

# Pacific Journal of Mathematics

## **SUPPORT POINT FUNCTIONS AND THE LOEWNER VARIATION**

RICHARD PELL

## SUPPORT POINT FUNCTIONS AND THE LOEWNER VARIATION

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1. **Introduction.** Let  $U = \{z: |z| < 1\}$  and  $\mathcal{S}$  the set of functions  $f, f(z) = z + a_2z^2 + \dots$ , that are analytic and 1:1 in  $U$ . Denote by  $\sigma$  the collection of support point functions of  $\mathcal{S}$ , i.e., functions  $f \in \mathcal{S}$  that satisfy

$$\operatorname{Re} L(f) = \max_{g \in \mathcal{S}} \operatorname{Re} L(g)$$

for some nonconstant continuous (in the topology of local uniform convergence) linear functional on  $\mathcal{S}$ . Finally, denote by  $E(\mathcal{S})$  the set of extreme point functions of  $\mathcal{S}$ .

It is well known that if  $f \in \sigma \cup E(\mathcal{S})$ , then  $f(U)$  is the complement of a single Jordan arc extending from some finite point to  $\infty$  and along which  $|w|$  is strictly increasing. Indeed, this has been demonstrated for the class  $E(\mathcal{S})$  by L. Brickman [1] and for the class  $\sigma$  by A. Pfluger [5] (see also L. Brickman and D. Wilken [2]). Consequently, if  $f \in \sigma \cup E(\mathcal{S})$ , there is a Loewner chain

$$f(z, t) = e^t \left[ z + \sum_{n=2}^{\infty} a_n(t) z^n \right] \quad (0 \leq t < \infty)$$

with  $f(z, 0) = f(z)$  and  $f(z, t_1)$  subordinate to  $f(z, t_2)$  if  $0 \leq t_1 < t_2 < \infty$  (see [6, p. 157]). Note that  $e^{-t}f(z, t) \in \mathcal{S}$ . Let  $w(z, t) = e^{-t}(z + \hat{b}_2(t)z^2 + \hat{b}_3(t)z^3 + \dots)$  be analytic for  $t \in \{t: 0 \leq t < \infty\}$  and  $z \in U$ , 1:1 in  $U$  with  $|w(z, t)| < 1$ , and such that  $f(z) = f(w(z, t), t)$  for each  $t \in \{t: 0 \leq t < \infty\}$  and all  $z \in U$ . Observe that we define  $\hat{w}(z, t) \equiv e^t w(z, t) = z + \hat{b}_2(t)z^2 + \dots \in \mathcal{S}$  and that  $|\hat{w}(z, t)| < e^t$  for  $z \in U$ .

In §2 it is shown that if  $f \in E(\mathcal{S})$ , then  $e^{-t}f(z, t) \in E(\mathcal{S})$  and also that if  $f \in \sigma$ , then  $e^{-t}f(z, t) \in \sigma$ . This latter result is a generalization of a theorem due to S. Friedland and M. Schiffer [3, p. 143]. Also, in the process of generalizing this theorem a fairly easy method is established for finding for each  $t, 0 \leq t < \infty$ , a continuous linear functional which  $e^{-t}f(z, t)$  maximizes.

2. **Preservation of the sets  $\sigma$  and  $E(\mathcal{S})$  under the Loewner variation.** It is easy to show that if  $f \in E(\mathcal{S})$ , then  $e^{-t}f(z, t) \in E(\mathcal{S})$  also. Indeed, if this were not the case, then there would exist distinct functions  $f_1, f_2 \in \mathcal{S}$  and  $\lambda_1, \lambda_2 > 0$  with  $\lambda_1 + \lambda_2 = 1$  for which  $\lambda_1 f_1(z) + \lambda_2 f_2(z) = e^{-t}f(z, t)$ . This would imply that  $e^t \lambda_1 f_1(w(z, t)) + e^t \lambda_2 f_2(w(z, t)) = f(w(z, t), t) = f(z)$ . Since  $e^t f_1(w(z, t))$  and  $e^t f_2(w(z, t))$  are in  $\mathcal{S}$ , the fact that  $f(z) \in E(\mathcal{S})$  is contradicted and therefore

$e^{-t}f(z, t) \in E(\mathcal{S})(0 \leq t < \infty)$ .

The following theorem contains the analogous result for the class  $\sigma$ .

