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AN ALGEBRAIC EXTENSION OF THE LAX-MILGRAM THEOREM

GABRIEL MICHAEL MILLER OBI

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In this work a Lax-Milgram type theorem is proved for quadratic spaces over a division ring K with involution $*$, say, whose center contains an ordered domain P such that for every element a in K , $aa^* = |a|^2$, (where $|a|$, the absolute value of a , is in P^+ which is the set of positive elements of P), and for every element b in P^+ there exists an element c in P^+ , denoted by $b^{1/2}$, such that $c^2 = b$. Specifically, with the above assumptions on K , the following is proved:

Let (H_i, Φ_i) $i = 1, 2$ be quadratic spaces over K such that for each u in H_2 $\sup |\Phi_2(u, v)| (|\Phi_2(v, v)|^{1/2})^{-1}$ exists and equals $|\Phi_2(u, u)|^{1/2}$. Let $B: H_1 \times H_2 \rightarrow K$ be an orthocontinuous bilinear form satisfying:

(i) $\inf_{x \neq 0} \sup_{y \neq 0} |B(x, y)| (|\Phi_1(x, x)|^{1/2} |\Phi_2(y, y)|^{1/2})^{-1} = \gamma$ exists and $\gamma - \delta$ is in P^+ for some δ in P^+ .

(ii) $\sup |B(x, y)|$ exists and is in P^+ for all $y \neq 0$ $x \in H_1$.

Then given any orthocontinuous linear functional ϕ on H_2 whose kernel is splitting there exists a unique element x_0 in H_1 such that $\phi(y) = B(x_0, y)$ for all y in H_2 .

Moreover

$$\delta^{-1} \sup_{y \neq 0} |\phi(y)| (|\Phi_2(y, y)|^{1/2})^{-1} - |\Phi_1(x_0, x_0)|^{1/2} \in P^+ \cup \{0\}.$$

1. Introduction. Motivated by what he referred to as "happy accidents in the Hilbert space theory that correlate algebraic and topological considerations" Piziak [3] proposed an algebraic approach to the study of sesquilinear forms in infinite dimensions. In this approach he introduced the notion of quadratic spaces by the means of which he obtained an algebraic generalization of some Hilbert space results. He proved an algebraic version of the Riesz-Frechet Representation Theorem and discussed continuity all in the algebraic context of a vector space over a division ring in which no natural topology is present.

We here consider an algebraic extension of the Lax-Milgram Theorem, a variant of the Riesz Representation Theorem.

Our results are of pure algebra. We have not assumed a topology either on the division ring or on the vector space which we considered. It is thus interesting to note that these results imply their analogous standard topological results in the context of Hilbert space.

2. Preliminaries.

DEFINITION 2.1. [2], [3]. A quadratic space is a triple (K, H, Φ) where K is a division ring with involution, $*$, H is a left vector space over K , and Φ is a nondegenerate orthosymmetric $*$ -sesquilinear form on H with respect to the involutive anti-automorphism $*$ of K .

Where there is no confusion we shall denote a quadratic space simply by H or (H, Φ) .

DEFINITION 2.2. Let H be a quadratic space. For x, y in H we say x is orthogonal to y and write $x \perp y$ if $\Phi(x, y) = 0$.

We note that since Φ is orthosymmetric $x \perp y$ implies $y \perp x$ and vice-versa.

NOTATION 2.3. For any subset M of H , put

$$M^\perp = \{y \text{ in } H: \Phi(x, y) = 0 \text{ for all } x \text{ in } M\}.$$

It is easy to see that M^\perp is a subspace of H for every subset M of H . A vector x in H is said to be isotropic if $x \perp x$ and anisotropic otherwise. If every nonzero vector in H is anisotropic then H is called an anisotropic quadratic space.

DEFINITION 2.4. We call a subspace M of H \perp -closed iff $M = M^{\perp\perp}$. We say M splits H if $H = M \oplus M^\perp$ and we say M is semi-simple if $M \cap M^\perp = \{0\}$.

