

# Pacific Journal of Mathematics

**SYMMETRIC TWINS AND COMMON TRANSVERSALS**

MEIR KATCHALSKI AND ANDREW CHIANG-FUNG LIU

## SYMMETRIC TWINS AND COMMON TRANSVERSALS

M. KATCHALSKI AND A. LIU

**In this paper, we study the properties of certain families of sets on the circle and use the result to obtain a theorem on common transversals for sets in the plane.**

1. **Introduction.** The standard Helly type results (see [2]) are essentially of the following nature:

If each subfamily of a given size of a family of sets has a certain property, then the whole family has the same property.

Our results in this paper are in a different form:

Let  $\mathcal{F}$  be a family of  $n$  sets where  $n$  is sufficiently large. For any constant  $c$ ,  $0 < c < 1$ , there exists an integer  $k = k(c)$ ,  $1 < k < n$ , such that if each subfamily of  $\mathcal{F}$  of size  $k$  has a certain property, then some subfamily of  $\mathcal{F}$  of size at least  $cn$  has the same property.

A symmetric twin (see [3] for other kinds of twins) is a subset of a circle which consists of two closed arcs symmetric about the center of the circle. We shall also consider the whole circle as a degenerate symmetric twin. The property of interest here is that of having nonempty intersection. Our result is:

**THEOREM A.** *Let  $\mathcal{F}$  be a family on  $n$  symmetric twins on the same circle and let  $k$  be an integer,  $1 < k < n$ . If each subfamily of  $\mathcal{F}$  of size  $k$  has nonempty intersection, then some subfamily of  $\mathcal{F}$  of size at least  $n(k - 2)/(k + 1)$  has nonempty intersection.*

We point out that given  $0 < c < 1$ , we can choose  $k$  so that  $(k - 2)/(k + 1) > c$  provided that  $n$  is sufficiently large.

For families of connected closed sets in the plane, the property of interest here is that of having a common transversal (see [4]), which is a straight line intersecting all members of the family. Our result is:

**THEOREM B.** *Let  $\mathcal{F}$  be a family of  $n$  connected closed sets in the plane where  $n$  is sufficient large. For any constant  $c$ ,  $0 < c < 1$ , there exists an integer  $k = k(c)$ ,  $1 < k < n$ , such that if each subfamily of  $\mathcal{F}$  of size  $k$  has a common transversal, then some subfamily of  $\mathcal{F}$  of size at least  $n$  has  $ca$  common transversal.*

To prove Theorem B, we shall make use of Theorem A as well as yet another result of similar nature, proved in different terms

nology by Abbott and Katchalski ([1]):

**THEOREM C.** *Let  $\mathcal{S}$  be a family of  $n$  closed intervals on the line where  $n$  is sufficiently large. Let  $\alpha$  be any constant,  $0 < \alpha < 1$ . If at least  $\alpha \binom{n}{2}$  of the pairs of intervals have nonempty intersections, then some subfamily of  $\mathcal{S}$  of size at least  $(1 - \sqrt{1 - \alpha})n$  has nonempty intersection.*

**2. Proof of Theorem A.** We may assume that  $k \geq 3$ . Since  $n(k-2)/(k+1)$  is an increasing function of  $k$ , we may assume that  $\mathcal{F}$  has a subfamily  $\mathcal{B} = \{B_1, B_2, \dots, B_{k+1}\}$  with empty intersection. We may also assume that none of the  $B$ 's is the whole circle.

For  $1 \leq i \leq k+1$ , choose antipodal points  $a_i$  and  $a_{i+k+1}$  on the circle belonging to  $\cap (\mathcal{B} - \{B_i\})$ . Relabelling if necessary, assume that  $a_1, a_2, \dots, a_{2k+2}$  are in clockwise order on the circle. The arc from  $a_u$  to  $a_v$  will be denoted by  $[a_u, a_v]$ , and all subscripts are to be reduced mod  $(2k+2)$ .

