SUPPORT POINT FUNCTIONS AND THE LOEWNER VARIATION

Richard Pell
1. Introduction. Let $U = \{z: |z| < 1\}$ and $\mathcal{S}$ the set of functions $f, f(z) = z + a_2z^2 + \cdots$, that are analytic and 1:1 in $U$. Denote by $\sigma$ the collection of support point functions of $\mathcal{S}$, i.e., functions $f \in \mathcal{S}$ that satisfy

$$\text{Re } L(f) = \max_{g \in \sigma} \text{Re } L(g)$$

for some nonconstant continuous (in the topology of local uniform convergence) linear functional on $\mathcal{S}$. Finally, denote by $E(\mathcal{S})$ the set of extreme point functions of $\mathcal{S}$.

It is well known that if $f \in \sigma \cup E(\mathcal{S})$, then $f(U)$ is the complement of a single Jordan arc extending from some finite point to $\infty$ and along which $|w|$ is strictly increasing. Indeed, this has been demonstrated for the class $E(\mathcal{S})$ by L. Brickman [1] and for the class $\sigma$ by A. Pfluger [5] (see also L. Brickman and D. Wilken [2]). Consequently, if $f \in \sigma \cup E(\mathcal{S})$, there is a Loewner chain

$$f(z, t) = e^{-t}[z + \sum_{n=2}^{\infty} a_n(t)z^n] \quad (0 \leq t < \infty)$$

with $f(z, 0) = f(z)$ and $f(z, t_1)$ subordinate to $f(z, t_2)$ if $0 \leq t_1 < t_2 < \infty$ (see [6, p. 157]). Note that $e^{-t}f(z, t) \in \mathcal{S}$. Let $w(z, t) = e^{-t}[z + \hat{b}_2(t)z^2 + \hat{b}_3(t)z^3 + \cdots]$ be analytic for $t \in \{t: 0 \leq t < \infty\}$ and $z \in U$, 1:1 in $U$ with $|w(z, t)| < 1$, and such that $f(z) = f(w(z, t), t)$ for each $t \in \{t: 0 \leq t < \infty\}$ and all $z \in U$. Observe that we define $\hat{w}(z, t) = e^t w(z, t) = z + \hat{b}_2(t)z^2 + \cdots \in \mathcal{S}$ and that $|\hat{w}(z, t)| < e^t$ for $z \in U$.

In §2 it is shown that if $f \in E(\mathcal{S})$, then $e^{-t}f(z, t) \in E(\mathcal{S})$ and also that if $f \in \sigma$, then $e^{-t}f(z, t) \in \sigma$. This latter result is a generalization of a theorem due to S. Friedland and M. Schiffer [3, p. 143]. Also, in the process of generalizing this theorem a fairly easy method is established for finding for each $t, 0 \leq t < \infty$, a continuous linear functional which $e^{-t}f(z, t)$ maximizes.

2. Preservation of the sets $\sigma$ and $E(\mathcal{S})$ under the Loewner variation. It is easy to show that if $f \in E(\mathcal{S})$, then $e^{-t}f(z, t) \in E(\mathcal{S})$ also. Indeed, if this were not the case, then there would exist distinct functions $f_1, f_2 \in \mathcal{S}$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$ for which $\lambda_1 f_1(z) + \lambda_2 f_2(z) = e^{-t}f(z, t)$. This would imply that $e^{\lambda_1}f_1(w(z, t)) + e^{\lambda_2}f_2(w(z, t)) = f(w(z, t), t) = f(z)$. Since $e^t f_1(w(z, t))$ and $e^t f_2(w(z, t))$ are in $\mathcal{S}$, the fact that $f(z) \in E(\mathcal{S})$ is contradicted and therefore
Theorem. Let \( f \in \sigma \subset \mathcal{H} \). Then \( e^{-t}f(z, t) \in \sigma \) for all \( t \) such that \( 0 \leq t < \infty \).