3. Lax-Milgram theorem. The Lax-Milgram states, [4]:

Let H be a Hilbert space (real or complex) and B a bilinear form on H such that

(i) $|B(x, y)| \leq \rho \|x\| \|y\|$ for all x, y in H and some positive real constant ρ .

(ii) There exists a positive real number δ such that $B(x, x) > \delta \|x\|^2$ for every x in H .

Then there exists a unique bounded linear operator T on H such that

(a) $\langle x, y \rangle = B(x, Ty)$ for all x, y in H .

(b) $\|T\| \leq \delta^{-1}$.

(Here $\langle \cdot, \cdot \rangle$ denotes the inner product on H .)

An observation of this theorem shows that condition (i) implies that B is continuous and (ii) implies that

(ii) (a) B is positive definite.

(ii) (b) The maps $B_x(\cdot), B^y(\cdot): H \rightarrow C$ defined by $B_x(y) = B(x, y)$;

$B^y(x) = B(x, y)$ are nontrivial for each x (resp. each y) in H i.e., $B^y(x) = 0$ for all x in H implies $y = 0$ and $B_x(y) = 0$ for all y in H implies $x = 0$.

Also the conclusion of the theorem can be reframed thus:

Then given any bounded linear functional f on H there exists a unique vector u_0 in H such that $f(u) = B(u_0, u)$ for all u in H and $\|u_0\| \leq \delta^{-1} \|f\|$.

We shall now generalize this result to quadratic spaces.

DEFINITION 3.1. Let (H_i, Φ_i) , $i = 1, 2$ be quadratic spaces. Let $L: H_1 \rightarrow H_2$ be a linear transformation such that $L(M^{\perp\perp}) \subseteq L(M)^{\perp\perp}$ for every subspace M of H_1 . Then L is said to be orthocontinuous.

PROPOSITION 3.2. [2], [3]. Let (H_i, Φ_i) , $i = 1, 2$ be quadratic spaces and $L: H_1 \rightarrow H_2$ be a linear transformation. Then the following are equivalent:

(i) $M = M^{\perp\perp}$ implies $L^{-1}(M) = L^{-1}(M)^{\perp\perp}$ for all subspaces M of H_2 .

(ii) If M is a \perp -closed subspace of H_2 then $L^{-1}(M)$ is a \perp -closed subspace of H_1 .

(iii) $L(M^{\perp\perp}) \subseteq L(M)^{\perp\perp}$ for all subspaces M of H_1 .

(iv) $L^{-1}(M)^{\perp\perp} \subseteq L^{-1}(M^{\perp\perp})$ for all subspaces M of H_2 .

(v) L is orthocontinuous.

DEFINITION 3.3. [2], [3]. Let (K, H, Φ) be a quadratic space. A linear map $\phi: H \rightarrow K$ is called an orthocontinuous linear functional if $(\text{Ker } \phi)^{\perp\perp} = \text{Ker } \phi$.

Now, in Hilbert space a bilinear form $B(x, y)$ is continuous iff there exists a continuous linear transformation L on H such that $B(x, y) = \langle Lx, y \rangle$ for all x, y in H . Motivated by this fact we make the following definition:

DEFINITION 3.4. Let (H_i, Φ_i) , $i = 1, 2$ be quadratic spaces. A bilinear form $B: H_1 \times H_2 \rightarrow K$ is said to be orthocontinuous if there exists an orthocontinuous linear operator $L: H_1 \rightarrow H_2$ such that $B(x, y) = \Phi_2(Lx, y)$ for every (x, y) in $H_1 \times H_2$.

PROPOSITION 3.5. Let (H_i, Φ_i) , $i = 1, 2$ be quadratic space and $B: H_1 \times H_2 \rightarrow K$ be an orthocontinuous bilinear form. Then the mappings $B_x(\cdot): H_2 \rightarrow K$, $B^y(\cdot): H_1 \rightarrow K$ are orthocontinuous.

