3$ is infinite dimensional and complicated.

The recurrence formulas for $\{\alpha_k\}$, $\{\beta_k\}$ and $\{\gamma_k\}$ can be applied only to $k > n_1 + n_2$, $2(n_3 + n_4)$, $3(n_3 + n_4) + 4$ respectively, when we know the sequences in lower dimensions. Of course $\{\beta_k\}$ and $\{\gamma_k\}$ correspond to the number of $Z_2$ summands. The sequences of $\{\alpha_k\}$ and $\{\gamma_k\}$ in the lower dimensions were computed in $[3]$, essentially by
the use of the formulas of §2. \( \{\beta_k\} \) can be computed similarly.

We summarize these results in the following table:

<table>
<thead>
<tr>
<th></th>
<th>(\alpha)</th>
<th>(\beta)</th>
<th>(\gamma)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>8</td>
<td>16</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>18</td>
<td>58</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>26</td>
<td>104</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>36</td>
<td>192</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>56</td>
<td>356</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>82</td>
<td>652</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>118</td>
<td>1200</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>176</td>
<td>2210</td>
</tr>
<tr>
<td>14</td>
<td>0</td>
<td>258</td>
<td>4062</td>
</tr>
<tr>
<td>15</td>
<td>2</td>
<td>376</td>
<td>7472</td>
</tr>
<tr>
<td>16</td>
<td>0</td>
<td>554</td>
<td>13746</td>
</tr>
<tr>
<td>17</td>
<td>0</td>
<td>812</td>
<td>25280</td>
</tr>
<tr>
<td>18</td>
<td>2</td>
<td>1188</td>
<td>46498</td>
</tr>
</tbody>
</table>

Where the numbers beneath the heavy lines can be computed by the recursion formulas as will be seen presently.

The recursion formulas are the following:

\[
q_1(x) = 1 - x^6
\]
\[
q_2(x) = 1 - x^2 - 2x^4 - x^6
\]
\[
q_3(x) = 1 - x^6
\]
\[
(q_3q_3)(x) = 1 - x^3 \cdot 2x^2 - 3x + x^4 + 2x^6 + x^8 + x^{12}
\]
\[
q_4(x) = 1 - x^2 - 2x^3 - 3x^4 - 2x^5 - x^6
\]
\[
q_5(x) = 1 - x^6
\]
\[
(q_3q_5)(x) = 1 - x^3 \cdot 2x^2 - 3x^4 - 2x^5 - 2x^6 + x^8 + 2x^9 + 3x^{10} + 2x^{11} + x^{12}.
\]

As to that \(q_1(x) = q_3(x)\), \((q_3q_3)(x)\) expresses the recursion formula for \(\gamma\).

For example, to obtain \(\gamma_{17}\) we substitute into:

\[
\gamma_{17} = \gamma_{15} + 2\gamma_{14} + 3\gamma_{13} + 2\gamma_{12} + 2\gamma_{11} - \gamma_9 - 2\gamma_8 - 3\gamma_7 - 2\gamma_6 - \gamma_5 = 25280.
\]

**References**

3. G. Dula and E. Katz, *Computation of the homology of \( \Phi(X \vee Y) \)*, submitted.

Received January 21, 1978.

UNIVERSITY OF HAIFA
MOUNT CARMEL, HAIFA, 31999, ISRAEL

Current address: Tel-Aviv University
Tel-Aviv, Israel
PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)
University of California
Los Angeles, CA 90024

HUGO ROSSI
University of Utah
Salt Lake City, UT 84112

C. C. MOORE and ANDREW OGG
University of California
Berkeley, CA 94720

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, CA 90007

R. FINN and J. MILGRAM
Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

E. F. BECKENBACH
B. H. NEUMANN
F. WOLF
K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $84.00 a year (6 Vols., 12 issues). Special rate: $42.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1980 by Pacific Journal of Mathematics
Manufactured and first issued in Japan
Pacific Journal of Mathematics
Vol. 86, No. 2 December, 1980

Graham Donald Allen, David Alan Legg and Joseph Dinneen Ward, *Hermitian liftings in Orlicz sequence spaces* .................................................. 379
George Bachman and Alan Sultan, *On regular extensions of measures* .............. 389
Bruce Alan Barnes, *Representations Naimark-related to *-representations; a correction: “When is a representation of a Banach *-algebra Naimark-related to a *-representation?”* .................................................. 397
Earl Robert Berkson, *One-parameter semigroups of isometries into H* .......................... 403
M. Brodmann, *Piecewise catenarian and going between rings* .......................... 415
Joe Peter Buhler, *A note on tamely ramified polynomials* .................................. 421
William Lee Bynum, *Normal structure coefficients for Banach spaces* .................. 427
Lung O. Chung, *Biharmonic and polyharmonic principal functions* .................... 437
Vladimir Drobot and S. McDonald, *Approximation properties of polynomials with bounded integer coefficients* ............................................ 447
Giora Dula and Elyahu Katz, *Recursion formulas for the homology of* Ω(X ∨ Y) .......................................................... 451
John A. Ernest, *The computation of the generalized spectrum of certain Toeplitz operators* .......................................................... 463
Kenneth R. Goodearl and Thomas Benny Rushing, *Direct limit groups and the Keesling-Mardešić shape fibration* ............................................. 471
Raymond Heitmann and Stephen Joseph McAdam, *Good chains with bad contractions* .......................................................... 477
Patricia Jones and Steve Chong Hong Ligh, *Finite hereditary near-ring-semigroups* .......................................................... 491
Yoshikazu Katayama, *Isomorphisms of the Fourier algebras in crossed products* ........ 505
Meir Katchalski and Andrew Chiang-Fung Liu, *Symmetric twins and common transversals* .......................................................... 513
Mohammad Ahmad Khan, *Chain conditions on subgroups of LCA groups* ............. 517
Helmut Kröger, *Padé approximants on Banach space operator equations* .............. 535
Gabriel Michael Miller Obi, *An algebraic extension of the Lax-Milgram theorem* .......... 543
S. G. Pandit, *Differential systems with impulsive perturbations* .......................... 553
Richard Pell, *Support point functions and the Loewner variation* ......................... 561
J. Hyam Rubinstein, *Dehn’s lemma and handle decompositions of some 4-manifolds* .......................................................... 565
James Eugene Shirey, *On the theorem of Helley concerning finite-dimensional subspaces of a dual space* .................................................. 571
Oved Shisha, *Tchebycheff systems and best partial bases* .................................... 579
Michel Smith, *Large indecomposable continua with only one composant* .............. 593