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A new notion of simplicity for recursively enumerable (r.e.) sets is introduced, that of d-simplicity or simplicity with respect to arrays of differences of r.e. sets (d.r.e. sets). This notion arose from the method used to generate automorphisms of $\mathscr{E}^*$, the lattice of r.e. sets modulo finite sets, and is a further step toward finding a complete set of invariants for the automorphism types of $\mathscr{E}^*$. The d-simple sets are closely related to the small sets defined by Lachlan as a key part of his decision procedure for the $\forall\exists$-theory of $\mathscr{E}^*$. Finally, the degrees $\boldsymbol{D}$ of d-simple sets form a new invariant class of r.e. degrees, since $\boldsymbol{H}_1 \subseteq \boldsymbol{D}$ but $\boldsymbol{D}$ splits $\boldsymbol{L}_1$ (where $\boldsymbol{H}_1$ and $\boldsymbol{L}_1$ are the high and low r.e. degrees respectively). This refutes conjectures of Martin and Shoenfield which imply that degrees $\boldsymbol{C}$ of any class of r.e. sets invariant under automorphisms of $\mathscr{E}$ can be characterized by a finite set of equalities or inequalities involving the jump of degrees in $\boldsymbol{C}$.

0. **Introduction.** Let $\mathscr{E}$ denote the lattice of r.e. sets under inclusion. If $\mathscr{L}$ is a sublattice of $\mathscr{E}$ closed under finite differences, let $\mathscr{L}^*$ denote the quotient lattice of $\mathscr{L}$ modulo the ideal $\mathscr{F}$ of finite sets. Post's program [11] which has predominated for thirty years has been to classify an r.e. set A by its lattice of supersets $\mathscr{L}(A) = \{W: W \in \mathscr{E}$ and $A \subseteq W\}$. Further evidence for this approach was the automorphism result by Soare [17] that if A and B are maximal sets (i.e., $\mathscr{L}^*(A)$ and $\mathscr{L}^*(B)$ are isomorphic to the two element Boolean algebra) then A and B are automorphic, i.e., there exists $\Phi \in \mathrm{Aut}\ \mathscr{E}$ (the group of automorphisms of $\mathscr{E}$) such that $\Phi(A) = B$.

However, more recent results [9] show that $\mathscr{L}^*(A) \cong \mathscr{L}^*(B)$ does not necessarily imply that A is automorphic to B when $\mathscr{L}^*(A)$ is infinite, even if $\mathscr{L}^*(A)$ is a very well-behaved lattice such as the countable atomless Boolean algbera. To characterize the automorphism type of $A \in \mathscr{E}$ new invariants are needed which, unlike $\mathscr{L}^*(A)$, relate the structure of $\bar{A}$ to that of A. (*Warning*: all sets and degrees mentioned will be r.e.)

A second automorphism result [20] demonstrating uniformity of $\mathscr{E}^*$ is that if A is coinfinite and low (i.e., $A' \equiv_T \phi'$) then $\mathscr{L}^*(A) \cong \mathscr{E}^*$, and in fact the isomorphism is effective on indices. If A and B are low simple sets are they necessarily automorphic? In order to extend an automorphism $\Psi: \mathscr{L}^*(A) \to \mathscr{L}^*(B)$ to an automorphism

Φ of $\mathscr{E}^*$ such that $\Phi(A) = \Phi(B)$, the automorphism method uses a certain covering property [17, Theorem 2.2] of which the notion of d-simplicity, defined below, is a weak version. We prove that there are low simple sets A and B such that A is d-simple but B is not, and hence A is not automorphic to B. Thus, d-simplicity is a new lattice invariant property of sets $A \in \mathscr{E}$ not definable in terms of $\mathscr{L}^*(A)$.

A class $\mathscr{C} \subseteq \mathscr{E}$ is *invariant* if it is invariant under Aut $\mathscr{E}$. A class $\boldsymbol{C}$ of degrees is *invariant* if $\boldsymbol{C} = \{\deg(W): W \in \mathscr{C}\}$ for some invariant $\mathscr{C} \subseteq \mathscr{E}$. A fundamental open question relating the structure of a set to its degree is to determine which classes of degrees are invariant. Let $\boldsymbol{R}$ denote the (r.e.) degrees and define

$$\boldsymbol{H}_n = \{\boldsymbol{a}: \boldsymbol{a} \in \boldsymbol{R} \text{ and } \boldsymbol{a}^{(n)} = \boldsymbol{0}^{(n+1)}\} .$$
$$\boldsymbol{L}_n = \{\boldsymbol{a}: \boldsymbol{a} \in \boldsymbol{R} \text{ and } \boldsymbol{a}^{(n)} = \boldsymbol{0}^{(n)}\} ,$$

