INCREASING SEQUENCES OF BETTI NUMBERS

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We study the sequence of Betti numbers \( \{\beta_i(M)\}_{i \geq 1} \) of an arbitrary finitely generated nonfree module \( M \) over a commutative noetherian local ring \( R \) and show that for a certain class of rings this sequence is always nondecreasing, while for a certain subclass of rings, the subsequence \( \{\beta_i(M)\}_{i \geq 2} \) is strictly increasing.

In [3], a class of commutative noetherian local rings \((R, m)\) called BNSI rings was introduced. These rings have the property that for every finitely generated nonfree module \( M \), the sequence of Betti numbers \( \{\beta_i(M)\}_{i \geq 1} \) is strictly increasing. Recall that \( \beta_i(M) \) is the dimension of the \( R/m \)-vector space \( \text{Tor}_i^R(M, R/m) \); equivalently, it is the rank of the free module \( F_i \) where

\[
\cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0
\]

is a minimal \( R \)-free resolution of \( M \). A class of BNSI rings was given in [3, Theorem 3.2A]: Let \((S, n)\) be a noetherian local ring and let \( J \) be an ideal which is not contained in any prime ideal of grade 1. If \( S \) is a domain, then \( S/nJ \) is a BNSI ring.

In this note, using a result of G. Levin [2] we prove:

**Theorem 1.1.** Let \((S, n)\) be a noetherian local ring of Krull dimension \( d \geq 2 \). Then for \( n \) sufficiently large, the local ring \((R, m) = (S/n^n, n/n^n)\) has the property that for all finitely generated nonfree \( R \)-modules \( M \), the sequence \( \{\beta_i(M)\}_{i \geq 2} \) is strictly increasing. In fact, for all \( i \geq 2 \), \( \beta_{i+1}(M) - \beta_i(M) \geq d - 1 \).

Thus \( R \) is nearly a BNSI ring, except that our proof gives no estimate for \( \beta_2(M) - \beta_1(M) \). Another drawback is that we can not estimate how large \( n \) must be, since it comes, indirectly, from the Artin-Rees Lemma. To fill these gaps (at least partially) we offer the weaker, but more general:

**Corollary 2.2.** Let \((S, n)\) be a noetherian local ring of Krull dimension \( \geq 1 \), and let \( R = S/n^n \), with \( n \geq 1 \). Then for all finitely generated \( R \)-modules \( M \), the sequence \( \{\beta_i(M)\}_{i \geq 1} \) is nondecreasing.

It should be pointed out that if \( S \) is assumed to be a domain and grade \( n \geq 2 \), then by the theorem from [3] cited above, \( S/n^n \) is a BNSI ring for all \( n \geq 2 \).
1. We begin with:

**Proof of Theorem 1.1.** Let \( 0 \to K \to R^n \to M \to 0 \) be exact, with \( K \subset mR^n \), and let

\[
\cdots \longrightarrow R^{n_i} \longrightarrow \cdots \longrightarrow R^{n_1} \longrightarrow K \longrightarrow 0
\]

be a minimal \( R \)-free resolution of \( K \). Then \( n_{i+1} = \beta_i(K) = \beta_{i+1}(M) \), and since \( K \subset mR^n \), \( \text{ann}(m) \cdot K = 0 \). Similarly, all the higher syzygies of \( M \) are annihilated by \( \text{ann}(m) \). Thus it suffices to prove that for any finitely generated \( R \)-module \( N \) which is annihilated by \( \text{ann}(m) \), \( \beta_i(N) - \beta_1(N) \geq d - 1 \).

By [2, Formula (8), p. 9], for \( n \) sufficiently large we have

\[
P^N_R(t) = \frac{P^S_R(t)}{1 - t(P^S_R(t) - 1)}
\]

where for any noetherian local ring \( Q \) and finitely generated \( Q \)-module \( X \), \( P^S_R(t) \) is the Poincaré series \( \sum_{i=0}^{\infty} \beta_i(X) t^i \). Now \( P^S_R(t) = 1 + b_2 t + \cdots \). Since

\[
S^{b_2} \longrightarrow S \longrightarrow R \longrightarrow 0
\]

is part of a minimal \( S \)-resolution of \( R \), \( b_2 = \) the minimal number of generators of \( u^n \). But \( u^n \) is \( u \)-primary, and so by Krull's Generalized Ideal Theorem [1, Theorem 152], \( b_2 \geq \text{height } u = d \). Now

\[
1 - t(P^S_R(t) - 1) = 1 - b_2 t^2 - \cdots
\]

and so if \( (1 - b_2 t^2 - \cdots)^{-1} = \sum_{i=0}^{\infty} c_i t^i \), then \( c_0 = 1, c_1 = 0, \) and \( c_2 = c_0 b_2 = b_2 \).

