GOOD CHAINS WITH BAD CONTRACTIONS

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Let $R \subseteq T$ be commutative rings with $T$ integral over $R$. In the study of chains of prime ideals, it is often of interest to know about primes $q \subseteq q'$ of $T$ such that height $(q'/q) < \text{height} (q' \cap R/q \cap R)$. In this paper we will consider a chain of primes $q_1 \subseteq q_2 \subseteq \cdots \subseteq q_m$ in $T$ which is well behaved in that height $(q_m/q_1) = \sum_{i=2}^{m-1} \text{height} (q_i/q_{i-1})$, but which suffers the pathology that height $(q_i \cap R/q_{i-1} \cap R) > \text{height} (q_i/q_{i-1})$ for each $i = 2, \cdots, m$. Our goal is to find a bound on how large $m$ can be.

Our main result is that if $T$ is generated as an $R$-module by $n$ elements, then there is a bound $b_n$ such that $m \leq b_n$; moreover $b_2 = 2$ and in general $b_n \leq b_{n-1}^2 + b_{n-1}^2 + \cdots + b_{n-1} + 2$. Let us quickly add that we do not claim that this formula gives the best bound possible. (We rather suspect not.) If $c = b_{n-1} + 2$, we also have, as part of our main result, that $m \leq \text{height} (q_c/q_1) + b_{n-1}$. (If $m > b_{n-1}$, so that $q_c$ exists.) Finally, if we have the added assumption that height $(q_i/q_{i-1}) \leq r$ for $i = 2, \cdots, m$, then $m \leq 2(r+1)^{n-2}$.

The bulk of our effort is needed to discuss the case that $T = R[u]$ is a simple integral extension of $R$. This is done in § 3. That section also introduces a new “going down” technique of some interest. Section 2 treats a highly special situation in which we obtain a much sharper bound. This case has some interest in its own right and also starts an induction needed in § 3. The fourth section gives the main result mentioned above. Lastly, in § 5, we present some examples. These illustrate the point that there is no bound in general, even in the case of Noetherian domains, on $m$ which is independent of the size of the integral extension $R \subseteq T$. Specifically, we show that $b_n \to \infty$ as $n \to \infty$. Thus our bounds, while presumably not sharp, have the proper form.

DEFINITION. The chain of primes $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_m$ is taut if height $(P_m/P_i) = \sum_{i=2}^{m-1} \text{height} (P_i/P_{i-1})$.

NOTATION. The following notation will be standard throughout except when specifically indicated otherwise. $R \subseteq T$ will be an integral extension of domains, $q_1 \subseteq \cdots \subseteq q_m$ will be a taut chain of primes in $T$ lying over $p_1 \subseteq \cdots \subseteq p_m$ in $R$. Height$(p_m/p_i)$ will be finite and height$(p_i/p_{i-1}) > \text{height}(q_i/q_{i-1})$, $i = 2, \cdots, m$. Finally, $x$ will be an indeterminate.
2. Split simple extensions. In this section, as well as the next, we will assume, in addition to the standard assumptions mentioned in the introduction, that $T$ is a simple integral extension of $R$. In order to be more specific, we make a definition.

**Definition.** Let the domain $T = R[u]$ be a simple integral extension of $R$ with $u$ a root of a monic polynomial $f(x) \in R[x]$. We will say that $T$ is a simple integral extension of $R$ via $f(x)$. Throughout §§ 2 and 3, without further mention, we will assume that $T = R[u]$ is a simple integral extension of $R$ via $f(x)$ with $f(x)$ having degree $n$ and $f(u) = 0$. Furthermore, in the present section we add one more assumption, namely that $f(x)$ is split.

**Definition.** The polynomial $f(x) \in R[x]$ is said to be split if $R[u] = R[u']$ for any two roots $u$ and $u'$ of $f(x)$.

Notice that if $f(x) = x^2 + ax + b = (x - u)(x - u') \in R[x]$, then $-u - u' = a \in R$ so that $R[u] = R[u']$. Thus if $n = 2$, $f(x)$ is split. We will show in this section that when $f(x)$ is split, $m$ is bounded by $\deg f(x)$. Our first lemma is well known. We state it explicitly because it is frequently used in what follows.

**Lemma 2.1.** (a) Let $p$ be prime in a ring $A$. Let $g(x)$ be a monic polynomial in $A[x]$ with $\deg g(x) = d$. Then there are at most $d$ primes of $A[x]$ which lie over $p$ and contain $g(x)$.

(b) Let $T = R[u]$ be a simple integral extension of $R$ via $f(x)$ with $\deg f(x) = n$. Let $p$ be prime in $R$. Then at most $n$ primes in $T$ lie over $p$.

**Proof.** (a) follows from standard facts such as [3, §§ 1-5] and the fact that taken modulo $p$, $g(x)$ has at most $d$ irreducible factors. (b) follows from (a) by considering preimages under the map $R[x] \rightarrow R[u] = T$.

**Theorem 2.2.** Let $f(x)$ be split. Let $q$ be prime in $T$ with $p = q \cap R$. In $R[x]$, let $P$ be prime with $P \cap R = p$ and suppose that $f(x) \in P$. Then for some root $u$ of $f(x)$, $q$ is the image of $P$ under the homomorphism $R[x] \rightarrow R[u] = T$.

**Proof.** As is well known, there is a $g(x) \in P$ such that $P = \{ h(x) \in R[x] \mid sh(x) \in (p, g(x))R[x] \text{ for some } s \in R - p \}$. Since $R[x] \subset T[x]$ is integral and $qT[x] \cap R[x] = pR[x]$, by going up we can find a prime $Q$ of $T[x]$ with $Q \cap T = q$ and $Q \cap R[x] = P$. Thus $f(x) \in P \subset Q$ and as $f(x)$ splits in $T[x]$, for some root $u$ of $f(x)$ we have
Now $g(x) \in P \subset Q$ and as $x \equiv u \mod Q$, $g(u) \in Q \cap T = q$. Thus the preimage of $q$ under the map $R[x] \to R[u] = T$ contains $g(x)$, and so is easily seen to be $P$.