where $\boldsymbol{d}^{(0)} = \boldsymbol{0}$, and $\bar{\boldsymbol{L}}_n = \boldsymbol{R} - \boldsymbol{L}_n$. The degrees in $\boldsymbol{H}_1$ and $\boldsymbol{L}_1$ are called *high* and *low* respectively. Martin [9] showed that $\boldsymbol{H}_1 = \boldsymbol{M}$, the degrees of maximal sets and Lachlan [4] and Shoenfield [16] proved that $\bar{\boldsymbol{L}}_2 = \boldsymbol{A}$, the degrees of atomless sets. Given this progression of invariant classes, $\bar{\boldsymbol{L}}_0$, $\boldsymbol{H}_1$, $\bar{\boldsymbol{L}}_2$, Shoenfield conjectured that these exhausted the invariant classes while Martin conjectured that the invariant classes are precisely $\bar{\boldsymbol{L}}_{2n}$ and $\boldsymbol{H}_{2n+1}$ for $n \geqq 0$.

The major achievement of this paper is to prove that $\boldsymbol{D}$, the class of degrees of d-simple sets, is a new invariant class not of the form $\boldsymbol{H}_n$ or $\bar{\boldsymbol{L}}_n$ for any n. This is accomplished by showing that $\boldsymbol{H}_1 \subseteq \boldsymbol{D}$, but that $\boldsymbol{D}$ splits $\boldsymbol{L}_1$ and in fact that there is a simple set S with no d-simple set recursive in S. The other known classes of r.e. sets which contain members of some degree $\boldsymbol{d} \in \bar{\boldsymbol{L}}_1 - \boldsymbol{L}_0$ (such as simple or hypersimple sets) can be shown to contain members of every r.e. degree $\boldsymbol{d} > \boldsymbol{0}$ using the permitting method of Yates [22]. Such methods fail here because d-simplicity is defined in terms of certain arrays of *differences* of r.e. sets (d.r.e. sets) rather than arrays of r.e. sets.

The plan of the paper is as follows. In § 1 we define d-simplicity and prove that hyperhypersimple (hh-simple) sets are d-simple and that d-simple sets are simple. We prove that the class $\mathscr{D}$ of d-simple sets is closed upwards under inclusion (among the coinfinite sets) and that $\boldsymbol{D}$ is closed upwards and $\boldsymbol{H}_1 \subseteq \boldsymbol{D}$. Finally, we prove that there are low d-simple sets so $\boldsymbol{D} \cap \boldsymbol{L}_1 \neq \phi$.

In § 2 we review the small sets introduced by Lachlan [3] in his decision procedure for the $\forall\exists$-theory of $\mathscr{E}^*$. We prove that no d-simple set is small, that there is a simple small (and hence not

d-simple) set in every degree $\boldsymbol{d} > \boldsymbol{0}$, and that the d-simple sets do not coincide with any of the well-known classes of simple sets. There is a coinfinite r.e. set with no d-simple superset and the class of degrees of such sets is exactly $\boldsymbol{H}_1$.

In § 3 we prove that the d-simple sets are not closed under intersection and that the relation "d-simple in" is not transitive. In § 4 we prove our most important and difficult result that there is a degree $\boldsymbol{d} \in \boldsymbol{L}_1$ such that all sets of degree $\leqq \boldsymbol{d}$ are small and hence not d-simple. Thus D splits $\boldsymbol{L}_1$.

We use the standard notation in Rogers [14]. In addition let $A = {}^*B$ denote that the symmetric difference of A and B is finite, and $A \subseteq^* B$ denote that $A \cap \bar{B} = {}^*\phi$. Let $B \subset_\infty A$ denote that $A \subseteq B$ and $A - B$ is infinite. A *simultaneous enumeration* of a given recursive sequence $\{U_n\}_{n \in N}$ is a $1:1$ recursive function g with range $\{\langle m, n\rangle : m \in U_n\}$. Thus, at each stage s, $g(s) = \langle m, n\rangle$ causes one element m to be enumerated in one set U_n. Fixing g let U_n^s denote those elements enumerated in U_n by the end of stage s, and

$$U_n \backslash U_m = \{x : (\exists s)[x \in U_n^s - U_m^s]\} ,$$

those elements appearing in U_n before U_m. (The notation $X \backslash Y$ should not be confused with $X - Y$ which denotes $X \cap \bar{Y}$.) Let $U_n \searrow U_m = (U_n \backslash U_m) \cap U_m$, those elements enumerated first in U_n and later in U_m.

We identify a set with its characteristic function and let $A[x]$ denote the restriction of A to arguments $\leqq x$. We write $\{e\}_s^{A[z]}(x) = y$ if the eth Turing procedure with argument x and oracle $A[z]$ halts in $\leqq s$ steps and yields output y. We assume that $e, x, y, z \leqq s$, that $x, y, \leqq z$, and that if $z_1 \leqq z$ is used in the computation then all $z_2 \leqq z_1$ are also used.

1. d-simple sets.

