DEHN’S LEMMA AND HANDLE DECOMPOSITIONS OF SOME 4-MANIFOLDS

J. HYAM RUBINSTEIN
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OF SOME 4-MANIFOLDS

J. H. RUBINSTEIN

We give two short proof of a weak version of the
theorem of Laudenbach, Poenaru [3]. Also we show that
an embedded $S^1 \times S^2$ in $S^4$ bounds a copy of $B^2 \times S^1$. Finally
we establish that if $W$ is a smooth 4-manifold with $\partial W = \#_n S^1 \times S^2$ and $W$ is built from $\#_{n-1} B^2 \times S^1$ by attaching a
2-handle, then $W$ is homeomorphic to $\#_n B^2 \times S^2$.

1. 4-Dimensional handlebodies. Let $X$, $Y$ be the following
smooth 4-manifolds:

$$X = \#_n B^3 \times S^1 \quad \text{and} \quad Y = \#_n B^2 \times S^2.$$ 

In [3] it is proved that if $h: \partial X \to \partial Y$ is a diffeomorphism, then the
smooth closed 4-manifold $X \cup_h Y$ which is obtained by gluing along $h$, is
diffeomorphic to $S^4$.

We begin with two brief proofs, one using the Dehn's lemma
in [5] and the other employing unknotting in codimension 3, of the
following result:

THEOREM. Let $X$, $Y$, $h$ be as above. Then $X \cup_h Y$ is homeomor-
phic to $S^4$.

Proof. (1) Let $\{x_i\} \times S^1$ be a circle in the boundary of the
$i$th copy of $B^3 \times S^1$ in the connected sum $X = \#_n B^3 \times S^1$, for $1 \leq i \leq n$. Without loss of generality, all the loops $\{x_i\} \times S^1$ can be
assumed to miss the cells which are used to construct $X$ as a con-
nected sum. By the Dehn's lemma in [5], it follows that all of the
circles $h(\{x_i\} \times S^1)$ bound disjoint smooth embedded disks $D_i$ in $Y$,
for $1 \leq i \leq n$.

Let $N(D_i)$ denote a small tubular neighborhood of $D_i$ in $Y$.
Clearly $X \cup_h (N(D_1) \cup \cdots \cup N(D_n))$ is diffeomorphic to $B^4$, since $N(D_i)$
can be thought of as a 2-handle which geometrically cancels a 1-
handle of $X$. On the other hand, let $W$ denote the closure of $Y - N(D_1) - \cdots - N(D_n)$. Then $\partial W = S^3$ and $W$ is contained in $Y$ which
can be embedded in $S^4$. By the topological Schoenflies theorem [1], $W$ is homeomorphic to $B^4$. Consequently $X \cup_h Y$ is homeomorphic to $B^4 \cup B^4 = S^4$.

(2) By Van Kampen's theorem, $\pi_1(X \cup_h Y) = \{1\}$. Let $Z$ be
a bouquet of $n$ circles which is embedded in $X$ and is a deformation
retract of $X$. By isotopic unknotting in codimension 3, $Z$ is con-
tained in the interior of a PL 4-cell $B$ in $X \cup_h Y$. Therefore, by an isotopy we can shrink $X$ down towards $Z$ until $X$ is included in $\text{int} B$. Exactly as in (1), by the topological Schoenflies theorem we obtain that $X \cup_h Y - \text{int} B$ is homeomorphic to $B^4$ and so the result follows.

**REMARK.** Note that if the PL or smooth 4-dimensional Schoenflies theorem was known, then these arguments would establish that $X \cup_h Y$ is PL isomorphic or diffeomorphic to $S^4$.

2. **Embeddings of $S^1 \times S^2$ in $S^4$.** The following result was first proved by I. Aitchison (unpublished). We present a simplification of his method, which again uses the Dehn’s lemma in [5].

**THEOREM.** Let $h: S^1 \times S^2 \to S^4$ be a smooth embedding. Then $h$ extends to a topological embedding of $B^2 \times S^2$ in $S^4$.

