THE FULL $C^*$-ALGEBRA OF THE FREE GROUP ON TWO GENERATORS

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$C^*(F_2)$ is a primitive $C^*$-algebra with no nontrivial projection. $C^*(F_2)$ has a separating family of finite-dimensional representations.

1. Introduction. We present a "generic" $C^*$-algebra in illustration of several peculiar phenomena that may occur in the theory of representations.

Let $F_2$ denote the free group on two generators. If $\pi$ is the universal unitary representation of $F_2$ on a Hilbert space $H$, then the full group $C^*$-algebra $C^*(F_2)$ is the $C^*$-subalgebra of $B(H)$ generated by the set $\{\pi(g) : g \in F_2\}$ (see [4, §13.9]). Alternatively, we can re-define $C^*(F_2)$, in an operator-theoretical setting, as follows:

**Definition.** Let $U, V$ be two unitary operators on a Hilbert space $H$. We say that $(U, V)$ is a universal pair of unitaries iff for each pair of unitary operators $(U_i, V_i)$ on a Hilbert space $H$, the assignment

\[
\begin{align*}
U & \mapsto U_i \\
V & \mapsto V_i
\end{align*}
\]

extends to a $*$-homomorphism from $C^*(U, V)$ onto $C^*(U_i, V_i)$.

**Definition.** We let $C^*(F_2)$ denote the abstract $C^*$-algebra which is $*$-isomorphic with the $C^*$-algebra generated by a universal pair of unitaries.

Obviously, the universal pairs of unitaries are unique up to algebraic $*$-isomorphic equivalence. To see the existence of a universal pair of unitaries, we may simply let

\[
U = \bigoplus U_\nu, \quad V = \bigoplus V_\nu
\]

where $(U_\nu, V_\nu)$ runs through all possible pairs of unitary operators on a fixed separable Hilbert space. By some judicious selection, it suffices to let $\nu$ run through only a countable index set. [In fact, for a general separable $C^*$-algebra $\mathcal{A} \subseteq B(H)$. There is always a projection $P$ of countable dimension, such that $A \mapsto PAP$ is a $*$-isomorphism from $\mathcal{A}$ onto $P\mathcal{A}P \subseteq B(PH)$.]

The main result of this paper is concerned with various expres-
sions for the universal pairs of unitaries. On one hand, we can just take the expression (*) above, such that each \((U, V)\) is a pair of finite-dimensional unitary matrices (Theorem 7). On the other hand, there exist a universal pair of unitary operators that do not have a common nontrivial reducing subspace (Theorem 6). These two apparently opposite constructions induce two important representations of \(C^*(F_2)\).

Now, we examine \(C^*(F_2)\) in regard to its operator-algebraic structure. First and foremost, \(C^*(F_2)\) is a primitive \(C^*\)-algebra; i.e., \(C^*(F_2)\) has a faithful irreducible representation (Theorem 6). This yields the key information that is indispensable to the study of \(\text{Prim}(F_2)\) (cf. [7, Proposition 6.1]). Furthermore, by the universal property of \(C^*(F_2)\), all \(C^*\)-algebras generated by two unitaries (including all \(C^*\)-algebras generated by single operators) are \(*\)-homomorphic images of \(C^*(F_2)\). It may be surprising to see that such a "tremendous" \(C^*\)-algebra has no nontrivial projection (Theorem 1), and no nonnormal hyponormal element (Corollary 8); indeed, \(C^*(F_2)\) even has a faithful tracial state (Corollary 9).

In short, \(C^*(F_2)\), being faithfully irreducibly represented, serves as an example for each of the following unusual conditions:

(i) an irreducible \(C^*\)-algebra with (the most) abundant ideals.

(ii) an irreducible \(C^*\)-algebra with no nontrivial projection (cf. another example given by Philip Green [5]).

(iii) an irreducible \(C^*\)-algebra that admits a separating family of finite-dimensional representations.

Finally, we remark that all results of this paper can be extended to \(C^*(F_n)\) (for \(n > 1\)) and \(C^*(F_\infty)\). Readers are also referred to [2, Lemma 4.4-Theorem 4.5, pp. 1108-1109; 9, Proposition 2.7, p. 250; 10, §3; 11, Theorem 12] for some other unusual aspects of \(C^*(F_2)\).

2. The author is indebted to Robert Powers for helpful communication leading to the following theorem. (The idea of the proof is actually originated by Joel Cohen [3].)