We begin with a motivation of d-simplicity from the point of view of generating automorphisms of $\mathscr{E}$. Suppose that A and B are coinfinite low sets (so $\mathscr{L}(A) \cong \mathscr{L}(B)$ by [20]). Let us try to construct $\Phi \in \operatorname{Aut} \mathscr{E}$ such that $\Phi(A) = B$ by enumerating an array $\{\hat{W}_e\}_{e \in N}$ such that $\Phi(W_e) = \hat{W}_e$. Now if B is simple and A is not, say $W_e \cap A = \phi$, for W_e infinite, then we must fail since $\hat{W}_e \cap B \neq \phi$ for any choice of $\hat{W}_e$ infinite. To avoid this problem suppose that A and B are simple.

Given $W_1 \supseteq W_2$ which intersect $\bar{A}$ we must choose $\hat{W}_1 \supseteq \hat{W}_2$ which intersect $\bar{B}$. If we choose $\hat{W}_1$ and $\hat{W}_2$ such that $(\hat{W}_1 - \hat{W}_2) \cap B \neq \phi$ then we must be sure that A is sufficiently "large" so that $A \cap (W_1 - W_2) \neq \phi$ also, i.e., if B is simple with respect to certain arrays of d.r.e. sets then A must also be. What is the right definition of simplicity with respect to d.r.e. sets? Clearly we cannot ask that A intersect every infinite d.r.e. set since $\bar{A}$ is d.r.e. and infinite.

The definition is motivated by the hypotheses of the Extension Theorem [17, Theorem 2.2] for generating automorphisms of $\mathscr{E}$, where the problem of defining $\hat{X} = \Phi(X)$ is split into 2 parts corresponding to $X \cap \bar{A}$ and $X \cap A$. First consider $X \cap \bar{A}$ and choose a set $Y \subseteq X$ as small as possible such that $Y \cap \bar{A} = X \cap \bar{A}$. Then consider $X \cap A$ and let $\hat{X}$ be sufficiently large so that $\hat{X} \cap B \supseteq \hat{Y} \cap B$.

DEFINITION 1.1. (a) A coinfinite set A is *d-simple* if for all X there exists $Y \subseteq X$ such that

(1.1) $X \cap \bar{A} = Y \cap \bar{A}$, and

(1.2) $(\forall Z)[(Z - X) \text{ infinite} \Rightarrow (Z - Y) \cap A \neq \phi]$.

(b) Furthermore, A is *uniformly* d-simple if an index for Y can be found uniformly effectively from one for X.

Note that (1.2) asserts that A is simple with respect to the r.e. array of d.r.e. sets $\{(W_e - Y)\}_{e \in \omega}$. (Of course, d-simplicity is definable in the elementary theory of $\mathscr{E}$ and hence invariant under Aut $\mathscr{E}$.)

PROPOSITION 1.2. *If A is d-simple then A is simple.*

Proof. In Definition 1.1 set $X = \phi$. Hence $Y = \phi$ and (1.2) asserts that $Z \cap A \neq \phi$ for every infinite Z.

PROPOSITION 1.3. *If A is hh-simple then A is d-simple.*

Proof. Lachlan [2, Theorem 3] has shown that a coinfinite set A is hh-simple iff $\mathscr{L}(A)$ is a Boolean algebra or equivalently iff for every X there is a recursive $R \subseteq X$ such that $X \cap \bar{A} = R \cap \bar{A}$. In (1.1) set $Y = R$. Now if Z violates (1.2) then $Z - R$ is infinite but $(Z - R) \cap A = \phi$, so $Z \cap \bar{R}$ is an infinite r.e. subset of $\bar{A}$ violating the simplicity of A.

Notice that Proposition 1.3 does not establish that hh-simple sets are uniformly d-simple since Lachlan's procedure gives us no uniform way of finding R from X.

PROPOSITION 1.4. *Let $\mathscr{D} \subseteq \mathscr{E}$ be the class of d-simple sets. Then $\mathscr{D}$ is closed upwards among the coinfinite sets, namely if $A \in \mathscr{D}$ and $A \subseteq B \subset_\infty N$ then $B \in \mathscr{D}$ also.*

Proof. If (1.1) and (1.2) hold for A then they hold a fortiori for B since (1.1) for $\bar{A}$ and $\bar{B} \subseteq \bar{A}$ imply (1.1) for $\bar{B}$, while (1.2) for

A and $A \subseteq B$ imply (1.2) for B.