Now let \( P^S_R(t) = \sum_{j=0}^{\infty} p_j t^j \). Thus

\[
S^{p_2} \longrightarrow S^{p_1} \longrightarrow S^{p_0} \longrightarrow N \longrightarrow 0
\]

is part of a minimal \( S \)-free resolution of \( N \). We claim that \( p_2 + p_0 \geq p_1 \). To see this, localize at a minimal prime of \( S \) to obtain an artin ring \( T \). Then the sequence

\[
T^{p_2} \overset{f}{\longrightarrow} T^{p_1} \overset{g}{\longrightarrow} T^{p_0}
\]

is exact, so \( l(T^{p_1}) = l(\text{im } f) + l(\text{im } g) \leq l(T^{p_2}) + l(T^{p_0}) \), where \( l(X) \) denotes the length of \( X \). Therefore \( p_1 \leq p_2 + p_0 \). Now from (*) we have

\[
P^N_R(t) = \left( \sum_{i=1}^{\infty} c_i t^i \right) \left( \sum_{j=0}^{\infty} p_j t^j \right) = \sum_{k=0}^{\infty} \beta_k t^k.
\]

Thus \( \beta_1 = c_0 p_1 + c_1 p_0 = p_1 \), and \( \beta_2 = c_0 p_2 + c_1 p_1 + c_2 p_0 = p_2 + b_2 p_0 \). Since \( b_2 \geq d \geq 2, \beta_2 \geq p_2 + p_0 + (d-1)p_0 \geq p_1 + (d-1)p_0 = \beta_1 + \)


\[(d - 1)p_0. \text{ So } \beta_2 - \beta_1 \geq (d - 1)p_0 \geq d - 1 \geq 1.\]

2. We now remove the restriction that \(n\) be "sufficiently large". Our starting point is [3, Theorem 3.4]: Let \((S, n)\) be a noetherian local domain and let \(J\) be any nonzero ideal. Let \(R = S/nJ\). Then for any finitely generated \(R\)-module \(M\), the sequence \(\{\beta_i(M)\}_{i \geq 1}\) is nondecreasing.

The proof of this result was a minor modification of the proof of [3, Theorem 3.2]. A further modification yields:

**Proposition 2.1.** Let \((S, n)\) be a noetherian local ring and let \(J\) be a nonnilpotent ideal. Let \(R = S/nJ\). Then for any finitely generated \(R\)-module \(M\), the sequence \(\{\beta_i(M)\}_{i \geq 1}\) is nondecreasing.

**Proof.** Following the proof of [3, Theorem 3.4] we obtain an \(S\)-module \(A\) such that \(JS^p \subset A \subset S^p\), where \(p = \beta_1(M)\), and \(\beta_2(M)\) = the minimal number of generators of \(A\). Thus we must show that \(A\) can not be generated by \(p - 1\) elements.

Let \(x \in J\) be a nonnilpotent element, and let \(T\) be the localization of \(S\) at the multiplicative set \(\{x^i \mid i \geq 0\}\). Then

\[JS^p \otimes_S T \subset A \otimes_S T \subset S^p \otimes_S T = T^p.\]

Since \(J\) meets the multiplicative set, \(JS^p \otimes_S T = T^p\). Hence \(A \otimes_S T = T^p\). Now the minimal number of generators of \(A\) as an \(S\)-module is at least the minimal number of generators of \(A \otimes_S T\) as a \(T\)-module, and since a free module of rank \(p\) can not be generated by \(p - 1\) elements, we are done.

As an easy consequence we have:

**Corollary 2.2.** Let \((S, n)\) be a noetherian local ring of Krull dimension \(\geq 1\), and let \(R = S/n^\infty\). Then for any finitely generated \(R\)-module \(M\), the sequence \(\{\beta_i(M)\}_{i \geq 1}\) is nondecreasing.

**Proof.** When \(n = 1\), \(R\) is a field and all the Betti numbers in the sequence are 0. For \(n \geq 2\), let \(J = n^{n-1}\). Since Krull dim \(S \geq 1\), \(J\) is not nilpotent, and the preceding proposition applies.

**References**

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Spiros Argyros, *A decomposition of complete Boolean algebras* 

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