**Theorem 1.** \(C^*(F_2)\) has no nontrivial projection.

**Proof.** By a faithful representation, we may write \(C^*(F_2) = C^*(U, V)\), where \(U, V\) are a universal pair of unitary operators on a Hilbert space \(\mathcal{H}\). Let

\[
\mathcal{V} = \left\{ \text{all norm-continuous functions } \Phi: [0, 1] \longrightarrow \mathcal{B}(\mathcal{H}) \mid \text{such that } \Phi(0) \text{ are scalar operators} \right\}.
\]

Then \(\mathcal{V}\) is a \(C^*\)-algebra with no nontrivial projections. In fact, if \(\Phi \in \mathcal{V}\) is a projection, then \(\Phi(0)\) is 0 or 1 and by continuity, the
projections \((\Phi(t))_{t \in [0,1]}\) must be all 0 or all \(I\). Now we claim that \(C^*(F_2)\) can be imbedded into \(\mathcal{A}\) as a \(C^*\)-subalgebra and consequently, \(C^*(F_2)\) has no nontrivial projection either.

To see the claim, we first choose, by the spectral theorem, two hermitian operators \(A, B \in \mathcal{B}(\mathcal{H})\) such that \(U = e^{itA}, V = e^{itB}\). Next, define two unitary elements \(\Phi_\sigma, \Phi_\nu \in \mathcal{A}\) by

\[
\Phi_\sigma(t) = e^{itA}, \quad \Phi_\nu(t) = e^{itB}.
\]

Then obviously, the evaluation map \(\Phi \mapsto \Phi(1)\) is a \(*\)-homomorphism from \(C^*(\Phi_\sigma, \Phi_\nu)\) onto \(C^*(F_2)\). On the other hand, by the universal property of \(C^*(F_2)\), the assignment \(U \mapsto \Phi_\sigma, \quad V \mapsto \Phi_\nu\) determines a \(*\)-homomorphism from \(C^*(F_2)\) onto \(C^*(\Phi_\sigma, \Phi_\nu)\). Hence, the two \(*\)-homomorphisms above, being inverse to each other, must be \(*\)-isomorphisms. Therefore, \(C^*(F_2)\) can be imbedded into \(\mathcal{A}\) as claimed.

**Corollary 2.** If \(\pi\) is a faithful representation of \(C^*(F_2)\) on a Hilbert space \(\mathcal{H}\), then \(\pi(C^*(F_2))\) contains no nonzero compact operator.

**Proof.** Any \(C^*\)-algebra, containing a nonzero compact operator \(K\), must also contain \(K^*K\) and, thus, the finite-rank spectral projections of \(K^*K\). Since \(\pi(C^*(F_2)) \simeq C^*(F_2)\) contains no nontrivial projection, we have that \(\pi(C^*(F_2))\) contains no nonzero compact operator, either.

We proceed to construct a universal pair of unitary operators that do not have a common nontrivial reducing subspace. The main technique below is a variant of Radjavi-Rosenthal's treatment on nonexistence of common invariant subspace [8, Theorem 7.10, p. 121; Theorem 8.30, p. 162].

**Lemma 3.** Let \(A, B\) be two infinite matrices standing for operators on a separable Hilbert space \(\mathcal{H}\) endowed with a fixed orthonormal basis \(\{e_n\}_{n=1}^\infty\). If \(A\) is a diagonal operator with all distinct diagonal entries, and if all first-column entries of \(B\) are nonzero, then \(A, B\) do not have a common nontrivial reducing subspace.

**Proof.** By simple evaluation on infinite matrices, we deduce that the commutant of \(A\) consists of diagonal operators only, and, diagonal operators commuting with \(B\) must be scalar operators. Hence, the projections commuting with both \(A\) and \(B\) are trivial projections. Therefore, \(A, B\) do not have a common nontrivial reducing subspace.
In the following two lemmas, we deal with the compact perturbations of unitary operators.

**Lemma 4.** Let $U$ be a unitary operator on a separable Hilbert space $\mathcal{H}$. Then there exists a compact operator $K$, and a unitary diagonal operator $D$ with respect to an orthonormal basis $\{e_n\}_{n=1}^{\infty}$, such that $U = D + K$ and all diagonal entries of $D$ are distinct.