PROPOSITION 1.5. *If $\mathscr{C}' \subseteq \mathscr{E}$ is a class of coinfinite sets, and is closed upwards among the coinfinite sets, and $\mathscr{C}'$ contains all hh-simple sets then $\boldsymbol{C} = \{\deg(W): W \in \mathscr{C}'\}$ is closed upwards and $\boldsymbol{C} \supseteq \boldsymbol{H}_1$.*

Proof. Martin [10] shows that every $\boldsymbol{d} \in \boldsymbol{H}_1$ contains a maximal (and hence hh-simple) set. Hence $\boldsymbol{H}_1 \subseteq \boldsymbol{C}$. Now suppose $\boldsymbol{a} > \boldsymbol{b}$, and $\boldsymbol{b} \notin \boldsymbol{H}_1$ where $\boldsymbol{b} = \deg B$ and $B \in \mathscr{C}'$. Then by Martin [10] B is not hh-simple so by Lachlan [4, Theorem 1] there exists $A \supseteq B$ of degree $\boldsymbol{a}$. Now $A \in \mathscr{C}'$ by upward closure so $\boldsymbol{a} \in \boldsymbol{C}$.

COROLLARY 1.6. *Let $\boldsymbol{D} = \{\deg(W): W \in \mathscr{D}\}$. Then $\boldsymbol{H}_1 \subseteq \boldsymbol{D}$ and $\boldsymbol{D}$ is closed upwards.*

Proof. By Propositions 1.5., 1.4 and 1.3.

Next we prove that there is a low d-simple set, and hence $\boldsymbol{D} \cap \boldsymbol{L}_1 \neq \phi$. The construction is very similar to the usual construction [18, Theorem 4.1] of a low simple set A except that A must now intersect certain infinite d.r.e. sets instead of certain infinite r.e. sets. Let $\{\langle X_e, Z_e\rangle\}_{e \in \omega}$ be a recursive listing of all pairs of r.e. sets, and fix a simultaneous enumeration of $\{X_e, Z_e\}_{e \in \omega}$. To make A d-simple it suffices to make $\bar{A}$ infinite and to meet for each e the positive requirement,

$$(1.3) \quad P_e\colon (Z_e - X_e) \text{ infinite} \Longrightarrow (\exists x)(\exists s)[x \in (Z_e^s - X_e^s) \cap (A^{s+1} - A^s)]\,,$$

because we can let $Y_e = X_e \backslash A$. (Recall that $U \backslash V = \{x\colon (\exists s)[x \in U^s - V^s]\}$.) Thus Y_e satisfies (1.1) because $X_e \cap \bar{A} \subseteq X_e \backslash A$, and (1.2) holds because an element $x \in Z_e^s - X_e^s$ enumerated in A at stage $s+1$ is never later enumerated in Y_e so $(Z_e - Y_e) \cap A \neq \phi$.

Each P_e contributes at most one element to A, and the lowness requirements N_e defined below involve finite restraint, so the usual construction succeeds. Finally, the uniformity condition (b) of Definition 1.1 is satisfied by the definition of Y_e.

THEOREM 1.7. *There is a low uniformly d-simple set A.*

Proof. To make A low it suffices [18, p. 523] to meet for each e the negative requirement,

$$N_e\colon \{e\}_s^{A^s}(e) \text{ defined for infinitely many } s \Longrightarrow \{e\}^A(e) \text{ defined.}$$

Let $r(e, s)$ be the greatest integer used in the computation $\{e\}_s^{A_s}(e)$

if the latter is defined and $= -1$ otherwise. Set $A^0 = \phi$.

Stage $s+1$. Choose e minimal such that P_e has never received attention and such that

$$(1.4)\quad (\exists x)[x \in Z_e^s - (X_e^s \cup A^s)\ \&\ 2e < x\ \&\ (\forall i \leqq e)[r(i, s) < x]]\ .$$

Choose x minimal for e. Enumerate x in A and say that P_e *receives attention*. (If e fails to exist do nothing.) Let $A = \bigcup_s A^s$.

The second clause in (1.4) guarantees that $\bar{A}$ is infinite. Each requirement N_e is met because each P_i, $i < e$, contributes at most one element to A. Thus, $\lim\sup_s r(i, s)$ exists for all i and each requirement P_e is met.

COROLLARY 1.8. $\boldsymbol{D} \cap \boldsymbol{L}_1 \neq \phi$.

Our results yield new negative information on the question of what conditions on A and B guarantee that

$$(1.5)\qquad \mathscr{L}(A) \cong \mathscr{L}(B) \Longrightarrow A \text{ is automorphic to } B\ .$$

Let A be low and d-simple, and B be the low simple set of Corollary 2.7 which by Proposition 2.3 is not d-simple. Now $\mathscr{L}(A) \cong \mathscr{L}(B) \cong \mathscr{E}$ by [19] but A and B are not automorphic, because d-simplicity is clearly invariant under Aut $\mathscr{E}$. Hence, (1.5) is false for low simple sets and the d-simplicity of an r.e. set A is not definable as a property of $\mathscr{L}(A)$. In [9] it is shown that (1.5) is false for atomless hh-simple sets. It is unknown whether (1.5) holds when A and B are both low and d-simple but this seems unlikely. Maximal sets satisfy (1.5) because they possess a stronger covering property than d-simplicity [17, § 3]. It is unknown whether this stronger property is invariant under Aut $\mathscr{E}$ or under what conditions it is implied by d-simplicity. However, the construction of Theorem 1.7 can easily be modified to produce low simple sets with the stronger property.