Proof. Let L be as in 3.4.

$$\text{Ker } B_x(\cdot) = \{y: B(x, y) = 0\}$$

$$\begin{aligned}
 &= \{y: \Phi_2(Lx, y) = 0\} \\
 &= (Lx)^\perp \text{ a } \perp\text{-closed subspace.}
 \end{aligned}$$

Also

$$\begin{aligned}
 \text{Ker } B^y(\cdot) &= \{x: B(x, y) = 0\} \\
 &= \{x: \Phi_2(Lx, y) = 0\} \\
 &= \{x: Lx \in y^\perp\} \\
 &= L^{-1}(y^\perp) \text{ a } \perp\text{-closed subspace} \\
 &\text{since } L \text{ is orthocontinuous.}
 \end{aligned}$$

We note that the orthocontinuity of B implies that of B_x and B^y . The converse is not true in general, however. This is a consequence of [3, Theorem 3].

PROPOSITION 3.6. *Let (H_i, Φ_i) , $i = 1, 2$ be quadratic spaces. Let $M = \{0\} \times H_2 \cong H_2$ and $\pi: H_1 \times H_2 \rightarrow H_2$ be the projection $\pi((x, y)) = y$ for all (x, y) in $H_1 \times H_2$. Then π maps \perp -closed subspaces of $H_1 \times H_2$ onto \perp -closed subspaces of H_2 .*

Proof. We first note that any subspace of $H_1 \times H_2$ of the form $A \times B$ where A is a subspace of H_1 and B a subspace of H_2 is \perp -closed in $H_1 \times H_2$ iff A is \perp -closed in H_1 and B is \perp -closed in H_2 . Hence a subspace B of H_2 is \perp -closed iff $\pi^{-1}(B)$ is \perp -closed. We also note that $(H_1 \times H_2, \Phi_1 \oplus \Phi_2)$ is a quadratic space [2] and that M is a semi-simple splitting subspace. Let Φ_M be the restriction of $\Phi_1 \oplus \Phi_2$ to M . Then (M, Φ_M) is a quadratic space. Since π is a homomorphism there exists an isomorphism $\phi: H_1 \times H_2/H_1 \times \{0\} \rightarrow M$. For u, v in $H_1 \times H_2/H_1 \times \{0\}$ define

$$\Psi(u, v) = \Phi_M(\phi(u), \phi(v)).$$

Then since ϕ is 1-1 onto we have that Ψ is a nondegenerate orthosymmetric and *-sesquilinear form on $H_1 \times H_2/H_1 \times \{0\}$. Let $A \subseteq H_1 \times H_2/H_1 \times \{0\}$. If u is in A^\perp then

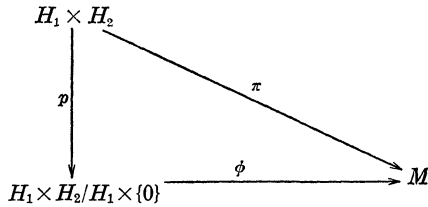
$$\begin{aligned}
 \Phi_M(\phi(u), \phi(v)) &= 0 \text{ for all } v \text{ in } A \\
 \iff \phi(u) &\in \phi(A)^\perp.
 \end{aligned}$$

Thus $\phi(A^\perp) = \phi(A)^\perp$. Also if $B \subseteq M$ then there exists $A \subseteq H_1 \times H_2/H_1 \times \{0\}$ such that $\phi(A) = B$. Suppose $B = B^{\perp\perp}$ then

$$\begin{aligned}
 A &= \phi^{-1}(B) \\
 &= \phi^{-1}(B^{\perp\perp}) \\
 &= \phi^{-1}(\phi(A)^{\perp\perp})
 \end{aligned}$$

$$\begin{aligned}
 &= \phi^{-1}(\phi(A^{\perp\perp})) \text{ by the above argument} \\
 &= A^{\perp\perp}.
 \end{aligned}$$

Hence ϕ, ϕ^{-1} map \perp -closed subspaces onto \perp -closed subspaces. Now consider the following diagram where p is the canonical projection.