**Corollary 2.3.** Let $f(x)$ be split. Let $p$ be prime in $R$.
(a) If $P_1$ and $P_2$ are prime in $R[x]$ with $P_1 \cap R[x] = p = P_2 \cap R[x]$ and $f(x) \in P_1 \cap P_2$ then $R[x]/P_1 \approx R[x]/P_2$, this isomorphism fixing $R/p$.
(b) Let $q_1$ and $q_2$ be primes in $T$ both lying over $p$. Then $T/q_1 \approx T/q_2$, this isomorphism fixing $R/p$.

**Proof.** (a) Let $q$ be a prime of $T$ lying over $p$. By Theorem 2.2, for roots $u_1$ and $u_2$ of $f(x)$, $q$ is the image of $P_i$ under $R[x] \to R[u_i] = T$, $i = 1, 2$. Thus $R[x]/P_1 \approx R[u_1]/q = R[u_2]/q \approx R[x]/P_2$.
(b) If $P$ is prime in $R[x]$ with $P \cap R = p$ and $f(x) \in P$, and if $q$ is any prime in $T$ lying over $p$, then the proof of (a) shows that $T/q \approx R[x]/P$. Thus $T/q_1 \approx R[x]/P \approx T/q_2$.

**Theorem 2.4.** Let $f(x)$ be split. Then $m \leq \deg f(x)$.

**Proof.** We first claim that there are distinct primes $Q_1, \ldots, Q_m$ lying over $p_m$ satisfying $q_i \subset Q_j$ and height$(Q_j/q_i) \geq$ height$(q_m/q_i)$, $j = 1, \ldots, m$. To do this, we induct on $m$. For $m = 2$, by going up there is a prime $q_2$ of $T$ with $q_1 \subset q_2'$ and $q_2' \cap R = p_2$ and height$(q_2'/q_1) = \text{height}(p_2/p_1) > \text{height}(q_2/q_1)$. Let $Q_1 = q_2$ and $Q_2 = q_2'$.

For $m > 2$ take $q_2'$ as above. The isomorphism in Corollary 2.3 between $T/q_2'$ and $T/q_2$ carries $q_2' \subset \cdots \subset q_m$ isomorphically to a chain $q_1' \subset \cdots \subset q_m'$ which also lies over $p_2 \subset \cdots \subset p_m$ (since $R/p_2$ is fixed).

By induction there are distinct primes $Q_1, \ldots, Q_{m-1}$ of $T$ lying over $p_m$ with $q_i \subset Q_j$ and height$(Q_j/q_i) \geq$ height$(q_m/q_i)$, $j = 1, \ldots, m - 1$. Since $q_2 \subset \cdots \subset q_m$ and $q_2' \subset \cdots \subset q_m'$ are "isomorphic", height$(q_i'/q_i) = \text{height}(q_i'/q_i)$. Recall also height$(q_i'/q_i) > \text{height}(q_i/q_i)$. By the tautness of $q_1 \subset \cdots \subset q_m$ we have for $j = 1, \ldots, m - 1$, height$(Q_j/q_i) \geq$ height$(Q_j/q_i) + \text{height}(q_i'/q_i) \geq$ height$(q_j/q_i)$ + height$(q_i'/q_i) >$ height$(q_m/q_i) + \text{height}(q_i'/q_i) = \text{height}(q_j/q_i)$. That is, height$(Q_j/q_i) >$ height$(q_m/q_i)$, for $j = 1, \ldots, m - 1$. Letting $Q_m = q_m$ proves our claim.

Finally, as the number of primes in $T$ contracting to any given prime in $R$ cannot exceed $\deg f(x)$, the existence of $Q_1, \ldots, Q_m$ shows that $m \leq \deg f(x)$.

The final result in this section discusses the situation when the bound given by Theorem 2.4 is obtained.

**Proposition 2.5.** Let $f(x)$ be split and let $m = \deg f(x)$. Suppose that $p \subset p_1 \subset p_m \subset p'$ with $p, p'$ primes in $R$ and that $q \cap R=$
Proof. The proof of Theorem 2.4 shows that there are primes $Q_1, \ldots, Q_m = q_m$ lying over $p_m$, each of which contains $q_1$. By going up, find a prime $q'_1$ of $T$ with $q \subset q'_1$ and $q'_1 \cap R = p$. Now $q_i$ is contained in $m$ primes lying over $p_m$ (namely $Q_1, \ldots, Q_m$) and so by Corollary 2.3 $q'_1$ is also contained in $m$ primes lying over $p_m$. However, since $\deg f(x) = m$, $Q_1, \ldots, Q_m$ are the only primes lying over $p_m$ and so $q \subset q'_1 \subset Q_1 \cap \cdots \cap Q_m$.

Now consider $R[x] \to R[u] = T$ and let $Q^*_1, Q^*_i, \ldots, Q^*_m$ be the preimages of $q'_1, Q_1, \ldots, Q_m$ respectively. Obviously $Q^*_1 \cap R = p'_1, Q^*_1 \cap R = p_m, j = 1, \ldots, m$ and $f(x) \in Q^*_1 \cap Q^*_i \cap \cdots \cap Q^*_m$ since $f(u) = 0$. By [4, Lemma 3] (applied to $R/p_m$) we easily see that there is a prime $P$ of $R[x]$ with $P \cap R = p_m$, and $f(x) \in P \subset Q^*$. However since $\deg f(x) = m$, at most $m$ primes in $R[x]$ can contain $f(x)$ and also contract to $P_m$. As each of $Q^*_1, \ldots, Q^*_m$ do just that, obviously $P = Q^*_j$ for some $j = 1, \ldots, m$. Thus $Q^*_j = P \subset Q^*$ from which we see that $Q_j \subset q'$. Thus $q \subset Q_1 \cap \cdots \cap Q_m \subset Q_j \subset q'$ and we are done.

3. Arbitrary simple extensions. We now drop the "split" assumption and just assume that $T$ is a simple integral extension of $R$ via $f(x)$ with $\deg f(x) = n$. We will show that there is a number $b_n$ such that $m \leq b_n$. We do not identify the best such bound although we do give an inequality limiting the size of the best such bound. To be explicit, let us use $b_n$ to denote the smallest number such that $m \leq b_n$ for all such $m$.