2. Small sets. A second notion which relates the structure of A to that of $\bar{A}$ (and is not merely a property of $\mathscr{L}(A)$) is the notion of a small set introduced by Lachlan [3, Theorem 3] as an important ingredient in his decision procedure. In this section we prove that no d-simple set is small, that there is a simple small (and hence not d-simple) set in every degree $\boldsymbol{d} > \boldsymbol{0}$, and that the d-simple sets do not coincide with any other well-known classes of simple sets.

DEFINITION 2.1. (a) If $B \subset A \subset_\infty N$ then B is *small in* $A (B \subset_s A)$ if for all U and V

(2.1) $$V \supseteq U \cap (A - B) \Longrightarrow (U - A) \cup V \text{ is r.e.}$$

(b) B is *small* if $B \subset_s A$ for some A.

The intuition is that B is sufficiently smaller than A so that any V satisfying the hypothesis of (2.1) must include enough of U so that the union of the d.r.e. set $(U - A)$ with V is r.e. Notice that $\phi \subset_s A$ for every $A \subset_\infty N$ (because $(U - A) \cup V = U \cup V$), and for A nonrecursive no $B = {}^*A$ is small in A. (If so set $U = N$, and $V = A - B$ implying that $\bar{A}$ is r.e.) The terminology "small" was introduced by M. Stob [21] after he observed,

PROPOSITION 2.2. (*Stob*). (a) *If* $A \subset B \subset C \subset_\infty N$ *and either* $A \subset_s B$ *or* $B \subset_s C$ *then* $A \subset_s C$.

(b) *If* $A \subset_\infty C \subset_\infty N$ *and* $A \cup \bar{C}$ *is not r.e., and* A *is not recursive, then there exists* B *such that* $A \subset B \subset C$, $A \not\subset_s B$, *and* $B \not\subset_s C$.

Part (a) asserts not only that $\subset_s$ is transitive but also that small sets are closed downwards under inclusion, while (b) implies that no notion of A being "close" to C can force all intermediate sets B to be small in C.

PROPOSITION 2.3. *If* B *is small then* B *is not d-simple.*

Proof. Let $B \subset_s A \subset_\infty N$. Suppose that B is d-simple. Then by Definition 1.1 with $X = A$ there must exist $Y \subseteq X$ such that

(2.2) $$X \cap \bar{B} = Y \cap \bar{B}, \text{ and}$$

(2.3) $$(\forall Z)[(Z - X) \text{ infinite} \Longrightarrow (Z - Y) \cap B \neq \phi] .$$

Now in Definition 2.1 set $U = N$ and $V = Y$. By (2.2), $V \supseteq U \cap (A - B)$ and hence $(U - A) \cup Y$ is r.e. by (2.1). But then $Z = (U - A) \cup Y$ violates (2.3) because $(Z - Y) = (N - A)$ is infinite but fails to intersect B.

M. Stob [21] and, independently, E. Herrmann have shown the converse to be false by producing a simple set which is neither small nor d-simple. Essentially the same proof as in Proposition 2.3 establishes the following alternate characterization of d-simple sets which emphasizes their relationship to small sets.

PROPOSITION 2.4. *If* $A \subset_\infty N$ *then* A *is d-simple if and only if for all* $X \supseteq A$ *there exists* Y *such that*

(2.4) $$X \cap \bar{A} = Y \cap \bar{A}, \textit{ and}$$

(2.5) $\neg(\exists W)[(W - X)$ *infinite and* $(W - X) \cup Y$ *is* r.e.$]$.

To show that every degree $\boldsymbol{d} > \boldsymbol{0}$ contains a simple small (and hence not d-simple) set we recall some well-known results.

PROPOSITION 2.5. *For any simple set S and degree $\boldsymbol{d} > \boldsymbol{0}$ there is a simple set $A \subseteqq S$ of degree $\boldsymbol{d}$.*

Proof. See either [18, Theorem 3.10] or [8, Theorem 3.1].

DEFINITION 2.6. If $B \subset_{\infty} A$ then B is a *major subset of* A ($B \subset_m A$) if for all W,

$$W \cup A =^* N \Longrightarrow W \cup B =^* N .$$

Lachlan [3, Theorem 3] proved that every nonrecursive r.e. set A has a major subset B such that $B \subset_s A$ (written $B \subset_{sm} A$). Notice that the requirements of majoricity and smallness tend to conflict because $B \subset_m A$ requires B "close" to A while $B \subset_s A$ requires B "far away" from A.