A subspace B of $H_1 \times H_2 / H_1 \times \{0\}$ is \perp -closed iff $p^{-1}(B)$ is \perp -closed. Indeed let B be \perp -closed then $\phi(B)$ is \perp -closed and $\pi^{-1}(\phi(B)) = p^{-1}(B)$ is \perp -closed since π being a projection is orthocontinuous. Also if $p^{-1}(B)$ is \perp -closed then since $p^{-1}(B) = \pi^{-1}(\phi(B))$ we must have that $\phi(B)$ is \perp -closed by the observation at the beginning of the proof. Hence $B = \phi^{-1}(\phi(B))$ is \perp -closed. Finally if A is \perp -closed in $H_1 \times H_2$, let B be such that $p^{-1}(B) = A$. Then

$$\begin{aligned}
 \pi(A) &= \phi(p(A)) \\
 &= \phi(p(p^{-1}(B))) \\
 &= \phi(B)
 \end{aligned}$$

which is \perp -closed.

LEMMA 3.7. *Let B be an orthocontinuous bilinear form on $H_1 \times H_2$. Let L be such that $B(x, y) = \Phi_2(Lx, y)$ by Definition 3.4. If $B_x(\cdot), B_y(\cdot)$ are nontrivial for each x in H_1 and y in H_2 , then L is 1-1 and onto.*

Proof. Suppose $Lx_1 = Lx_2$. Then

$$\Phi_2(Lx_1 - Lx_2, v) = 0 \text{ for all } v \text{ in } H_2,$$

therefore

$$\begin{aligned}
 B(x_1 - x_2, v) &= \Phi_2(L(x_1 - x_2), v) \\
 &= \Phi_2(Lx_1 - Lx_2, v) \\
 &= 0
 \end{aligned}$$

for all v in H_2 . Since $B_x(\cdot)$ is nontrivial we must have that $x_1 - x_2 = 0$ or $x_1 = x_2$. Thus L is 1-1. To show that L is onto we note that since L is orthocontinuous its graph, $G(L)$, is a \perp -closed

subspace of $(H_1 \times H_2, \Phi_1 \oplus \Phi_2)$. Hence $\pi(G(L))$ is \perp -closed in H_2 . Suppose $L(H_1) = \pi(G(L)) \neq H_2$. Then there exists a nonzero vector x in H_2 such that for all x in H_1 , $\Phi_2(Lx, z) = 0$ [4]. Thus

$$\begin{aligned} 0 &= \Phi_2(Lx, z) \\ &= B(x, z) \\ &= B^z(x) \end{aligned}$$

for every x in H_1 . Since $B^y(\cdot)$ is nontrivial we must have $z = 0$. This contradiction establishes the result.

LEMMA 3.8. *Let $S: H_1 \rightarrow H_2$ be an orthocontinuous linear transformation from a quadratic space H_1 to a quadratic space H_2 . If S^{-1} exists then $(S^*)^{-1}$ exists and $(S^*)^{-1} = (S^{-1})^*$.*

Proof. Since $\text{Im}(S^*) = \text{Ker}(S)^\perp$ and $\text{Im}(S) = \text{Ker}(S^*)^\perp$ [3], and since S is 1-1 onto because of the existence of S^{-1} , we must have that S^* is 1-1 and onto and hence that $(S^*)^{-1}$ exists. Also since S is orthocontinuous the graph of S^{-1} is \perp -closed in $(H_2 \times H_1, \Phi_2 \oplus \Phi_1)$. So S^{-1} is a CDD transformation [2]. Let x be in the domain of S^{-1} and y in the domain of $(S^*)^{-1}$. Then since S is CDD we have that

$$\begin{aligned} \Phi_2(x, (S^*)^{-1}y) &= \Phi_2(Su, (S^*)^{-1}y) \quad \text{for some } u \text{ in } H_1 \\ &= \Phi_1(u, S^*(S^*)^{-1}y) \\ &= \Phi_1(u, y) \\ &= \Phi_1(S^{-1}Su, y) \\ &= \Phi_1(S^{-1}x, y). \end{aligned}$$