We have already seen at the start of §2 that if $n = 2$ then $f(x)$ is split, and so by Theorem 2.4 we have $b_2 = 2$. (This is best possible, [5, Example 2, pp. 203-205].) We will now assume inductively that $b_{n-1}$ exists.

In our next lemma we start a chain at $P_z$ rather than $P_1$, since that will be the situation when we apply the lemma.

**Lemma 3.1.** Let $P_z = \cdots \subset P_m$ be a taut chain of primes in $R[x]$ contracting to $p_z \subset \cdots \subset p_m$ in $R$. Let $P'_z \neq P_z$ with $P'_z \cap R = p_z$. Let $f(x)$ be a monic polynomial of degree $n$ with $f(x) \in P_z \cap P'_z$. Let $s > 0$ be an integer with $m > b_{n-1}(s - 1) + 1$. Then for some $i \in \{1, \ldots, m - s\}$ there is a taut chain $P_{i+1} \subset \cdots \subset P_{i+s}$ in $R[x]$ lying over $p_{i+1} \subset \cdots \subset p_{i+s}$ with $\text{height}(P_{i+j}/P_{i+j-1}) = \text{height}(P_{i+j}/P_{i+j-1})$, $j = 2, \ldots, s$ and with $P_z = P_{i+1}$ and $\text{height}(P_{i+s}/P_z) \leq \text{height}(P_{i+s}/P_z)$.

**Proof.** Obviously we may work modulo $p_z$; so assume that $p_z = 0$. Since $f(x) \in P_z \cap P'_z, R[x]/P_z$ and $R[x]/P'_z$ are simple integral extensions of $R$ via $f(x)$. Let $R[x]/P_z \approx R[u]$ and $R[x]/P'_z \approx R[u']$.
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with \(u\) and \(u'\) distinct roots of \(f(x)\) (distinct since \(P_2 \neq P'_2\)). Taken modulo \(P_2, P_2 \subset \cdots \subset P_m\) becomes a taut chain \(0 = q_2 \subset \cdots \subset q_m\) in \(R[u]\) lying over \(0 = p_2 \subset \cdots \subset p_m\). As \(R[u] \subset R[u, u']\) is integral, we lift \(0 = q_2 \subset \cdots \subset q_m\) to a taut chain \(0 = q^*_2 \subset \cdots \subset q^*_m\) in \(R[u, u']\), with height \(q^*_n = \text{height } q_m\).

Since \(f(u') = 0, f(x) = (x - u')g(x)\) with \(g(x)\) monic in \(R[u'][x]\). As \(u \neq u'\), we have \(g(u) = 0\) so that \(R[u, u']\) is a simple integral extension of \(R[u']\) via \(g(x)\). Since \(\deg g(x) = n - 1\), the induction assumption concerning the existence of \(b_{n-1}\) applies to \(R[u']\).

Let \(b = b_{n-1}\) and consider a subchain of \(q^*_n \subset \cdots \subset q^*_m\), namely \(q^*_n \subset q^*_{n+(s-1)} \subset q^*_{n+2(s-1)} \subset \cdots \subset q^*_{n+b(s-1)}\), which, being a subchain of a taut chain, is taut. (Note \(q^*_{n+b(s-1)}\) exists since \(m > b(s - 1) + 1\).) Because this taut (sub)-chain contains \(b + 1\) primes, by the induction assumption for some \(s > 0\) we must have height \(q^*_n \subset \cdots \subset q^*_s\) implies that \(q^*_{n+1} \cap R[u'] \subset \cdots \subset q^*_{s+1} \cap R[u']\) is taut, and that height \(q^*_r / q^*_r \cap R[u']/q^*_r \cap R[u']\) which in turn equals height \(q^*_{r+j} / q^*_{r+j-1}\) where the manner in which \(q^*_r \subset \cdots \subset q^*_m\) was constructed. Also height \(q^*_{r+1} \cap R[u']\) \(\geq\) height \(q^*_{r+1}\) since height \(q^*_{r+1} =\) height \(q^*_{r+1}\).

Finally, recalling that \(R[u'] \sim R[x]/P_r'\), the chain \(q^*_r \cap R[u'] \subset \cdots \subset q^*_{s+1} \cap R[u']\) gives rise to a chain \(P^*_{r+1} \subset \cdots \subset P^*_{s+1}\) in \(R[x]\) with \(P^*_r \subset P^*_{r+1}\). This chain satisfies the lemma follows easily from what we know about \(q^*_r \cap R[u'] \subset \cdots \subset q^*_{s+1} \cap R[u']\).

**COROLLARY 3.2.** Let the domain \(T\) be a simple integral extension of \(R\) via \(f(x)\) with \(\deg f(x) = n\). Let \(q_2 \subset \cdots \subset q_m\) be a taut chain in \(T\) lying over \(p_2 \subset \cdots \subset p_m\) in \(R\). Let \(q'_2 \neq q_2\) be prime in \(T\) with \(q'_2 \cap R = p_2\). Let \(s > 0\) be an integer with \(m > b(n-1) + 1\). Then for some \(i \in \{1, \cdots, m - s\}\), there is a taut chain \(q^*_{i+1} \subset \cdots \subset q^*_{i+s}\) in \(T\) lying over \(p_{i+1} \subset \cdots \subset p_{i+s}\) with height \(q^*_{i+j} / q^*_{i+j-1}\) = height \(q^*_{i+j} / q^*_{i+j-1}\), \(j = 2, \cdots, s\), and with \(q^*_{i+1} \leq q^*_{i+1}\) and height \(q^*_{i+1} / q^*_{i}\) \(\geq\) height \(q^*_{i+1} / q^*_{i}\).

**Proof.** Let \(P_2 \subset \cdots \subset P_m\) and \(P'_2\) be, respectively, the preimages of \(q_2 \subset \cdots \subset q_m\) and \(q'_2\) under \(R[x] \to R[u] = T\). Then, since \(f(x) \in P_2 \cap P'_2\), the hypothesis of Lemma 3.1 is satisfied. We complete the proof by letting \(q^*_{i+1} \subset \cdots \subset q^*_{i+s}\) be the images of \(P^*_{i+1} \subset \cdots \subset P^*_{i+s}\) given by Lemma 3.1.