Let  $1 \leq i \leq k+1$ . Since  $B_i$  is a symmetric twin, we have

$$[a_{i+1}, a_{i+k}] \cup [a_{i+k+2}, a_{i-1}] \subset B_i.$$

Thus  $x \in B_i$  if  $x \notin [a_{i-1}, a_{i+1}] \cup [a_{i+k}, a_{i+k+2}]$ . Consequently,

$$\cap (\mathcal{B} - \{B_{i+1}, B_{i+2}\}) \subset [a_i, a_{i+3}] \cup [a_{i+k+1}, a_{i+k+4}].$$

For any  $F \in \mathcal{F} - \mathcal{B}$ ,  $\{F\} \cup (\mathcal{B} - \{B_{i+1}, B_{i+2}\})$  is a subfamily of  $\mathcal{F}$  of size  $k$  and has nonempty intersection. Hence for  $1 \leq i \leq k+1$ ,

$$F \cap [a_i, a_{i+3}] \neq \emptyset$$

as  $F$  is a symmetric twin.

It follows that each  $F \in \mathcal{F} - \mathcal{B}$ , being a symmetric twin, contains all of the points  $a_1, a_2, \dots, a_{2k+2}$  with the possible exception of 6. Hence one of these points, say  $a$ , belongs to at least

$$\frac{(2k+2) - 6}{2k+2} |\mathcal{F} - \mathcal{B}| = \frac{k-2}{k+1} (n - k - 1)$$

members of  $\mathcal{F} - \mathcal{B}$ . The point  $a$  also belongs to  $k$  members of  $\mathcal{B}$ . The theorem follows since  $(k-2)/(k+1)(n-k-1) + k > n(k-2)/(k+1)$ .

**3. Proof of Theorem B.** Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ . For  $0 < c < 1$ , choose  $k$  so that

$$c = 1 - \sqrt{1 - \alpha}$$

with

$$\alpha = \left( \left\lfloor \frac{k}{2} \right\rfloor - 2 \right) / \left( \left\lfloor \frac{k}{2} \right\rfloor + 1 \right).$$

Let  $C$  be a fixed circle in the plane. For  $1 \leq i, j \leq n, i \neq j$ , let  $A_{ij}$  be the set of all points on  $C$  which lie on straight lines which pass through the center of  $C$  and are parallel to some common transversal of  $F_i$  and  $F_j$ . Clearly  $A_{ij}$  is a symmetric twin on  $C$ . Let  $\mathcal{A}$  denote the collection of all these  $A$ 's.

Since every subfamily of  $\mathcal{F}$  of size  $k$  has a common transversal, every subfamily of  $\mathcal{A}$  of size  $\lfloor k/2 \rfloor$  has nonempty intersection. By Theorem A,  $\mathcal{A}$  has a subfamily of size at least  $\alpha \binom{n}{2}$  with nonempty intersection. Let  $x$  be a point in this intersection.

Let  $L$  be a fixed straight line perpendicular to the straight line joining  $x$  and the center of  $C$ . For  $1 \leq i \leq n$ , let  $G_i$  be the projection of  $F_i$  onto  $L$ . Clearly  $G_i$  is a closed interval on  $L$ . Let  $\mathcal{G}$  denote the collection of all these  $G$ 's.

For  $1 \leq i, j \leq n, i \neq j, G_i \cap G_j \neq \emptyset$  if  $x \in A_{ij}$ . Hence at least  $\alpha \binom{n}{2}$  of the pairs of intervals have nonempty intersection. By Theorem C,  $\mathcal{G}$  has a subfamily of size at least  $cn$  with nonempty intersection. Let  $y$  be a point in this intersection.

The theorem now follows as the straight line passing through  $y$  and perpendicular to  $L$  is a common transversal of a subfamily of  $\mathcal{F}$  of size at least  $cn$ .

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