Therefore $(S^{-1})^* \subseteq (S^*)^{-1}$ [2], [3]. Now let R and U be the mappings defined on $H_1 \times H_2 \rightarrow H_2 \times H_1$ by

$$\begin{aligned} R((x, y)) &= (y, x) \\ U((x, y)) &= (-y, x). \end{aligned}$$

Then, as can be easily checked, we have that R, U are 1-1, onto; $R^{-1}(A^\perp) = R^{-1}(A)^\perp$ and $U(A^\perp) = U(A)^\perp$. Now

$$\begin{aligned} G((S^*)^{-1}) &= \{(x, (S^*)^{-1}x): x \in H_1\} \\ &= \{(S^*u, u): u \in H_2\} \\ &= R^{-1}(G(S^*)) \\ &= R^{-1}(U(G(S)^\perp)) \\ &= R^{-1}(U(G(S)))^\perp \end{aligned}$$

and

$$\begin{aligned}
 G((S^{-1})^*) &= \{(S^{-1}y, y): y \in H_2\}^\perp & [2] \\
 &= \{(-x, Sx): x \in H_1\}^\perp \\
 &= \{-R^{-1}(-Sx, x): x \in H_1\}^\perp \\
 &= \{-R^{-1}U(x, Sx): x \in H_1\}^\perp \\
 &= R^{-1}(U(G(S)))^\perp.
 \end{aligned}$$

Therefore the domain of $(S^*)^{-1}$ and that of $(S^{-1})^*$ coincide. Hence [2], [3], we have that $(S^{-1})^* = (S^*)^{-1}$.

LEMMA 3.9. *Let (H_i, Φ_i) , $i = 1, 2$ be quadratic spaces. If S is a 1-1, onto orthocontinuous transformation on H_1 to H_2 then S^{-1} is orthocontinuous.*

Proof. By 3.8 $(S^*)^{-1}$ exists and equals $(S^{-1})^*$. This shows that S^{-1} is an everywhere defined linear transformation which is such that $(S^{-1})^*$ everywhere defined. Therefore S^{-1} is orthocontinuous [2], [3].

THEOREM 3.10. *Let (H_i, Φ_i) , $i = 1, 2$ be quadratic spaces and B an orthocontinuous bilinear form on $H_1 \times H_2$. If $B_x(\cdot)$, $B^y(\cdot)$ are nontrivial then there exists an orthocontinuous linear transformation, T , on H_2 to H_1 such that*

$$\Phi_2(u, v) = B(Tu, v)$$

for all u, v in H_2 . Further if H_2 is such that every \perp -closed subspace is splitting then T is unique.

Proof. Let L be the linear map on H_1 to H_2 associated with B as in Definition 3.4. Then by 3.7, 3.8, and 3.9 $T = L^{-1}$ is orthocontinuous and

$$B(Tu, v) = \Phi_2(u, v) \quad \text{for all } u, v \text{ in } H_2.$$

Now suppose every \perp -closed subspace of H_2 splits H_2 . Then the mapping $B_x: H_2 \rightarrow K$ being orthocontinuous we have by [3, Theorem 3.1], that there exists a unique vector u_0 in H_2 such that $B_x(y) = \Phi_2(u_0, y)$ for all y in H_2 . For each x define $\tilde{L}x = u_0$. Then \tilde{L} is linear and the uniqueness of u_0 implies that $\tilde{L} = L$. Hence $L^{-1} = T$ is unique.

