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## **DEHN'S LEMMA AND HANDLE DECOMPOSITIONS OF SOME 4-MANIFOLDS**

J. HYAM RUBINSTEIN

## DEHN'S LEMMA AND HANDLE DECOMPOSITIONS OF SOME 4-MANIFOLDS

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**We give two short proof of a weak version of the theorem of Laudenbach, Poenaru [3]. Also we show that an embedded  $S^1 \times S^2$  in  $S^4$  bounds a copy of  $B^2 \times S^2$ . Finally we establish that if  $W$  is a smooth 4-manifold with  $\partial W = \#_n S^1 \times S^2$  and  $W$  is built from  $\#_{n-1} B^2 \times S^2$  by attaching a 2-handle, then  $W$  is homeomorphic to  $\#_n B^2 \times S^2$ .**

1. **4-Dimensional handlebodies.** Let  $X, Y$  be the following smooth 4-manifolds:

$$X = \#_n B^3 \times S^1 \quad \text{and} \quad Y = \#_n B^2 \times S^2 .$$

In [3] it is proved that if  $h: \partial X \rightarrow \partial Y$  is a diffeomorphism, then the smooth closed 4-manifold  $X \mathbf{U}_h Y$  which is obtained by gluing along  $h$ , is diffeomorphic to  $S^4$ .

We begin with two brief proofs, one using the Dehn's lemma in [5] and the other employing unknotting in codimension 3, of the following result:

**THEOREM.** *Let  $X, Y, h$  be as above. Then  $X \mathbf{U}_h Y$  is homeomorphic to  $S^4$ .*

*Proof.* (1) Let  $\{x_i\} \times S^1$  be a circle in the boundary of the  $i$ th copy of  $B^3 \times S^1$  in the connected sum  $X = \#_n B^3 \times S^1$ , for  $1 \leq i \leq n$ . Without loss of generality, all the loops  $\{x_i\} \times S^1$  can be assumed to miss the cells which are used to construct  $X$  as a connected sum. By the Dehn's lemma in [5], it follows that all of the circles  $h(\{x_i\} \times S^1)$  bound disjoint smooth embedded disks  $D_i$  in  $Y$ , for  $1 \leq i \leq n$ .

Let  $N(D_i)$  denote a small tubular neighborhood of  $D_i$  in  $Y$ . Clearly  $X \mathbf{U}_h (N(D_1) \cup \dots \cup N(D_n))$  is diffeomorphic to  $B^4$ , since  $N(D_i)$  can be thought of as a 2-handle which geometrically cancels a 1-handle of  $X$ . On the other hand, let  $W$  denote the closure of  $Y - N(D_1) - \dots - N(D_n)$ . Then  $\partial W = S^3$  and  $W$  is contained in  $Y$  which can be embedded in  $S^4$ . By the topological Schoenflies theorem [1],  $W$  is homeomorphic to  $B^4$ . Consequently  $X \mathbf{U}_h Y$  is homeomorphic to  $B^4 \cup B^4 = S^4$ .

(2) By Van Kampen's theorem,  $\pi_1(X \mathbf{U}_h Y) = \{1\}$ . Let  $Z$  be a bouquet of  $n$  circles which is embedded in  $X$  and is a deformation retract of  $X$ . By isotopic unknotting in codimension 3,  $Z$  is con-

tained in the interior of a PL 4-cell  $B$  in  $X \mathbf{U}_h Y$ . Therefore, by an isotopy we can shrink  $X$  down towards  $Z$  until  $X$  is included in  $\text{int } B$ . Exactly as in (1), by the topological Schoenflies theorem we obtain that  $X \mathbf{U}_h Y - \text{int } B$  is homeomorphic to  $B^4$  and so the result follows.

**REMARK.** Note that if the PL or smooth 4-dimensional Schoenflies theorem was known, then these arguments would establish that  $X \mathbf{U}_h Y$  is PL isomorphic or diffeomorphic to  $S^4$ .

**2. Embeddings of  $S^1 \times S^2$  in  $S^4$ .** The following result was first proved by I. Aitchison (unpublished). We present a simplification of his method, which again uses the Dehn's lemma in [5].

**THEOREM.** *Let  $h: S^1 \times S^2 \rightarrow S^4$  be a smooth embedding. Then  $h$  extends to a topological embedding of  $B^2 \times S^2$  in  $S^4$ .*

*Proof.* Let  $V, W$  be the closures of the components of  $S^4 - h(S^1 \times S^2)$  (by Alexander duality there are two such components). By the Mayer-Vietoris sequence, without loss of generality the inclusion  $h(S^1 \times S^2) \rightarrow V$  induces an isomorphism  $H_1(h(S^1 \times S^2)) \rightarrow H_1(V)$  and  $H_1(W) = 0$ .