**PROPOSITION 3.3.** Let \(b = b_{n-1}\). Let \(l \geq 0\) be an integer and let \(m \geq b + b^{r-1} + \cdots + b + 2\). Then for some \(r = 1, \cdots, m, p\) has lying over it distinct primes \(Q_1, \cdots, Q_{r+1}\) in \(T\) such that \(q_1 \subset Q_1 \cap \cdots \cap Q_{r+1}\).
and \( \text{height}(Q_j/q_j) > \text{height}(q_j/q_i) \) for \( j = 1, \ldots, l + 1 \).

**Proof.** We induct on \( l \). First, since \( \text{height}(p_3/p_r) > \text{height}(q_3/q_r) \), by going up there is a prime \( q' \) of \( T \) with \( q_1 \subset q' \) and \( \text{height}(q'/q_j) = \text{height}(p_3/p_r) \). If \( l = 0 \) then \( r = 2 \) and \( Q_1 = q' \) satisfy the proposition.

For \( l > 0 \), we apply Corollary 3.2 with \( s = b^{l-1} + b^{l-2} + \cdots + b + 2 \). Since \( m > b(s - 1) + 1 \) we have for some \( i \in \{1, \ldots, m - s\} \) a taut chain \( q_{i+1} \subset \cdots \subset q_{i+s} \) in \( T \) lying over \( p_{i+1} \subset \cdots \subset p_{i+s} \) with height \( (q_{i+s}/q_{i+j-1}) = \text{height}(q_{i+j}/q_{i+j-1}) \) which is less than height \( (q_i/q_{i+1}) \) for \( j = 2, \ldots, s \).

We apply the case \( l - 1 \) of the induction assumption to the chain \( q_{i+1} \subset \cdots \subset q_{i+s} \) (recalling that \( s = b^{l-1} + b^{l-2} + \cdots + b + 2 \)), to produce an \( r \in \{i + 1, \ldots, i + s\} \) and distinct primes \( Q_i, \ldots, Q_l \) of \( T \) lying over \( p_r \), with \( q_{i+1} \subset Q_1 \cap \cdots \cap Q_l \) and \( \text{height}(Q_j/q_{i+1}) > \text{height}(q_j/q_{i+1}) \) for \( j = 1, \ldots, l \). If we now let \( Q_{i+1} = q' \), obviously \( Q_i \) is distinct from \( Q_1, \ldots, Q_l \) and we now have \( q_{i+1} \subset Q_1 \cap \cdots \cap Q_l \) and height \( (Q_j/q_{i+1}) \geq \text{height}(q_j/q_{i+1}) \) for \( j = 1, \ldots, l + 1 \).

We have \( q_1 \subset q' \subset q_{i+1} \) by Corollary 3.2. To complete the proof, we must only show that \( \text{height}(Q_j/q_i) > \text{height}(q_j/q_i) \) for \( j = 1, \ldots, l + 1 \). To do this, we collect various facts.

(i) \( \text{height}(q_j/q_{j+1}) = \text{height}(q_j/q_{j+1}) \). This follows from the fact that \( \text{height}(q_{i+j}/q_{i+j-1}) = \text{height}(q_{i+j}/q_{i+j-1}) \) for \( j = 2, \ldots, s \) by Corollary 3.2 and the tautness of \( q_{i+1} \subset \cdots \subset q_{i+s} \) and \( q_{i+1} \subset \cdots \subset q_{i+s} \).

(ii) \( \text{height}(Q_j/q_{i+1}) \geq \text{height}(q_j/q_{i+1}) \). This follows from (i) and the previously noted fact that \( \text{height}(Q_j/q_{i+1}) \geq \text{height}(q_j/q_{i+1}) \).

(iii) \( \text{height}(q_{i+1}/q_i) \geq \text{height}(q_{i+1}/q_i) \) by Corollary 3.2.

(iv) \( \text{height}(q_j/q_i) > \text{height}(q_j/q_i) \) by choice of \( q'_i \).

Finally, from the tautness of \( q_1 \subset \cdots \subset q_r \) and (ii), (iii), and (iv), we have \( \text{height}(q_j/q_i) = \text{height}(q_j/q_{i+1}) + \text{height}(q_{i+1}/q_i) + \text{height}(q_3/q_1) < \text{height}(Q_j/q_{i+1}) + \text{height}(q_{i+1}/q_i) + \text{height}(q_3/q_1) \leq \text{height}(Q_j/q_i) \) for \( j = 1, \ldots, l + 1 \) to complete the proof.

At this point we can prove that \( b^n \) exists and show that \( b^n \leq b^{n-1} + b^{n-2} + \cdots + b + 1 \) with \( b = b_{n-1} \). To see this, with the notation of Proposition 3.3, if \( m > b^{n-1} + b^{n-2} + \cdots + b + 1 \) we would have primes \( q_r, Q_1, \ldots, Q_s \) lying over \( p_r \) which are distinct (by the inequality in that proposition). However, as \( \text{deg } f(x) = n \), at most \( n \) primes can lie over \( p_r \), a contradiction. Thus \( m \leq b^{n-1} + \cdots + b + 1 \).

We wish to introduce a "going down" technique which will let us improve this inequality somewhat, giving \( b^n \leq b^{n-1} + b^{n-2} + \cdots + b + 2, b = b_{n-1} \), and which, in certain circumstances, allows us to give a more substantial improvement on the bound on \( b^n \).

**Definition.** Let \( p \) be a prime in the ring \( R \). Let \( I \) be an
ideal in \( R[x] \). Define \( k(p, I) = n \) if \( IR_p[x] \) contains a monic polynomial of degree \( n \) but no monic polynomial of lesser degree. (If \( IR_p[x] \) contains no monic polynomial let \( k(p, I) = \infty \).)

**Lemma 3.4.** Let \( p \) be prime in a ring \( R \) and let \( I \) be an ideal in \( R[x] \). Suppose that \( k(p, I) = n < \infty \).

(a) If \( g(x) \in I \) and \( \deg g(x) < n \) then \( g(x) \in pR[x] \).

(b) Let \( h(x) \in I \) with \( \deg h(x) = n \) and the leading coefficient of \( h(x) \) outside of \( p \). Let \( P \) be prime in \( R[x] \) with \( P \cap R = p \). Then \( I \subseteq P \) if and only if \( h(x) \in P \).