THEOREM 3.11. *Let (H_i, Φ_i) be quadratic spaces and $B: H_1 \times H_2 \rightarrow K$ an orthocontinuous bilinear form such that $B_x(\cdot)$, $B^y(\cdot)$ are nontrivial. Then given any orthocontinuous linear functional ϕ on H_2 such that $\text{Ker } \phi$ splits H_2 there exists a unique vector u_0 in H_1*

such that

$$\phi(y) = B(u_0, y) \quad \text{for all } y \text{ in } H_2.$$

Proof. Since $\text{Ker } \phi$ splits H_2 we have by [3, Theorem 3.1] that there exists a unique vector y_0 in H_2 such that $\phi(y) = \Phi_2(y_0, y)$ for all y in H_2 . Let L be the orthocontinuous linear transformation associated with B as in 3.4. Then by 3.7, 3.8, and 3.9 L^{-1} exists and is orthocontinuous. Put $u_0 = L^{-1}y_0$. Then the uniqueness of y_0 and the fact that L is a bijection implies that y_0 is unique. Also

$$\begin{aligned} \phi(y) &= \Phi_2(y_0, y) \\ &= \Phi_2(LL^{-1}y_0, y) \\ &= \Phi_2(Lu_0, y) \\ &= B(u_0, y) \end{aligned}$$

for all y in H_2 .

In what follows we shall assume that the center of K contains an ordered domain P [1] and that

(i) for every a in K $aa^* = |a|^2$,

(ii) for any b in P^+ there exists an element c in P^+ such that $c^2 = b$. Put $c = b^{1/2}$.

Here P^+ denotes the set of positive elements of P . $|a|$ is as defined in [2]. We also assume that $\sup_{v \neq 0} |\Phi_2(u, v)| (|\Phi_2(v, v)|^{1/2})^{-1}$ exists for each u in H_2 and that it is equal to $|\Phi_2(u, u)|^{1/2}$.

THEOREM 3.12. *Let (H_i, Φ_i) , $i = 1, 2$ be quadratic spaces. With the above assumption on K , Φ_1, Φ_2 let B be an orthocontinuous bilinear form satisfying:*

$$3.12.1 \quad \sup_{x \in H_1} |B(x, y)| \text{ exists and is in } P^+ \text{ for all } y \neq 0.$$

$$3.12.2 \quad \inf_{x \neq 0} \sup_{y \neq 0} |B(x, y)| (|\Phi_1(x, x)|^{1/2} |\Phi_2(y, y)|^{1/2})^{-1} = \gamma$$

exists and $\gamma - \delta$ is in P^+

for some δ in P^+ .

Then given any orthocontinuous linear functional ϕ on H_2 such that $\text{Ker } \phi$ splits H_2 there exists a unique element x_0 in H_1 such that

$$\phi(y) = B(x_0, y) \quad \text{for all } y \text{ in } H_2.$$

Moreover

$$\delta^{-1} \sup_{y \neq 0} |\phi(y)| (|\Phi_2(y, y)|^{1/2})^{-1} - |\Phi_1(x_0, x_0)|^{1/2} \in P^+ \cup \{0\}.$$

Proof. By 3.12.1, 3.12.2 we have that $B_x(\cdot)$, $B^y(\cdot)$ are non-

trivial for each x in H_1 and y in H_2 . Hence by 3.11 there exists a unique vector x_0 in H_1 such that $\phi(y) = B(x_0, y)$ for all y in H_2 . By condition 3.12.2 we have

$$\begin{aligned} |\Phi_1(x_0, x_0)|^{1/2} &< \delta^{-1} \sup_{y \neq 0} |B(x_0, y)| (|\Phi_2(y, y)|^{1/2})^{-1} \\ &= \delta^{-1} \sup_{y \neq 0} |\phi(y)| (|\Phi_2(y, y)|^{1/2})^{-1}. \end{aligned}$$

REMARK 3. We note that a continuous linear functional on a pre-Hilbert space H , say, is orthocontinuous and if H is a Hilbert space then the kernel of a continuous functional, which is a closed subspace, is splitting. Also if H_1, H_2 are pre-Hilbert spaces with H_2 complete then a continuous bilinear form on $H_1 \times H_2$ is orthocontinuous. Hence the corollary that follows is an immediate consequence of the theorem.