Proof. Since \( f \in \sigma \), there exists a nonconstant continuous linear functional, \( L \), for which
\[
\text{Re } L(f) = \max_{g \in \mathcal{H}} \text{Re } L(g) .
\]

At this point we need a representation theorem due to O. Toeplitz [7].

Theorem (Toeplitz). Let \( f(z) = z + a_2 z^2 + \cdots \in \mathcal{H} \). Then \( L(f) \)
is a continuous linear functional on \( \mathcal{H} \) if and only if there exists
a sequence \( \{b_n\} \) with \( \limsup_{n \to \infty} |b_n|^{1/n} < 1 \) such that
\[
L(f) = \sum_{n=1}^{\infty} a_n b_n.
\]

Now, \( f(z) = f(w(z, t), t) \) where \( e^t w(z, t) = \hat{w}(z, t) = z + \hat{b}_2(t) z^2 + \cdots \in \mathcal{H} \) and \( |\hat{w}(z, t)| < e^t \) for \( z \in U \). Since
\[
f(w(z, t), t) = e^t [w(z, t) + a_2(t) w^2(z, t) + \cdots + a_n(t) w^n(z, t) + \cdots]
= \hat{w}(z, t) + a_2(t) e^{-t} \hat{w}^2(z, t) + \cdots
+ a_n(t) e^{-(n-1)t} \hat{w}^n(z, t) + \cdots,
\]
and if \( L(f) = \sum_{n=1}^{\infty} a_n b_n \), then it follows that
\[
\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} [\hat{b}_n^{(1)} + a_2(t) e^{-t} \hat{b}_n^{(2)} + a_3(t) e^{-2t} \hat{b}_n^{(3)} + \cdots]
+ a_n(t) e^{-(n-1)t} \hat{b}_n^{(n)}] b_n
= \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{n} a_k(t) e^{-(k-1)t} \hat{b}_n^{(k)} b_n \right]
\]
where \( \hat{b}_n^{(k)} \) is the \( n \)th coefficient of \( \hat{w}^k(z, t) = [z + \hat{b}_2(t) z^2 + \cdots]^k \).
However, since \( \hat{w}^k(z, t) \) is analytic in \( U \) and bounded by \( e^{kt} \), it follows from Cauchy's formula that
\[
|\hat{b}_n^{(k)}| = \left| \frac{1}{2\pi i} \int_{|z|=1} \frac{\hat{w}^k(z, t) dz}{z^{n+1}} \right| = \left| \frac{1}{2\pi} \int_0^{2\pi} \hat{w}^k(e^{i\theta}, t) d\theta \right|
\leq \frac{1}{2\pi} \int_0^{2\pi} |\hat{w}^k(e^{i\theta}, t)| d\theta \leq e^{kt}
\]
for all \( n = 1, 2, \cdots \). Also, since \( e^{-t}f(z, t) = z + a_2(t) z^2 + \cdots \in \mathcal{H} \),
it follows from Littlewood's theorem [4] that \( |a_k(t)| \leq ke \). Therefore,
\begin{align*}
\sum_{k=1}^{n} |a_k(t)e^{-(k-1)t} \hat{b}_n^{(k)} b_n| & \leq \sum_{k=1}^{n} |ke \cdot e^{-(k-1)t} \cdot e^{kt} \cdot b_n| \\
& = e^{(t+1)} |b_n| \left( \frac{n(n+1)}{2} \right).
\end{align*}

Notice also that \( \limsup_{n \to \infty} |e^{(t+1)} \cdot b_n \cdot n(n+1)/2|^{1/n} = \limsup_{n \to \infty} |b_n|^{1/n} < 1 \). Consequently, the double summation, \( \sum_{n=1}^{\infty} [\sum_{k=1}^{n} a_k(t)e^{-(k-1)t} \hat{b}_n^{(k)} b_n] \), converges absolutely and therefore the order of summation can be reversed and one obtains

\begin{align*}
\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{n} a_k(t)e^{-(k-1)t} \hat{b}_n^{(k)} b_n \right] \\
= \sum_{k=1}^{\infty} \left[ \sum_{n=1}^{\infty} a_k(t)e^{-(k-1)t} \hat{b}_n^{(k)} b_n \right] \\
= \sum_{k=1}^{\infty} \left[ \sum_{n=1}^{\infty} \hat{b}_n^{(k)} b_n e^{-(k-1)t} \right] a_k(t).
\end{align*}