(c) The number of primes \( P \) in \( R[x] \) satisfying \( P \cap R = p \) and \( I \subseteq P \) does not exceed \( n \).

**Proof.** Without loss we may localize at \( p \).

(a) Since \( k(p, I) = n < \infty \) and \( (R, p) \) is quasi-local, there is in \( I \) a monic polynomial \( h(x) \) of degree \( n \), and no monic polynomial of lesser degree. If the result is false, then for some \( g(x) = a_kx^k + \cdots + a_i x^i + \cdots + a_0 \in I \) with \( k < n \) we have \( a_i \notin p \) for some \( i \). Assume that \( g(x) \) and \( i \) have been chosen so as to make \( i \) as large as possible. Now \( a_k \in p \) since \( g(x) \) is not monic. We have \( a_k h(x) - x^{n-k} g(x) \in I \). Its degree is clearly less than \( n \) and its \( (i + n - k) \)th coefficient is not in \( p \). This is a contradiction since \( i + n - k > i \).

(b) Since \( h(x) \) (in part (a)) is monic, clearly \( I \) is generated by \( h(x) \) together with those polynomials in \( I \) having degree less than \( n \). By part (a), each of these latter polynomials is in \( pR[x] \subset P \). Thus \( I \subseteq P \) if and only if \( h(x) \in P \).

(c) This is immediate from Lemma 2.1 and (b).

**Proposition 3.5.** Let \( p < p' \) be primes in a ring \( R \). Let \( I \) be an ideal of \( R[x] \), and suppose that \( k(p, I) = k(p', I) < \infty \). If \( P' \) is prime in \( R[x] \) with \( P' \cap R = p' \) and \( I \subseteq P' \), then there is a prime \( P \) in \( R[x] \) with \( P \cap R = p \) and \( I \subseteq P \subseteq P' \).

**Proof.** We may localize at \( p' \). If \( k(p', I) = n \) then \( I \) contains a monic polynomial \( h(x) \) of degree \( n \). Thus \( h(x) \in I \subseteq P' \). By [4, Lemma 3] (applied to \( R/p \)) there is a prime \( P \) of \( R[x] \) with \( P \cap R = p \) and \( h(x) \in P \subseteq P' \). By Lemma 3.4, \( I \subseteq P \).

We apply Proposition 3.5 to our special situation of \( R \subset R[u] = T \) a simple integral extension of domains, \( u \) a root of the monic polynomial \( f(x) \).

**Corollary 3.6.** Let \( p \subset p' \) be primes in \( R \). Let \( I = \ker(R[x] \to R[u] = T) \) and suppose that \( k(p, I) = k(p', I) \). If \( q' \) is prime in \( T \)
with \( q' \cap R = p' \) then there is a prime \( q \) of \( T \) with \( q \cap R = p \) and \( q \subset q' \).

**Proof.** Since \( f(x) \in I, k(p', I) < \infty \). Let \( P' \) be the preimage of \( q' \) under \( R[x] \to R[u] \). Then \( P' \cap R = p' \) and \( I \subset P' \). With \( P \) as in Proposition 3.5 take \( q \) to be the image of \( P \) in \( T \).

**Theorem 3.7.** \( b_\infty \leq b_{n-2} + b_{n-3} + \cdots + b + 2 \) where \( b = b_{n-1} \).

**Proof.** Let \( B = b_{n-2} + b_{n-3} + \cdots + b + 2 \) and assume that \( m > B \). We will derive a contradiction. Applying Proposition 3.3 to the chain \( q_0 \subset \cdots \subset q_B \) we see that for some \( r \in \{1, \cdots, B\} \) there are distinct primes \( Q_1, \cdots, Q_{n-1} \) of \( T \) lying over \( p_r \) with \( q_0 \subset Q_1 \cap \cdots \cap Q_{n-1} \) and \( \text{height}(q_{j}/q_0) > \text{height}(q_{j}/q_{j-1}) \), \( j = 1, \cdots, n-1 \). Obviously \( q_r \) is distinct from \( Q_1, \cdots, Q_{n-1} \) and if we let \( Q_n = q_r \) then, since \( \deg f(x) = n, Q_1, \cdots, Q_n \) are all of the primes of \( T \) lying over \( p_r \) and we have \( \text{height}(Q_j/q_0) \geq \text{height}(Q_j/q_{j-1}) \), \( j = 1, \cdots, n \).

We claim that if \( p \) is prime in \( R \) with \( p_r \subset p \), then \( k(p, I) = n \) where \( I = \ker(R[x] \to R[u]) = T \). Since \( f(x) \in I, k(p, I) \leq n \). Also \( p_r \subset p \) implies \( k(p_r, I) \leq k(p, I) \) and so we must only show that \( k(p_r, I) \geq n \). That this is true follows from Lemma 3.4 (c) and the existence of \( Q_1, \cdots, Q_n \).

We now consider a chain of maximal length between \( p_r \) and \( p_m \). Since \( k(p, I) = n \) for each prime \( p \) in that chain, we can use Corollary 3.6 iteratively to find a prime \( q \) of \( T \) with \( q \cap R = p_r, q \subset q_m \) and \( \text{height}(q_m/q) = \text{height}(p_m/p_r) \). Since \( q_0 \subset \cdots \subset q_m \) is taut and \( \text{height}(q_i/q_{i-1}) > \text{height}(q_{i+1}/q_{i-1}) \), \( i = r + 1, \cdots, m \), obviously \( \text{height}(q_m/q) = \text{height}(p_m/p_r) > \text{height}(q_m/q_r) \), (here we use \( m > B \geq r \)). As \( Q_1, \cdots, Q_n \) are all of the primes which lie over \( p_r \), we must have \( q = Q_j \), some \( j = 1, \cdots, n \). Thus \( \text{height}(q_i/q_0) = \text{height}(Q_i/q_0) \geq \text{height}(q_r/q_0) \). Thus \( \text{height}(q_m/q_i) \geq \text{height}(q_m/q) + \text{height}(q_i/q_0) > \text{height}(q_m/q_r) + \text{height}(q_i/q_0) \), contradicting the tautness of \( q \subset \cdots \subset q_m \). This completes the proof.