**THEOREM.** *Let  $f \in \sigma \subset \mathcal{S}$ . Then  $e^{-t}f(z, t) \in \sigma$  for all  $t$  such that  $0 \leq t < \infty$ .*

*Proof.* Since  $f \in \sigma$ , there exists a nonconstant continuous linear functional,  $L$ , for which

$$\operatorname{Re} L(f) = \max_{g \in \mathcal{S}} \operatorname{Re} L(g).$$

At this point we need a representation theorem due to O. Toeplitz [7].

**THEOREM (Toeplitz).** *Let  $f(z) = z + a_2z^2 + \dots \in \mathcal{S}$ . Then  $L(f)$  is a continuous linear functional on  $\mathcal{S}$  if and only if there exists a sequence  $\{b_n\}$  with  $\limsup_{n \rightarrow \infty} |b_n|^{1/n} < 1$  such that  $L(f) = \sum_{n=1}^{\infty} a_n b_n$ .*

Now,  $f(z) = f(w(z, t), t)$  where  $e^t w(z, t) = \hat{w}(z, t) = z + \hat{b}_2(t)z^2 + \dots \in \mathcal{S}$  and  $|\hat{w}(z, t)| < e^t$  for  $z \in U$ . Since

$$\begin{aligned} f(w(z, t), t) &= e^t [w(z, t) + a_2(t)w^2(z, t) + \dots + a_n(t)w^n(z, t) + \dots] \\ &= \hat{w}(z, t) + a_2(t)e^{-t}\hat{w}^2(z, t) + \dots \\ &\quad + a_n(t)e^{-(n-1)t}\hat{w}^n(z, t) + \dots, \end{aligned}$$

and if  $L(f) = \sum_{n=1}^{\infty} a_n b_n$ , then it follows that

$$\begin{aligned} \sum_{n=1}^{\infty} a_n b_n &= \sum_{n=1}^{\infty} [\hat{b}_n^{(1)} + a_2(t)e^{-t}\hat{b}_n^{(2)} + a_3(t)e^{-2t}\hat{b}_n^{(3)} + \dots \\ &\quad + a_n(t)e^{-(n-1)t}\hat{b}_n^{(n)}] b_n \\ &= \sum_{n=1}^{\infty} \left[ \sum_{k=1}^n a_k(t)e^{-(k-1)t}\hat{b}_n^{(k)} b_n \right] \end{aligned}$$

where  $\hat{b}_n^{(k)}$  is the  $n$ th coefficient of  $\hat{w}^k(z, t) = [z + \hat{b}_2(t)z^2 + \dots]^k$ . However, since  $\hat{w}^k(z, t)$  is analytic in  $U$  and bounded by  $e^{kt}$ , it follows from Cauchy's formula that

$$\begin{aligned} |\hat{b}_n^{(k)}| &= \left| \frac{1}{2\pi i} \int_{|\varepsilon|=1} \frac{\hat{w}^k(\varepsilon, t) d\varepsilon}{\varepsilon^{n+1}} \right| = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\hat{w}^k(e^{i\theta}, t)}{e^{in\theta}} d\theta \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |\hat{w}^k(e^{i\theta}, t)| d\theta \leq e^{kt} \end{aligned}$$

for all  $n = 1, 2, \dots$ . Also, since  $e^{-t}f(z, t) = z + a_2(t)z^2 + \dots \in \mathcal{S}$ , it follows from Littlewood's theorem [4] that  $|a_k(t)| \leq ke$ . Therefore,

$$\begin{aligned} \sum_{k=1}^n |\alpha_k(t) e^{-(k-1)t} \widehat{b}_n^{(k)} b_n| &\leq \sum_{k=1}^n |k e^{-k t} \cdot e^{k t} \cdot b_n| \\ &= e^{(t+1)} |b_n| \left( \frac{n(n+1)}{2} \right). \end{aligned}$$