COROLLARY 2.7. *For any simple set A and degree $\boldsymbol{d} > \boldsymbol{0}$ there is a small simple set $B \subseteqq A$ of degree $\boldsymbol{d}$.*

Proof. Given A simple find $M \subset_{sm} A$ by Lachlan [3, Theorem 3] and simple $B \subset M$ of degree $\boldsymbol{d}$ by Proposition 2.5. Now $B \subset_s A$ by Proposition 2.2 (a).

(Of course the result for B merely non d-simple and not necessarily small follows by the same proof without Proposition 2.2 and the notion of smallness using the downward closure of non-d-simple sets of Proposition 1.4.)

Notice that $\mathscr{D}$ does not coincide with any of the well-known classes of simple sets such as simple, hypersimple, hh-simple, or r-maximal, etc. (A coinfinite set A is *r-maximal* if there is no recursive set R such that $R \cap \bar{A}$ and $\bar{R} \cap \bar{A}$ are both infinite.) Other classes of simple sets are discussed in [19, § 3]. All of these but the simple and hypersimple sets exist only in high degrees while Theorem 1.7 produced a d-simple set which is both low and not hypersimple [14, p. 138] by the second clause of (1.4). Thus $\mathscr{D}$ is contained in the simple sets but in no other of the usual classes.

Furthermore, $\mathscr{D}$ does not contain any of these classes except for the hh-simple sets. Proper r-maximal (r-maximal, nonmaximal) sets may be either d-simple or not. If A is r-maximal and $B \subset_{sm} A$

then B is also r-maximal but small and hence not d-simple. On the other hand the usual constructions [5, p. 300] or [12, Theorem 6] of atomless r-maximal sets can easily be combined with the positive requirements P_e of Theorem 1.7 to produce an atomless r-maximal d-simple set. (We say that A is *atomless* if A has no maximal superset.)

R.A. Shore and the authors have noted that the Robinson construction [12, Theorem 6] and the Lachlan small major subset construction [3, Theorem 3] may be combined to produce a "small tower" $\{A_n\}_{n \in \omega}$ of simple sets such that

$$(\forall i)[A_i \subseteqq_s A_{i+1} \subset_\infty N] , \tag{2.6}$$

and

$$(\forall W)[\bar{A}_0 \subset^* W \text{ or } (\exists i)[W \subseteqq A_i]] . \tag{2.7}$$

Condition (2.7) guarantees that A_0 is r-maximal and atomless while (2.6) (together with Proposition 2.2 (a)) guarantees that all coinfinite supersets of A_0 are small and hence not d-simple.

COROLLARY 2.8. (*Lerman, Shore, Soare*). *The set of degrees containing a coinfinite* r.e. *set with no d-simple superset is exactly* $\boldsymbol{H}_1$.

Proof. For any $\boldsymbol{b} \in \boldsymbol{H}_1$, there exists $B \subset_m A_0$ of degree $\boldsymbol{b}$ by Lerman [6]. Now every coinfinite superset C of B is small because $B \subset_m A_0$ implies $\bar{A}_0 \not\subseteq^* C$, so $C \subset A_i$ for some i by (2.7). Thus, by Proposition 2.2 (a) B has no d-simple (or even nonsmall) superset.

On the other hand if $\deg(B) \notin \boldsymbol{H}_1$ then B has a d-simple superset D. By [10, p. 306] there is a recursive array $\{W_{f(n)}\}_{n \in \omega}$ of disjoint finite sets with union N and such that $|W_{f(n)} \cap \bar{B}| > n$ for all n. We build $D \supseteqq B$ to satisfy each positive requirement P_n of Theorem 1.7 (and thus be d-simple) by allowing any $x \in \bigcup_{m \geqq n} W_{f(n)}$ to serve as the witness for P_n. Since $\bigcup_{m<n} W_{f(n)}$ is finite, almost every element $x \in \bar{B}$ may serve as a witness for P_n so P_n is satisfied, yet at most n members of $W_{f(n)}$ are enumerated in D so $W_{f(n)} \cap \bar{D} \neq \phi$ and $\bar{D}$ is infinite.

3. On closure properties of d-simple sets. Many classes of simple sets such as simple sets, hypersimple sets, and hh-simple sets (although not r-maximal sets) are closed under intersection [14, pp. 122, 156, 251] and thus together with the cofinite sets form a filter in $\mathscr{E}$. This is not true for the d-simple sets. Indeed we show that there are d-simple sets C and D such that $(C \cap D) \subset_s (C \cup D) \subset_\infty N$, and thus $C \cap D$ is not d-simple. We also show that the

relation "d-simple in" is not transitive in contrast to the relation "simple in."