**Proof.** Let $V$, $W$ be the closures of the components of $S^4 - h(S^1 \times S^2)$ (by Alexander duality there are two such components). By the Mayer-Vietoris sequence, without loss of generality the inclusion $h(S^1 \times S^2) \to V$ induces an isomorphism $H_i(h(S^1 \times S^2)) \to H_i(V)$ and $H_i(W) = 0$.

Let $G$ denote the group which is the pushout of the homomorphisms $\pi_i(h(S^1 \times S^2)) \to \pi_i(V)$ and $\pi_i(h(S^1 \times S^2)) \to \pi_i(W)$. By Van Kampen’s theorem, $G = \{1\}$. On the other hand there is a homomorphism of $G$ onto $\pi_i(W)$ induced by the epimorphism $\pi_i(V) \to H_i(V) \cong H_i(h(S^1 \times S^2)) \cong \pi_i(h(S^1 \times S^2))$. Consequently $\pi_i(W) = \{1\}$ follows.

Now we can apply the Dehn’s lemma in [5] to obtain that $h(S^1 \times *)$ bounds a smooth embedded disk $D$ in $W$. Let $N(D)$ be a small tubular neighborhood of $D$ in $W$. Then the closure of $W - N(D)$ is a topological 4-cell, by the topological Schoenflies theorem [1]. Therefore $W$ is homeomorphic to $B^2 \times S^2$ and $h$ extends to a topological embedding of $B^2 \times S^2$ as desired.

**REMARK.** This result is analogous to the classical theorem of Alexander that any smooth embedded $S^1 \times S^1$ in $S^2$ bounds a smooth solid torus $B^2 \times S^1$.

3. **Handle decompositions and slice links.** In [2], Kirby, Melvin proved that if a smooth 4-manifold $M$ has boundary $S^1 \times S^2$ and is constructed by attaching a 2-handle to $B^4$ along a curve $C$ with the 0-framing, then $M$ is homeomorphic to $B^2 \times S^2$ and $C$ is a slice knot. We prove the following generalization of their result:
THEOREM. Let $W$ be a smooth 4-manifold which is obtained by adding $n$ 2-handles to $B^4$ along the curves $C_1, \ldots, C_n$. The 2-handles induce a framing of the link $C_1 \cup \cdots \cup C_n$. Assume that framed surgery on the sublink $C_1 \cup \cdots \cup C_i$ in $S^3$ yields $#_n S^1 \times S^2$, for all $i$ with $1 \leq i \leq n$. Then $W$ is homeomorphic to $#_n B^2 \times S^2$ and $C_1 \cup \cdots \cup C_n$ is a slice link.

COROLLARY. Let $W$ be a smooth 4-manifold such that $\partial W$ is diffeomorphic to $#_n S^1 \times S^2$ and $W$ is built by attaching a 2-handle to $#_n B^2 \times S^2$. Then $W$ is homeomorphic to $#_n B^2 \times S^2$.

Proof of theorem. By the assumption that surgery on the link $C_1 \cup \cdots \cup C_n$ gives $#_n S^1 \times S^2$, it immediately follows that $\partial W$ is diffeomorphic to $#_n S^1 \times S^2$. If the handle decomposition of $W$ is turned upside down, then $W$ is constructed by attaching $n$ 2-handles to $(#_n S^1 \times S^2) \times I$ along some curves $C'_1 \times \{1\}, C'_2 \times \{1\}, \ldots, C'_n \times \{1\}$ and then adding a 4-handle. We will assume that the 2-handle glued along $C'_i \times \{1\}$ is dual to the 2-handle added along $C_i$ to $B^4$.

Let $W_i$ or $W'_i$ denote the 4-manifold which is obtained by adjoining $i$ 2-handles to $B^4$ or $(#_n S^1 \times S^2) \times I$ respectively along the curves $C_i$, $C_{i+1}$, or $(#_{n-i} S^1 \times S^2) \times I$ respectively. Then $\partial W_i$ is diffeomorphic to $#_n S^1 \times S^2$, since surgery on $C_1 \cup \cdots \cup C_i$ gives $#_n S^1 \times S^2$. Also $W_i$ is a cobordism between $\partial W_i$ and $W_{n-i}$ and therefore $W_i$ is free and $(C_i)$ is primitive, i.e., is contained in a free basis of the free group $\pi_1(#_n S^1 \times S^2)$.