Notice also that  $\limsup_{n \rightarrow \infty} |e^{(t+1)} \cdot b_n \cdot n(n+1)/2|^{1/n} = \limsup_{n \rightarrow \infty} |b_n|^{1/n} < 1$ . Consequently, the double summation,  $\sum_{n=1}^{\infty} [\sum_{k=1}^n \alpha_k(t) e^{-(k-1)t} \widehat{b}_n^{(k)} b_n]$ , converges absolutely and therefore the order of summation can be reversed and one obtains

$$\begin{aligned} \sum_{n=1}^{\infty} \alpha_n b_n &= \sum_{n=1}^{\infty} \left[ \sum_{k=1}^n \alpha_k(t) e^{-(k-1)t} \widehat{b}_n^{(k)} b_n \right] \\ &= \sum_{k=1}^{\infty} \left[ \sum_{n=k}^{\infty} \alpha_k(t) e^{-(k-1)t} \widehat{b}_n^{(k)} b_n \right] \\ &= \sum_{k=1}^{\infty} \left[ \sum_{n=k}^{\infty} \widehat{b}_n^{(k)} b_n e^{-(k-1)t} \right] \alpha_k(t). \end{aligned}$$

Now, for  $f \in \mathcal{S}$  define  $L_t(f) \equiv \sum_{k=1}^{\infty} (\sum_{n=k}^{\infty} \widehat{b}_n^{(k)} b_n e^{-(k-1)t}) \alpha_k$ . From the theorem of Toeplitz it follows that  $L_t$  will be a continuous linear functional on  $\mathcal{S}$  provided that

$$\limsup_{k \rightarrow \infty} \left| \sum_{n=k}^{\infty} \widehat{b}_n^{(k)} b_n e^{-(k-1)t} \right|^{1/k} < 1.$$

Since  $\limsup_{k \rightarrow \infty} |b_k|^{1/k} = \rho < 1$ , there exists an  $N$  and an  $r$  such that  $\rho < r < 1$  and  $|b_k| \leq r^k$  for all  $k \geq N$ . Therefore,  $|\sum_{n=k}^{\infty} \widehat{b}_n^{(k)} b_n e^{-(k-1)t}|^{1/k} \leq (e^{kt} \cdot e^{-(k-1)t} \sum_{n=k}^{\infty} r^n)^{1/k} = e^{t/k} r / (1-r)^{1/k}$  for all  $k \geq N$ . Since

$$\limsup_{k \rightarrow \infty} \left[ e^{t/k} \frac{r}{(1-r)^{1/k}} \right] = r < 1,$$

it follows that  $\limsup_{k \rightarrow \infty} |\sum_{n=k}^{\infty} \widehat{b}_n^{(k)} b_n e^{-(k-1)t}|^{1/k} \leq r < 1$ .

Since  $\operatorname{Re} L(f) = \operatorname{Re} (\sum_{n=1}^{\infty} \alpha_n b_n)$  is a maximum for the class  $\mathcal{S}$ , it follows easily that  $\operatorname{Re} L_t(e^{-t} f(z, t))$  is also a maximum for the class  $\mathcal{S}$ . In order to see this one needs only to observe that if  $f$  and  $\widehat{f}$  are any two functions in  $\mathcal{S}$  related by a relation of the form  $f(z) = e^t \widehat{f}(w(z, t))$ , then  $L(f) = L_t(\widehat{f})$ . This completes the proof of the theorem.

REMARKS. Since  $f(z) \equiv f(w(z, t), t)$  for some  $w(z, t)$ , one can express  $L_t(e^{-t} f(z, t)) = \sum_{k=1}^{\infty} (\sum_{n=k}^{\infty} \widehat{b}_n^{(k)} b_n e^{-(k-1)t}) \alpha_k(t)$  in terms of the coefficients of the functions  $f(z)$  and  $e^{-t} f(z, t)$ . This can easily be done provided that  $L(f)$  ( $L(f) = \sum_{n=1}^{\infty} \alpha_n b_n$ ) does not contain too many terms. Then for each  $t$ ,  $0 < t < \infty$ , the corresponding Schiffer differential equation which  $e^{-t} f(z, t)$  must satisfy can then be computed with little difficulty. Unfortunately, extracting useful information from these new equations is not an easy task.

Suppose, however, that it is known that  $\operatorname{Re} L(f)$  is a maximum for the class  $\mathcal{S}$  when  $f$  is one of the Koebe functions,  $f(z) = z/(1 - e^{i\theta}z)^2$  ( $0 \leq \theta < 2\pi$ ). Then since  $e^{-t}f(z, t) = f(z)$  in this case, it follows that  $\operatorname{Re} L_t(f)$  is a maximum for the class  $\mathcal{S}$  for all  $t$  ( $0 \leq t < \infty$ ). From this one can establish a one parameter family of new coefficient inequalities for the class  $\mathcal{S}$ . S. Friedland and M. Schiffer [3, p. 149] have done this for the case where  $L(f) = a_1$ .

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