To prepare for these proofs we review Lachlan's strategy [3, p. 134] for ensuring $B \subset_s A$. Fix a recursive listing $\{\langle U_i, V_i\rangle\}_{i\in\omega}$ of all pairs of (r.e.) sets, and a simultaneous enumeration of these. We must meet for each i the negative requirement,

$$(3.1)\qquad N_i\colon V_i \supseteq U_i \cap (A - B) \Longrightarrow (U_i - A) \cup V_i \text{ is r.e.}$$

To accomplish this we attempt to enumerate a set T_i such that if $V_i \supseteq U_i \cap (A - B)$ then

$$(3.2)\qquad T_i \subseteq U_i \ \&\ T_i \supseteq U_i - A \ \&\ (T_i - V_i) \cap B =^* \phi\ ,$$

so that $T_i \cup V_i =^* (U_i - A) \cup V_i$ and the conclusion of (3.1) is satisfied.

Since $B \subset A$ we may assume that every element $x \in B$ is enumerated in A first. To control the enumeration of T_i we have a movable marker Γ_i whose position at the end of stage s, Γ^i_s, is the least $x \in (T^s_i - V^s_i) \cap (A^s - B^s)$ if x exists, and s otherwise. At stage $s + 1$: (1) in defining B^{s+1} all $x \in (T^s_i - V^s_i) \cap (A^{s+1} - B^s)$ are restrained with priority N_i from entering B, and N_i is *injured* if some x which it restrains enters B (say because of a positive requirement of higher priority); (2) after A^{s+1} and B^{s+1} are defined then every $x \in U^s_i - A_{s+1}$ such that $x \leq \Gamma^s_i$ is enumerated T_i. Clearly $T_i \subseteq U_i$. Now if $V_i \not\supseteq T_i \cap (A - B)$ then $V_i \not\supseteq U_i \cap (A - B)$ so (3.1) is automatically satisfied and furthermore $\lim_s \Gamma^s_i < \infty$, so T_i is finite and finitely many x are ever restrained by N_i. If $V_i \supseteq T_i \cap (A - B)$ then $\lim_s \Gamma^s_i = \infty$, so the first two clauses of (3.2) are met and every element x is restrained by N_i for at most finitely many stages. Furthermore, if N_i is injured at most finitely often then the third clause of (3.2) is also met and thus requirement N_i is met.

THEOREM 3.1. *The d-simple sets are not closed under intersection.*

Proof. We shall construct d-simple sets C and D such that $(C \cap D) \subset_s (C \cup D) \subset_\infty N$, whence $C \cap D$ is small and hence not d-simple. Let $\{\langle X_e, Z_e\rangle\}_{e\in\omega}$ and $\{\langle U_e, V_e\rangle\}_{e\in\omega}$ be two listings of all pairs of (r.e.) sets, and fix a simultaneous enumeration of these. To make C and D d-simple it suffices to meet the positive requirement P_e of (1.3) with C in place of A namely,

$$P^C_e\colon (Z_e - X_e) \text{ infinite} \Longrightarrow (\exists x)(\exists s)[x \in (Z^s_e - X^s_e) \cap (C^{s+1} - C^s)]\ ,$$

and similarly P^D_e with D in place of C. Let $A = C \cup D$ and $B = C \cap D$. To insure $B \subset_s A$ we simply meet for each i requirement N_i of

(3.1). The priority ranking of requirements is $\cdots, N_e, P_e^C, P_e^D, \cdots$. There are no restrictions on an element x *first* entering $C \cup D$ but once there it may not enter $C \cap D$ for some positive requirement until it is unrestrained by all negative requirements of higher priority.

Stage $s = 0$. Set $C^0 = D^0 = \phi$.

Stage $s + 1$. Given C^s and D^s set $A^s = C^s \cup D^s$ and $B^s = C^s \cap D^s$, and define T_i^s as above. Choose the positive requirement of highest priority which has never received attention and such that

$$(3.3) \qquad (\exists x)[x \in (Z_e^s - X_e^s) \ \& \ 3e < x \\ \& \ \neg (\exists i \leqq e)[x \in (T_i^s - V_i^s) \cap (A^s - B^s)] \,.$$

Choose x minimal for e. Now P_e receives attention and we enumerate x in C if P_e is P_e^C and in D if P_e is P_e^D. If e fails to exist do nothing. Let $C = \bigcup_s C^s$ and $D = \bigcup_s D^s$.

LEMMA 1. $(C \cap D) \subset_s (C \cup D) \subset_\infty N$.

Proof. Note that $(C \cup D) \subset_\infty N$ by the second clause of (3.3) and the fact that P_e^C or P_e^D contributes at most one element to $C \cup D$. Now by the third clause of (3.3) N_i is injured by P_e only if $e < i$. Thus, for each i, N_i is injured only finitely often and T_i satisfies the third clause of (3.2). Now if $V_i \supseteqq U_i \cap (A - B)$ then T_i satisfies (3.2) and N_i is met.