Next, $\pi_1(W'_i)$ has a presentation consisting of a set of free generators of $\pi_1(W_i) \cong \pi_1(#_n S^1 \times S^2)/\langle \langle C'_{i+1}, \ldots, \{C'_n\} \rangle$ and the one relation $\langle \langle C'_{i+1}, \ldots, \{C'_n\} \rangle$. Hence by the results on p. 283 and p. 354 of [4] again, $\pi_1(W'_i)$ is free and $\{C'_n\}$ is primitive. Therefore we obtain that $(\{C'_n\}, C'_i)$ is contained in a free basis for $\pi_1(#_n S^1 \times S^2)$. Continuing on with this argument, we conclude that $(\{C'_n\}, \ldots, \{C'_n\})$ is a free basis of $\pi_1(#_n S^1 \times S^2)$. So by Lemma 2 of [3], there is a diffeomorphism $h: #_n S^1 \times S^2 \to #_n S^1 \times S^2$ such that $h(S^1 \times \{x_i\})$ is homotopic to $C'_i$ for
all $i$, $1 \leq i \leq n$, where $S^1 \times \{x_i\}$ is contained in the $i$th copy of $S^1 \times S^2$ used to form $\#_n S^1 \times S^2$ and is disjoint from the 3-cells employed for the connected sum.

Let $M$ be the smooth 4-manifold with $\partial M = S^3$ which is built by adding $n$ 3-handles and 4-handles to $W'_n$, using the component $(\#_n S^1 \times S^2) \times \{0\}$ of $\partial W'_n$. The 3-handles can be attached along the 2-spheres $h(\{y_i\} \times S^2) \times \{0\}$, for $1 \leq i \leq n$, where $\{y_i\} \times S^2$ is in the $i$th copy of $S^1 \times S^2$ used to obtain $\#_n S^1 \times S^2$ and $\{y_i\} \times S^2$ misses the 3-cells utilized for the connected sum. Turning the 3- and 4-handles of $M$ upside down, we find that $M$ can be constructed with a 0-handle, $n$ 1-handles and $n$ 2-handles. Note that each 2-handle of $M$ algebraically cancels one of the 1-handles, since $C_i'$ is homotopic to $h(S^1 \times \{x_i\})$.

The Mazur trick can now be applied. $M \times I$ is a 5-manifold composed of a 0-handle, $n$ 1-handles and $n$ 2-handles. By the Whitney trick, the 2-handles geometrically cancel the 1-handles. Consequently $M \times I$ is diffeomorphic to $B^5$ and $2M = \partial(M \times I)$ is diffeomorphic to $S^4$. By the topological Schoenflies theorem [1], $M$ is homeomorphic to $B^4$.

Let $N$ denote the smooth closed 4-manifold which is obtained by gluing a 4-cell to $M$ along $\partial M = S^3$. Then $N$ is homeomorphic to $S^4$. Since $N = W \cup \#_n B^2 \times S^1$ it follows that $W$ is homeomorphic to $\#_n B^2 \times S^2$, either by isotopic unknotting in codimension 3 or by using the Dehn’s lemma in [5] plus the topological Schoenflies theorem as in §2. This proves the first part of the theorem. Finally, exactly the same argument as in [2] applies to show that $C_1 \cup \cdots \cup C_n$ is a slice link.

**Proof of corollary.** If $W$ satisfies the conditions of the corollary, then $W$ can be constructed by adding $n$ 2-handles to $B^4$ along the curves $C_1, \cdots, C_n$ where $C_1 \cup \cdots \cup C_{n-1}$ is a trivial link of $n-1$ components in $S^3$. Hence $W$ satisfies the hypotheses of the theorem and so $W$ is homeomorphic to $\#_n B^2 \times S^2$.

**Note.** I would like to thank C. F. Miller for very helpful advice on the group theory in the above theorem.

**References**


Received March 30, 1979 and in revised form September 4, 1979.

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Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsuisha (International Academic Printing Co., Ltd.), 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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