We repeat that we doubt that equality holds in Theorem 3.7. Let us note that \( b_2 \leq b_3 \leq b_4 \leq \cdots \). To see this, observe that if \( T \) is a simple integral extension of \( R \) via \( f(x) \), then it is also a simple integral extension of \( R \) via \( xf(x) \). The examples at the end of this paper show that \( b_\infty \to \infty \) as \( n \to \infty \).

We now consider situations in which we can give other bounds on the size of \( m \).

**Lemma 3.8.** Suppose that \( m > b_{n-1} \). Let \( c = b_{n-1} + 1 \). If \( p \) is any prime of \( R \) containing \( p_r \), then \( k(p, I) = n \) where \( I = \ker(R[x] \to \cdots \subset q_m \).
Proof. Since $f(x) \in I$, obviously $k(p_c, I) \leq k(p, I) \leq n$. We must show $k(p_c, I) \geq n$. For this we may localize at $p_c$. If $k(p_c, I) < n$ then $I$ contains a monic polynomial $g(x)$ with $\deg g(x) = d < n$. Clearly $T$ is a simple integral extension of $R$ via $g(x)$. However the existence of the chain $q_1 \subset \cdots \subset q_e$ with $c > b_n - 1 \geq b_d$ contradicts the definition of $b_d$.

**Lemma 3.9.** Suppose that $m > b_n - 1$ and let $c = b_n - 1 + 1$. Let $p$ be any prime of $R$ containing $p_c$ and let $q$ be any prime in $T$ lying over $p$. Then $q_1 \subset q$.

**Proof.** Let $P_1$ and $P$ be the preimages of $q_1$ and $q$, respectively, under the map $R[x] \to R[u] = T$. We claim that $k(p, P_1) = n$. The result follows, since obviously $f(x) \in P_1 \cap P$ and so by Lemma 3.4(b) (with $h(x) = f(x)$ and $I = P_1$) $P_1 \subset P$. Thus $q_1 \subset q$.

To show that $k(p, P_1) = n$, we may work modulo $p_1$. That is we go to $R/p_1 \subset T/q_1$ and so assume that $p_1 = 0 = q_1$. Now $P_1 = \ker(R[x] \to T)$ and Lemma 3.8 gives $k(p, P_1) = n$.

**Theorem 3.10.** Suppose that $m > b_n - 1$ and let $c = b_n - 1 + 1$. Then $m \leq \text{height}(q_e/q_1) + b_n - 1$.

**Proof.** Consider a chain of maximal length between $p_c$ and $p_m$. By Lemma 3.8, for each prime $p$ in that chain, $k(p, I) = n$ with $I = \ker(R[x] \to T)$. By iteration of Corollary 3.6, we can find a prime $q$ of $T$ with $q \cap R = p_c$, $q \leq q_m$ and $\text{height}(q_m/q) = \text{height}(p_m/p_c)$. By Lemma 3.9, $q_1 \subset q$. Since $q_1 \subset \cdots \subset q_m$ is taut we have $\sum_{i=1}^m \text{height}(q_i/q_{i-1}) + \text{height}(q_e/q_i) = \text{height}(q_m/q) \geq \text{height}(q_m/q) + \text{height}(q/q_i) \geq \sum_{i=1}^m \text{height} (p_{i-c})/(p_{i-1}) + \text{height}(q/q_i)$. Thus $\text{height}(q_e/q_i) \geq \sum_{i=1}^m [\text{height}(p_{i-c}) - \text{height}(q_{i-1}/q_i)] + \text{height}(q/q_i)$. By our underlying assumption concerning how $q_1 \subset \cdots \subset q_m$ contracts to $p_i \subset \cdots \subset p_m$, each term in this last summation is at least one. Thus $\text{height}(q_e/q_i) \geq (m - c) + \text{height}(q/q_i) \geq m - c + 1 = m - b_{n-1}$. Thus $m \leq \text{height}(q_e/q_i) + b_{n-1}$.

**Corollary 3.11.** Suppose that $m > b_n - 1$ and that $\text{height}(q_i/q_{i-1}) \leq r$ for $j = 2, \ldots, b_n - 1$. Then $m \leq (r + 1)b_{n-1}$.

**Proof.** Immediate from Theorem 3.10 and the tautness of $q_1 \subset \cdots \subset q_e$.

Suppose that we fix $r > 0$ and restrict our attention to chains
$q_i \subseteq \cdots \subseteq q_m$ with height$(q_i/q_{i-1}) \leq r$, $i = 2, \ldots, m$. Let $b'_n$ denote the best possible bound on $m$ for such chains when $\deg f(x) = n$. Then Lemma 3.8 through Corollary 3.11 can be repeated, replacing $b_{n-1}$ with $b'_{n-1}$, thus showing that $b'_n \leq (r + 1)b'_{n-1}$. Since $b'_2 = 2$, by induction we get $b'_n \leq 2(r + 1)^{n-2}$.

**Theorem 3.12.** If $\text{height}(q_i/q_{i-1}) \leq r$ for $i = 2, \ldots, m$, then $m \leq 2(r + 1)^{n-2}$.

4. Finitely generated modules. We give our main result, assuming only that $T$ is a finitely generated $R$-module.

**Theorem 4.1.** Let $R \subseteq T$ be domains with $T$ a finitely generated $R$-module, generated by $n$ elements. Let $q_1 \subseteq \cdots \subseteq q_m$ be a taut chain of primes in $T$ lying over $p_1 \subseteq \cdots \subseteq p_m$ with height$(p_m/p_i)$ finite. Suppose that $\text{height}(p_i/p_{i-1}) > \text{height}(q_i/q_{i-1})$ for $i = 2, \ldots, m$. Then $m$ is subject to the following:

(i) $m \leq b_n$,

(ii) if $m > b_{n-1}$, then $m \leq \text{height}(q_c/q_i) + b_{n-1}$ with $c = b_{n-1} + 1$,

(iii) $m \leq 2(r + 1)^{n-2}$ with $r = \max \{\text{height}(q_i/q_{i-1}) | i = 2, \ldots, m\}$.