Let  $G$  denote the group which is the pushout of the homomorphisms  $\pi_1(h(S^1 \times S^2)) \rightarrow \pi_1(V)$  and  $\pi_1(h(S^1 \times S^2)) \rightarrow \pi_1(W)$ . By Van Kampen's theorem,  $G = \{1\}$ . On the other hand there is a homomorphism of  $G$  onto  $\pi_1(W)$  induced by the epimorphism  $\pi_1(V) \rightarrow H_1(V) \cong H_1(h(S^1 \times S^2)) \cong \pi_1(h(S^1 \times S^2))$ . Consequently  $\pi_1(W) = \{1\}$  follows.

Now we can apply the Dehn's lemma in [5] to obtain that  $h(S^1 \times *)$  bounds a smooth embedded disk  $D$  in  $W$ . Let  $N(D)$  be a small tubular neighborhood of  $D$  in  $W$ . Then the closure of  $W - N(D)$  is a topological 4-cell, by the topological Schoenflies theorem [1]. Therefore  $W$  is homeomorphic to  $B^2 \times S^2$  and  $h$  extends to a topological embedding of  $B^2 \times S^2$  as desired.

**REMARK.** This result is analogous to the classical theorem of Alexander that any smooth embedded  $S^1 \times S^1$  in  $S^3$  bounds a smooth solid torus  $B^2 \times S^1$ .

**3. Handle decompositions and slice links.** In [2], Kirby, Melvin proved that if a smooth 4-manifold  $M$  has boundary  $S^1 \times S^2$  and is constructed by attaching a 2-handle to  $B^4$  along a curve  $C$  with the 0-framing, then  $M$  is homeomorphic to  $B^2 \times S^2$  and  $C$  is a slice knot. We prove the following generalization of their result:

**THEOREM.** *Let  $W$  be a smooth 4-manifold which is obtained by adding  $n$  2-handles to  $B^4$  along the curves  $C_1, \dots, C_n$ . The 2-handles induce a framing of the link  $C_1 \cup \dots \cup C_n$ . Assume that framed surgery on the sublink  $C_1 \cup \dots \cup C_i$  in  $S^3$  yields  $\#_i S^1 \times S^2$ , for all  $i$  with  $1 \leq i \leq n$ . Then  $W$  is homeomorphic to  $\#_n B^2 \times S^2$  and  $C_1 \cup \dots \cup C_n$  is a slice link.*

**COROLLARY.** *Let  $W$  be a smooth 4-manifold such that  $\partial W$  is diffeomorphic to  $\#_n S^1 \times S^2$  and  $W$  is built by attaching a 2-handle to  $\#_{n-1} B^2 \times S^2$ . Then  $W$  is homeomorphic to  $\#_n B^2 \times S^2$ .*

*Proof of theorem.* By the assumption that surgery on the link  $C_1 \cup \dots \cup C_n$  gives  $\#_n S^1 \times S^2$ , it immediately follows that  $\partial W$  is diffeomorphic to  $\#_n S^1 \times S^2$ . If the handle decomposition of  $W$  is turned upside down, then  $W$  is constructed by attaching  $n$  2-handles to  $(\#_n S^1 \times S^2) \times I$  along some curves  $C'_1 \times \{1\}, C'_2 \times \{1\}, \dots, C'_n \times \{1\}$  and then adding a 4-handle. We will assume that the 2-handle glued along  $C'_i \times \{1\}$  is dual to the 2-handle added along  $C_i$  to  $B^4$ .

Let  $W_i$  or  $W'_i$  denote the 4-manifold which is obtained by adjoining  $i$  2-handles to  $B^4$  or  $(\#_n S^1 \times S^2) \times I$  respectively along the curves  $C_1, \dots, C_i$  or  $C'_{n-i+1} \times \{1\}, \dots, C'_n \times \{1\}$  respectively. Then  $\partial W_i$  is diffeomorphic to  $\#_i S^1 \times S^2$ , since surgery on  $C_1 \cup \dots \cup C_i$  gives  $\#_i S^1 \times S^2$ . Also  $W - \text{int } W'_i$  is diffeomorphic to  $W_{n-i}$  and therefore  $W'_i$  is a cobordism between  $\#_n S^1 \times S^2$  and  $\#_{n-i} S^1 \times S^2$ . Note that  $W'_i$  can also be constructed by adding  $n-i$  2-handles to  $(\#_{n-i} S^1 \times S^2) \times I$ .