**Proof.** From [6], every normal operator $U$ can be written as $D_0 + K_0$, where $K_0$ is a compact operator and $D_0$ is a diagonal operator with respect to an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be the diagonal entries of $D_0$. Since $U = D_0 + K_0$ is unitary, we derive that $\lim_{n \to \infty} |\alpha_n| = 1$. It is easy to choose a complex sequence $\{\beta_n\}_{n=1}^{\infty}$ such that

$$\begin{align*}
|\beta_n| &= 1 \text{ for all } n \\
\beta_i \neq \beta_j \text{ whenever } i \neq j \\
\lim_{n \to \infty} (\alpha_n - \beta_n) &= 0.
\end{align*}$$

Denoting by $D$ for the diagonal operator with the diagonal entries $\{\beta_n\}_{n=1}^{\infty}$, we have that $D_0 - D$ is a diagonal compact operator; thus $K = K_0 + D_0 - D$ is a compact operator and $U = D + K$ as desired.

**Lemma 5.** Let $U$ be a unitary operator on a separable Hilbert space $\mathcal{H}$ endowed with a fixed orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Then there exists a compact operator $K \in \mathcal{B}(\mathcal{H})$, such that $U - K$ has the infinite matrix expression with all first-column entries being nonzero and with $U - K$ unitary.

**Proof.** Let $v$ be a unit vector with all nonzero co-ordinates with respect to the orthonormal basis $\{e_n\}_{n=1}^{\infty}$, and let $\mathcal{S}$ denote the linear span of $v$ and $Ue_i (= \text{the first-column vector of } U)$. Choose any unitary operator $V \in \mathcal{B}(\mathcal{H})$ such that

$$\begin{align*}
[V(\mathcal{S})] &= \mathcal{S} \text{ with } V(Ue_i) = v, \\
[V]_{\mathcal{S}^\perp} &= I_{\mathcal{S}^\perp}.
\end{align*}$$

Then $V - I$ is an operator of rank $\leq 2$ and the first column vector of $VU$ is $v$; thus

$$VU = U + (V - I)U$$

is a compact perturbation of $U$ as desired.

**Theorem 6.** $C^*(F_2)$ is a primitive $C^*$-algebra; i.e., $C^*(F_2)$ has a faithful irreducible representation.
Proof. Since $C^*(F_2)$ is separable, we may write $C^*(F_2) = C^*(U, V)$, where $U, V$ are a universal pair of unitary operators on a separable Hilbert space $\mathcal{H}$. Applying Lemmas 4–5 to $U, V$, we have that

$$U = U_0 + \text{compact}, \quad V = V_0 + \text{compact},$$

and with respect to a suitable orthonormal basis, $U_0$ is a unitary diagonal operator with distinct diagonal entries, and $V_0$ is a unitary operator with all first-column entries nonzero. From the universal property, the assignment

$$U \mapsto U_0, \quad V \mapsto V_0$$

defines a representation

$$\pi: C^*(F_2) \to C^*(U_0, V_0) \subseteq \mathcal{B}(\mathcal{H}).$$

By Lemma 3, $U_0, V_0$ do not have a common nontrivial reducing subspace; thus $C^*(U_0, V_0)$ is an irreducible $C^*$-algebra, and $\pi$ is an irreducible representation.

It remains to show that $\pi$ is faithful. Letting $\mathcal{K}(\mathcal{H})$ be the ideal of compact operators and $\eta: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the natural quotient map, we have then

$$\eta(C^*(U_0, V_0)) = C^*(\eta(U_0), \eta(V_0)) = C^*(\eta(U), \eta(V)).$$

But from Corollary 2, $\eta$ restricted to $C^*(U, V)$ is an $^*$-isomorphism. The composition of the canonical $^*$-homomorphisms

$$C^*(F_2) \xrightarrow{\pi} C^*(U_0, V_0) \xrightarrow{\eta} C^*(\eta(U_0), \eta(V_0)) = C^*(\eta(U), \eta(V))$$

leads to the identity map on $C^*(F_2)$; therefore $\pi$ is a $^*$-isomorphism as desired.

For a general $C^*$-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ with separable $\mathcal{H}$, we can construct a "completely order injection" $\varphi$ from $\mathcal{A}$ into $\bigoplus_{n=1}^{\infty} M_n$, the direct sum of full matrix algebras, by letting

$$\varphi(A) = \bigoplus_{n=1}^{\infty} P_n AP_n$$

where $\{P_n\}$ is a sequence of finite-rank projections approaching strongly to $I$. In case $\mathcal{A} = C^*(F_2)$, we will modify $\varphi$ to get actually an "algebraic $^*$-isomorphism".