Now, for \( f \in \mathcal{S} \) define \( L_t(f) = \sum_{n=1}^{\infty} (\sum_{n=1}^{\infty} \hat{b}_n^{(k)} b_n e^{-(k-1)t}) a_k \). From the theorem of Toeplitz it follows that \( L_t \) will be a continuous linear functional on \( \mathcal{S} \) provided that

\[ \limsup_{k \to \infty} \left| \sum_{n=1}^{\infty} \hat{b}_n^{(k)} b_n e^{-(k-1)t} \right|^{1/k} < 1. \]

Since \( \limsup_{k \to \infty} |b_k|^{1/k} = \rho < 1 \), there exists an \( N \) and an \( r \) such that \( \rho < r < 1 \) and \( |b_k| \leq r^k \) for all \( k \geq N \). Therefore, \( |\sum_{n=k}^{\infty} \hat{b}_n^{(k)} b_n e^{-(k-1)t}|^{1/k} \leq e^{kt} e^{-(k-1)t} \sum_{n=k}^{\infty} \tau^n 1/k = e^{t/k} \tau/(1 - \tau)^{1/k} \) for all \( k \geq N \). Since

\[ \limsup_{k \to \infty} \left[ e^{t/k} \frac{\tau}{(1 - \tau)^{1/k}} \right] = \tau < 1, \]

it follows that \( \limsup_{k \to \infty} |\sum_{n=k}^{\infty} \hat{b}_n^{(k)} b_n e^{-(k-1)t}|^{1/k} \leq \tau < 1. \)

Since \( \text{Re } L(f) = \text{Re } (\sum_{n=1}^{\infty} a_n b_n) \) is a maximum for the class \( \mathcal{S} \), it follows easily that \( \text{Re } L_t(e^{-t}f(z, t)) \) is also a maximum for the class \( \mathcal{S} \). In order to see this one needs only to observe that if \( f \) and \( \hat{f} \) are any two functions in \( \mathcal{S} \) related by a relation of the form \( f(z) = e^t \hat{f}(w(z, t)) \), then \( L(f) = L_t(\hat{f}) \). This completes the proof of the theorem.

**REMARKS.** Since \( f(z) = f(w(z, t), t) \) for some \( w(z, t) \), one can express \( L_t(e^{-t}f(z, t)) = \sum_{n=1}^{\infty} (\sum_{n=k}^{\infty} \hat{b}_n^{(k)} b_n e^{-(k-1)t}) a_k(t) \) in terms of the coefficients of the functions \( f(z) \) and \( e^{-t}f(z, t) \). This can easily be done provided that \( L(f) \) \( (L(f) = \sum_{n=1}^{\infty} a_n b_n) \) does not contain too many terms. Then for each \( t, 0 < t < \infty \), the corresponding Schiffer differential equation which \( e^{-t}f(z, t) \) must satisfy can then be computed with little difficulty. Unfortunately, extracting useful information from these new equations is not an easy task.
Suppose, however, that it is known that \( \Re L(f) \) is a maximum for the class \( \mathscr{S} \) when \( f \) is one of the Koebe functions, \( f(z) = z/(1 - e^{i\theta}z) \quad (0 \leq \theta < 2\pi) \). Then since \( e^{-t}f(z, t) = f(z) \) in this case, it follows that \( \Re L_t(f) \) is a maximum for the class \( \mathscr{S} \) for all \( t \) \( (0 \leq t < \infty) \). From this one can establish a one parameter family of new coefficient inequalities for the class \( \mathscr{S} \). S. Friedland and M. Schiffer [3, p. 149] have done this for the case where \( L(f) = \alpha \).

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