COROLLARY 3.13. *Let H_1 be a pre-Hilbert space and H_2 be a Hilbert space (both real, complex or quaternion) and $B(\cdot, \cdot)$ a bilinear form on $H_1 \times H_2$ such that*

$$3.13.1 \quad |B(x, y)| \leq \delta' \|x\|_{H_1} \|y\|_{H_2}$$

for all x in H_1 and y in H_2 and for some positive real $\delta' < \infty$,

$$3.13.2 \quad \inf_{\|x\|_{H_1}=1} \sup_{\|y\|_{H_2} \leq 1} |B(x, y)| \geq \delta > 0,$$

$$3.13.3 \quad \sup_{x \in H_1} |B(x, y)| > 0 \quad \text{for all } y \neq 0.$$

Let ϕ be a bounded linear functional on H_2 . Then there exists a unique vector x_0 in H_1 such that

$$\phi(y) = B(x_0, y) \quad \text{for all } y \text{ in } H_2.$$

Moreover

$$\delta \|x_0\|_{H_1} = \|\phi\|.$$

REMARK 4. We note from the foregoing that the completeness of H_1 is not necessary for these results to hold unless $H_1 = H_2$; in which case we obtain the Lax-Milgram Theorem.

REFERENCES

1. B. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, Macmillan Company, New York, 1965.
2. R. Piziak, *An Algebraic Generalization of Hilbert Space Geometry*, Ph. D. Thesis, Univ. of Mass., 1969.

3. H. Piziak, *Sesquilinear forms in infinite dimensions*, Pacific J. Math., **43** (1972), 475-481.
4. K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

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Graham Donald Allen, David Alan Legg and Joseph Dinneen Ward, <i>Hermitian liftings in Orlicz sequence spaces</i>	379
George Bachman and Alan Sultan, <i>On regular extensions of measures</i>	389
Bruce Alan Barnes, <i>Representations Naimark-related to *-representations; a correction: "When is a representation of a Banach *-algebra Naimark-related to a *-representation?"</i>	397
Earl Robert Berkson, <i>One-parameter semigroups of isometries into H^p</i>	403
M. Brodmann, <i>Piecewise catenarian and going between rings</i>	415
Joe Peter Buhler, <i>A note on tamely ramified polynomials</i>	421
William Lee Bynum, <i>Normal structure coefficients for Banach spaces</i>	427
Lung O. Chung, <i>Biharmonic and polyharmonic principal functions</i>	437
Vladimir Drobot and S. McDonald, <i>Approximation properties of polynomials with bounded integer coefficients</i>	447
Giora Dula and Elyahu Katz, <i>Recursion formulas for the homology of $\Omega(X \vee Y)$</i>	451
John A. Ernest, <i>The computation of the generalized spectrum of certain Toeplitz operators</i>	463
Kenneth R. Goodearl and Thomas Benny Rushing, <i>Direct limit groups and the Keesling-Mardešić shape fibration</i>	471
Raymond Heitmann and Stephen Joseph McAdam, <i>Good chains with bad contractions</i>	477
Patricia Jones and Steve Chong Hong Ligh, <i>Finite hereditary near-ring-semigroups</i>	491
Yoshikazu Katayama, <i>Isomorphisms of the Fourier algebras in crossed products</i>	505
Meir Katchalski and Andrew Chiang-Fung Liu, <i>Symmetric twins and common transversals</i>	513
Mohammad Ahmad Khan, <i>Chain conditions on subgroups of LCA groups</i>	517
Helmut Kröger, <i>Padé approximants on Banach space operator equations</i>	535
Gabriel Michael Miller Obi, <i>An algebraic extension of the Lax-Milgram theorem</i>	543
S. G. Pandit, <i>Differential systems with impulsive perturbations</i>	553
Richard Pell, <i>Support point functions and the Loewner variation</i>	561
J. Hyam Rubinstein, <i>Dehn's lemma and handle decompositions of some 4-manifolds</i>	565
James Eugene Shirey, <i>On the theorem of Helley concerning finite-dimensional subspaces of a dual space</i>	571
Oved Shisha, <i>Tchebycheff systems and best partial bases</i>	579
Michel Smith, <i>Large indecomposable continua with only one composant</i>	593
Stephen Tefteller, <i>Existence of eigenvalues for second-order differential systems</i>	601