**Theorem 7.** $C^*(F_2)$ has a separating family of finite-dimensional representations.
Proof. Since $C^*(F_2)$ is separable, we may assume that $C^*(F_2) = C^*(U, V)$ where $U, V$ are a universal pair of unitary operators on a separable Hilbert space $\mathcal{H}$. Let $\{P_n\}_n$ be a sequence of increasing projections in $\mathcal{B}(\mathcal{H})$ approaching strongly to the identity operator $I$ with rank $P_n = n$. Write

\[
A_n = P_n U P_n, \quad B_n = P_n V P_n,
\]

\[
U_n = \begin{bmatrix} A_n & (P_n - A_n A_n^*)^{1/2} \\ (P_n - A_n A_n^*)^{1/2} & -A_n^* \end{bmatrix},
\]

\[
V_n = \begin{bmatrix} B_n & (P_n - B_n B_n^*)^{1/2} \\ (P_n - B_n B_n^*)^{1/2} & -B_n^* \end{bmatrix}.
\]

By identifying $P_n \mathcal{H} P_n$ with $M_n$, we may regard $P_n$ as the identity $n \times n$ matrix, and $U_n$, $V_n$ as $2n \times 2n$ unitary matrices. From the universal property of $C^*(F_2)$, the assignment

\[ U \mapsto U_n, \quad V \mapsto V_n \]

defines a representation $\pi_n: C^*(F_2) \to M_{2n}$. Now, we claim that $\{\pi_n\}_{n=1}^\infty$ is a separating family of representations; in other words, the $\ast$-homomorphism

\[ \pi: C^*(F_2) \to \bigoplus_{n=1}^\infty M_{2n}, \]

defined by

\[ \pi(A) = \bigoplus_{n=1}^\infty \pi_n(A), \]

is actually a $\ast$-isomorphism.

Note that in the strong topology, $U_n, U_n^*, V_n, V_n^*$ converge to

\[
\begin{bmatrix} U & 0 \\ 0 & -U^* \end{bmatrix}, \quad \begin{bmatrix} U^* & 0 \\ 0 & -U \end{bmatrix}, \quad \begin{bmatrix} V & 0 \\ 0 & -V^* \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} V^* & 0 \\ 0 & -V \end{bmatrix}
\]

respectively. Hence, if $F(\cdot, \cdot)$ is a finite linear combination of words in two free variables, then $F(U_n, V_n)$ also converges to

\[
\begin{bmatrix} F(U, V) & 0 \\ 0 & F(-U^*, -V^*) \end{bmatrix}
\]

in the strong topology. Therefore, for any $\varepsilon > 0$, and given $\|F(U, V)\| = 1$, we have that

\[ \|F(U_n, V_n)\| \geq 1 - \varepsilon \]

for all sufficiently large $n$; thus
\[ \| \pi(F(U, V)) \| \geq \| \pi_n(F(U, V)) \| = \| F(U_n, V_n) \| \geq 1 - \varepsilon . \]

Since \( \varepsilon \) is arbitrary, we conclude that \( \pi \), restricted to the pre-\( C^* \)-algebra generated by \( U, V \), is an isometry. By continuity, \( \pi \) is an isometry and, thus, a \( * \)-isomorphism as desired.

We say that an operator \( A \) is hyponormal iff \( A^* A \geq AA^* \).

**Corollary 8.** Every hyponormal operator in \( C^*(F_2) \) is normal.

**Proof.** From the theorem above, we may imbed \( C^*(F_2) \) into \( \bigoplus_{n=1}^\infty M_{2^n} \) as a \( C^* \)-subalgebra. Since every hyponormal matrix is normal, we have then for each \( A = \bigoplus A_n \in \bigoplus M_{2^n} \),

\[
\begin{align*}
A \text{ is hyponormal} & \implies A_n \text{ is hyponormal for each } n \\
& \implies A_n \text{ is normal for each } n \\
& \implies A \text{ is normal}
\end{align*}
\]

as desired.

**Corollary 9.** \( C^*(F_2) \) has a faithful tracial state.

**Proof.** By Theorem 7, we can imbed \( C^*(F_2) \) into \( \bigoplus_{n=1}^\infty M_{2^n} \) as a \( C^* \)-subalgebra. Let \( \tau_n \) be the faithful tracial state of \( M_{2^n} \). Then \( \tau: \bigoplus M_{2^n} \to C \), defined by

\[
\tau(\bigoplus A_n) = \sum (\tau_n(A_n)/2^n) ,
\]

is a faithful tracial state as desired.

**References**


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