Let  $\{C\}$  denote the homotopy class of a loop  $C$  relative to some base point and let  $\langle * \rangle$  denote the normal closure of the set of elements  $*$  in some group. By Van Kampen's theorem applied to the two handle decompositions of  $W'_i$ , we conclude that

$$\pi_1(W'_i) \cong \pi_1(\#_n S^1 \times S^2) / \langle \{C'_{n-i+1}\}, \dots, \{C'_n\} \rangle$$

and  $\pi_1(W'_i)$  has rank  $\leq n - i$ . Consider the case when  $i = 1$ . By a classical theorem of Whitehead (see Exercise 20 on p. 283 of [4]) and by Corollary 5.14.2 on p. 354 of [4], it follows that  $\pi_1(W'_1)$  is free and  $\{C'_n\}$  is primitive, i.e., is contained in a free basis of the free group  $\pi_1(\#_n S^1 \times S^2)$ .

Next,  $\pi_1(W'_2)$  has a presentation consisting of a set of free generators of  $\pi_1(W'_1) \cong \pi_1(\#_n S^1 \times S^2) / \langle \{C'_n\} \rangle$  and the one relation  $\{C'_{n-1}\}$ . Hence by the results on p. 283 and p. 354 of [4] again,  $\pi_1(W'_2)$  is free and  $\{C'_{n-1}\}$  is primitive. Therefore we obtain that  $\{\{C'_{n-1}\}, \{C'_n\}\}$  is contained in a free basis for  $\pi_1(\#_n S^1 \times S^2)$ . Continuing on with this argument, we conclude that  $\{\{C'_1\}, \dots, \{C'_n\}\}$  is a free basis of  $\pi_1(\#_n S^1 \times S^2)$ . So by Lemma 2 of [3], there is a diffeomorphism  $h: \#_n S^1 \times S^2 \rightarrow \#_n S^1 \times S^2$  such that  $h(S^1 \times \{x_i\})$  is homotopic to  $C'_i$  for

all  $i$ ,  $1 \leq i \leq n$ , where  $S^1 \times \{x_i\}$  is contained in the  $i$ th copy of  $S^1 \times S^2$  used to form  $\#_n S^1 \times S^2$  and is disjoint from the 3-cells employed for the connected sum.

Let  $M$  be the smooth 4-manifold with  $\partial M = S^3$  which is built by adding  $n$  3-handles and 4-handles to  $W'_n$ , using the component  $(\#_n S^1 \times S^2) \times \{0\}$  of  $\partial W'_n$ . The 3-handles can be attached along the 2-spheres  $h(\{y_i\} \times S^2) \times \{0\}$ , for  $1 \leq i \leq n$ , where  $\{y_i\} \times S^2$  is in the  $i$ th copy of  $S^1 \times S^2$  used to obtain  $\#_n S^1 \times S^2$  and  $\{y_i\} \times S^2$  misses the 3-cells utilized for the connected sum. Turning the 3- and 4-handles of  $M$  upside down, we find that  $M$  can be constructed with a 0-handle,  $n$  1-handles and  $n$  2-handles. Note that each 2-handle of  $M$  algebraically cancels one of the 1-handles, since  $C'_i$  is homotopic to  $h(S^1 \times \{x_i\})$ .

The Mazur trick can now be applied.  $M \times I$  is a 5-manifold composed of a 0-handle,  $n$  1-handles and  $n$  2-handles. By the Whitney trick, the 2-handles geometrically cancel the 1-handles. Consequently  $M \times I$  is diffeomorphic to  $B^5$  and  $2M = \partial(M \times I)$  is diffeomorphic to  $S^4$ . By the topological Schoenflies theorem [1],  $M$  is homeomorphic to  $B^4$ .

Let  $N$  denote the smooth closed 4-manifold which is obtained by gluing a 4-cell to  $M$  along  $\partial M = S^3$ . Then  $N$  is homeomorphic to  $S^4$ . Since  $N = W \cup \#_n B^3 \times S^1$  it follows that  $W$  is homeomorphic to  $\#_n B^2 \times S^2$ , either by isotopic unknotting in codimension 3 or by using the Dehn's lemma in [5] plus the topological Schoenflies theorem as in §2. This proves the first part of the theorem. Finally, exactly the same argument as in [2] applies to show that  $C_1 \cup \dots \cup C_n$  is a slice link.

*Proof of corollary.* If  $W$  satisfies the conditions of the corollary, then  $W$  can be constructed by adding  $n$  2-handles to  $B^4$  along the curves  $C_1, \dots, C_n$  where  $C_1 \cup \dots \cup C_{n-1}$  is a trivial link of  $n - 1$  components in  $S^3$ . Hence  $W$  satisfies the hypotheses of the theorem and so  $W$  is homeomorphic to  $\#_n B^2 \times S^2$ .

*Note.* I would like to thank C. F. Miller for very helpful advice on the group theory in the above theorem.

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