**Proof.** Since $T$ is a finitely generated $R$-module only finitely many primes of $T$ lie over $p_m$, and we may choose $u \in q_m$ but in no other prime lying over $p_m$. Obviously $q_m$ is the only prime of $T$ lying over $q_m \cap R[u]$ and so $\text{height}(q_m \cap R[u]/q_i \cap R[u]) = \text{height}(q_m/q_i)$ (by going up since $\text{height}(q_m \cap R[u]/q_i \cap R[u]) \leq \text{height}(p_m/p_i) < \infty$). Clearly we have $(q_i \cap R[u]) \subseteq \cdots \subseteq (q_m \cap R[u])$, a taut chain in $R[u]$ with $\text{height}(q_i \cap R[u]/q_{i-1} \cap R[u]) = \text{height}(q_i/q_{i-1}) < \text{height}(p_i/p_{i-1})$ for $i = 2, \ldots, m$. A standard determinant argument shows that $u$ satisfies a monic polynomial of degree $n$ over $R$, and our result follows from the existence of $b_n$ and Theorems 3.10 and 3.12.

**Corollary 4.2.** Let $R$ be a domain with integral closure $R'$. Suppose that $R'$ is a finitely generated $R$-module with $n$ generators. Let the domain $T$ be an integral extension of $R$. Let $0 = q_1 \subseteq \cdots \subseteq q_m$ be a taut chain of primes in $T$ lying over $0 = p_1 \subseteq \cdots \subseteq p_m$ in $R$ with height $p_m$ finite. Suppose that $\text{height}(p_i/p_{i-1}) > \text{height}(q_i/q_{i-1})$ for $i = 2, \ldots, m$. Then (i) $m \leq b_n$; (ii) if $m > b_{n-1}$, then $m \leq \text{height}(q_c/q_i) + b_{n-1}$; and (iii) $m \leq 2(r + 1)^{n-1}$ with $r = \max \{\text{height}(q_i/q_{i-1}) | i = 2, \ldots, m\}$.

**Proof.** If $T'$ is the integral closure of $T$, we may lift $0 = q_1 \subseteq \cdots \subseteq q_m$ to a taut chain $0 = q'_1 \subseteq \cdots \subseteq q'_m$ in $T'$ with height $q'_m = \text{height} q_m$. By going down in $R' \subseteq T'$, height $q' \cap R' = \text{height} q'$ and
we see that \(0 = (q'_1 \cap R') \subseteq \cdots \subseteq (q'_m \cap R')\) is taut in \(R'\) and height \((q'_i \cap R'|q'_{i-1} \cap R') = \text{height}(q'_i/q'_{i-1}) = \text{height}(p_i/p_{i-1}) i = 2, \ldots, m\). Applying Theorem 4.1 to \(0 = (q'_1 \cap R') \subseteq \cdots \subseteq (q'_m \cap R')\), we are done.

5. Examples. In this section, we construct a family of examples which demonstrate that \(b_n \to \infty\) as \(n \to \infty\). We also show that if \(R \subset T\) is an infinite integral extension, no bound need exist at all. This construction is a generalization of Nagata's Example 2 [5, pp. 203-205] and is very similar to [2]. However, except for the quotation of one key theorem, the presentation of the basic example will be self-contained.

**Example 5.1.** Retaining the previous notation, we show any \(m\) can be realized in some finite integral extension \(R \subset T\) (which depends on \(m\)). Moreover, our example is Noetherian.

Fix an integer \(m \geq 2\). Let \(K\) be a countable field of characteristic zero and let \(y_i, \ldots, y_{m-1}, z_1^{(1)}, \ldots, z_m^{(m-1)}\) be indeterminates. We iteratively define a sequence of Noetherian domains \(K = T_1 \subseteq T_2 \subseteq T_3 \subseteq \cdots \subseteq T_m = T\) as follows: Set \(T_{i+1} = T_i[y_i]\) for each \(i = 1, \ldots, m - 1\). Suppose \(Z \in K[[y_i]]\) is a formal power series, say \(Z = a^{(i)}_i y_i + a^{(i)}_{i+1} y_{i+1} + \cdots\). If we set \(z^{(i)}_n = (z^{(i)}_0 - \sum_{j=1}^n a^{(i)}_j y_j)/y^n\) for each \(n \geq 0\), then

\[
(*) \quad z^{(i)}_n = (z^{(i)}_{n+1} + a^{(i)}_{n+1}) y_i \in \overline{T}_{i+1}[z^{(i)}_{n+1}].
\]

Thus we may define a direct union of simple transcendental extensions of \(\overline{T}_{i+1}, T_{i+1} = \lim_{n \to \infty} \overline{T}_{i+1}[z^{(i)}_n]\), for each \(i = 1, \ldots, m - 1\). Moreover, by [2, Corollary 1.6], we may choose the formal power series \(Z_i\) in such a way that \(T_{i+1}\) will be Noetherian.

The nature of the construction makes it very easy to determine the primes; primes in the intermediate rings extend to primes in \(T\). Hence we easily see, for each \(i = 1, \ldots, m\), \(q_i = (y_i, \ldots, y_{i-1}) T\) is prime. Also, by \((*)\), \(z^{(i)}_n \in y_i T\) for each \(i, n\). By the Krull Altitude Theorem, height \(q_i \leq i - 1\). \((0) = q_1 \subset q_2 \subset \cdots \subset q_m\) is a taut chain and height\((q_{i+1}/q_i) = 1 i = 1, \ldots, m - 1\). Before leaving this chain, we make one additional observation, also apparent from the construction. The quotient \(T/q_i\) is canonically isomorphic to the subring \(S_i = K[y_i, \ldots, z^{(i)}_n, \ldots, y_{m-1}, \ldots, z^{(m-1)}_n, \ldots]\) for each \(i = 1, \ldots, m\).

Next we iteratively define a second chain \((0) = Q_1 \subset Q_2 \subset Q_3 \subset \cdots \subset Q_m\). First note that, using \((*)\) again, \(z^{(i)}_{n+1} = (z^{(i)}_n + a^{(i)}_n)(y_i - 1) + z^{(i)}_n + a^{(i)}_{n+1}\). Thus \(z^{(i)}_{n+1} \equiv z^{(i)}_n - a^{(i)}_{n+1}(\text{mod}(y_i - 1))\). So if we set, for each \(i = 1, \ldots, n - 1, \overline{Q}_{i+1} = Q_i + (y_i - 1) T\) and \(Q_{n+1} = \overline{Q}_{n+1} + z^{(i)}_n T\), we have (using equality to denote canonical isomorphism) \(T/\overline{Q}_{i+1} = S_i[z^{(i)}_n]\) and \(T/Q_{i+1} = S_i\). So these ideals are prime as required.
and another application of the Krull Altitude Theorem guarantees that this chain is taut.

Our next step is to construct $R$. Again we construct a chain of rings $T = R_1 \supset R_2 \supset \cdots \supset R_m = R$. For each $i = 1, \ldots, m - 1$, set $R_{i+1} = S_{i+1} + (q_{i+1} \cap Q_{i+1} \cap R_i)$. Since $S_{i+1} \subset S_i \subset R_i$, $R_{i+1} \subset R_i$ as desired. We claim that $R_i$ is an integral extension of $R_{i+1}$, generated by two elements as an $R_{i+1}$-module. To verify the claim, consider the canonical $R_{i+1}$-module homomorphism $\pi_i: R_i \rightarrow (R_i/q_{i+1} \cap R_i) \oplus (R_i/q_{i+1} \cap R_i) = S_{i+1} \oplus S_{i+1}$. Note $\pi_i(1) = (1, 1)$ and $\pi_i(y_i) = (0, 1)$ together generate $S_{i+1} \oplus S_{i+1} = \text{image}(\pi_i)$ and so $R_i = (1)R_{i+1} + (y_i)R_{i+1} + \ker(\pi_i)$. However, $\ker(\pi_i) = q_{i+1} \cap Q_{i+1} \subset R_i$ and so $R_i - R_{i+1} + y_iR_{i+1}$ proving our claim. Therefore $T$ is generated as an $R$-module by $2^{m-1}$ elements. Consequently, by Eakin’s Theorem [1, p. 281], $R$ is a Noetherian domain.

It now only remains to show $R \subset T$ exhibits the desired chain behavior. As $\dim T_{i+1} = (\dim T_i) + 1$ and $\dim T_{i+1} = (\dim T_i) + 1$ for each $i = 1, \ldots, m - 1$, $\dim T = 2(m - 1)$. So, by going up, $\dim R = 2(m - 1)$. Thus $(0) = Q_i \cap R \subset Q_i \cap R \subset Q_i \cap R \subset \cdots \subset Q_m \cap R$ is taut; then $Q_i \cap R \subset Q_i \cap R \subset \cdots \subset Q_m \cap R$ is likewise taut and $\text{height}(Q_{i+1} \cap R)/(Q_i \cap R) = 2$ for each $i = 1, \ldots, m - 1$. However, by construction, $Q_i \cap R = q_i \cap R$ and so $\text{height}(q_{i+1} \cap R)/(q_i \cap R) = 2$. As $(0) = q_i \subset \cdots \subset q_m$ is a taut chain in $T$ and $\text{height}(q_{i+1}/q_i) = 1$, we have the desired chain.

In particular, this example shows $b_{2^{m-1}} \geq m$ and so $b_n \rightarrow \infty$ as $n \rightarrow \infty$.

**Example 5.2.** There is an infinite integral extension $R \subset T$ and an infinite taut chain in $T$, $(0) = q_i \subset q_2 \subset \cdots$, such that $(0) = q_i \cap R \subset q_i \cap R \subset \cdots$ is taut and $\text{height}(q_{i+1}/q_i) = 1 < 2 = \text{height}(q_{i+1} \cap R/q_i \cap R)$ for each $i$. Necessarily, $R$ is not Noetherian.

Example 5.2 will be a direct union of domains constructed in the manner of (5.1). We begin as in (5.1) with a sequence of domains $K = T_1 \subset T_2 \subset \cdots \subset T_m \subset \cdots$, this time choosing an infinite sequence. For each fixed $m$, we perform the construction in (5.1), superscripting our symbols with $(m)$ when confusion is possible. Thus $T_m = T^{(m)}$ and we have $R^{(m)} \subset T^{(m)}$.

Noting $T^{(m)} \subset T^{(m+1)}$, $q_i^{(m)} \subset q_i^{(m+1)}$ and $Q_i^{(m)} \subset Q_i^{(m+1)}$ for each $i = 1, \cdots, m$, and $S_i^{(m)} \subset S_i^{(m+1)}$ for each $i = 1, \cdots, m$, we have direct unions $T = \cup T^{(m)}$, $q_i = \cup q_i^{(m)}$, $Q_i = \cup Q_i^{(m)}$, and $S_i = \cup S_i^{(m)}$ with $T/q_i = S_i = T/Q_i$. Next, using the fact that $S^{(m)} = K = S^{(m+1)}$, and some obvious containments, we have $R^{(m)} = R^{(m)} = S^{(m)} + (q_i^{(m)} \cap Q^{(m)} \cap R^{(m)} = S^{(m)} + (q_i^{(m)} \cap Q^{(m)} \cap R^{(m)}) \subset S^{(m+1)} + (q_i^{(m+1)} \cap Q^{(m+1)} \cap R^{(m+1)}) = \cdots$.
Thus, we also have a direct union $R = \bigcup R^{(m)}$. We claim $R \subset T$ is the desired extension.

If $y \in T, y \in T^{(m)}$ for some $m$ and so is integral over $R^{(m)}$ and consequently $R$. Thus we have an integral extension. Since $R$ and $T$ are direct unions, the statement about the $q_i$'s is valid because it holds in $R^{(m)} \subset T^{(m)}$ for each $m$.

**Example 5.3.** There is a Noetherian domain $R$ such that, for each $m$, we may find an integral extension $T$ of $R$ and a taut chain of primes $(0) = q_1 \subset q_2 \subset \cdots \subset q_m$ in $T$ such that $q_1 \cap R \subset \cdots \subset q_m \cap R$ is taut and $\text{height}(q_{i+1}/q_i) < \text{height}(q_{i+1} \cap R/q_i \cap R)$ for each $i = 1, \ldots, m - 1$.

This example will not be formally constructed. It is obtained by combining two construction ideas. One constructs a family of local Example (5.1)'s and combines them in the manner of Nagata's Example 1 [5, p. 203] (the Noetherian ring with infinite Krull dimension). This is a useful and straightforward way of obtaining this sort of infinite bad behavior.

**References**


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