LEMMA 2. *C and D are d-simple.*

Proof. Fix e. Define $I = \{i \leqq e: \lim_s \Gamma_i^s < \infty\}$. Now $T = \cup \{T_i: i \in I\}$ is finite. If $i \leqq e$ and $i \notin I$ then $V_i \supseteqq T_i \cap (A - B)$ so no x is restrained by N_i at more than finitely many stages. Thus, for any $x \notin T$ there is a stage s_x such that for all $s \geqq s_x$, x is not restrained by any N_i, $i \leqq e$. Hence, requirements P_e^C and P_e^D are met.

Certain notions of simplicity are transitive when considered in relativized form. For example, if A is simple in B (i.e., $A \subseteqq B$ and $B - A$ is infinite and immune) and B is simple in C then A is simple in C. This is also true when "simple in" is replaced by "hypersimple in" or "*hh*-simple in" but not for "*r*-maximal in" or for "*d*-simple in." For $B \subset_\infty A$ the definition of B *d*-simple in A is obtained from Definition 1.1 by restricting all the quantifiers X, Y, Z in Definition 1.1 to be subsets of A.

THEOREM 3.2. *There exist sets $B \subset A \subset_\infty N$ such that B is d-simple in A, and A is d-simple but B is not d-simple (indeed $B \subset_s A$).*

Proof. Combine the construction of the preceding theorem with the following extra positive requirements Q_e which guarantee that B is d-simple in A,

$$Q_e: (Z_e - X_e) \cap A \text{ infinite} \Longrightarrow (\exists x)(\exists s)[x \in (Z_e^s - X_e^s) \cap (B^{s+1} - B^s)] .$$

Now as in (1.3) we can take $Y_e = (X_e \cap A) \backslash B$. The strategy for meeting Q_e is to enumerate in B^{s+1} some $x \in (Z_e^s \cap A^s) - (X_e^s \cup B^s)$. Since Q_e contributes at most one element to B and since the negative requirements N_i permanently restrain only finitely many elements of $A - B$ from entering B, these requirements Q_e can clearly be combined with all the previous requirements in the proof of Theorem 3.1 to obtain the extra conclusion.

4. Degrees of d-simple sets. We know by Corollaries 1.6 and 1.8 that all high degrees and some low degrees contain d-simple sets. It is natural to conjecture that there is a d-simple set in every nonzero degree since this is true for the other known classes of simple sets which intersect the low degrees. To our surprise we discovered that there is a degree $\boldsymbol{d} > \boldsymbol{0}$ such that every set of lower degree is not d-simple (and indeed is small if coinfinite). Thus $\boldsymbol{H}_1 \subset \boldsymbol{D}$ and $\boldsymbol{D}$ splits $\boldsymbol{L}_1$, and likewise for $\boldsymbol{D}$ replaced by $\boldsymbol{N}$, the degrees containing non-small sets. We do not know whether $\boldsymbol{D} \supseteq \boldsymbol{L}_1$ or whether $\boldsymbol{N} = \boldsymbol{D}$.

THEOREM 4.1. *There exists a simple set S such that every coinfinite set A recursive in S is small and hence not d-simple.*

Proof. We begin with a broad sketch of the proof and then give the detailed construction. We must make $\bar{S}$ infinite and meet for all j, a, and e the requirements,

$$P_j: W_j \cap S \neq \phi \text{ if } W_j \text{ is infinite, and}$$

$$R_{a,e}: W_a = \{e\}^S \text{ and } \bar{W}_a \text{ infinite} \Longrightarrow (\exists x_{a,e})[W_a \subset_s X_{a,e} \subset_\infty N] .$$

Given the hypotheses of $R_{a,e}$ we attempt to satisfy its conclusion as in § 3 by meeting for every i the negative requirement N_i of (3.1) with B and A replaced by W_a and $X_{a,e}$ respectively. Namely, requirement

(4.1) $\quad N_{a,e,i}: V_i \supseteq U_i \cap (X_{a,e} - W_a) \Longrightarrow (U_i - X_{a,e}) \cup V_i$ is r.e., where $\{\langle U_i, V_i\rangle\}_{i\in\omega}$ is a recursive list of all pairs of (r.e.) sets as in § 3. We accomplish this as in (3.2) by attempting to enumerate $T_{a,e,i}$ so that if the hypothesis of (4.1) is satisfied then

$$(4.2)\quad T_{a,e,i} \subseteq U_i \;\&\; T_{a,e,i} \supseteq U_i - X_{a,e} \;\&\; (T_{a,e,i} - V_i) \cap W_a = {}^*\phi\ ,$$

so that $T_{a,e,i} \cup V_i = {}^*(U_i - X_{a,e}) \cup V_i$ and the conclusion of (4.1) is satisfied. Define the recursive functions,

$$u(e, x, s) = \begin{cases} \min\{z: \{e\}_s^{S_s[z]}(x) \text{ is defined}\} & \text{if } z \text{ exists}\ ,\\ -1 & \text{otherwise}\end{cases}$$

$$l(a, e, s) = \max\{x: (\forall y \leq x)[W_{a,s}(y) = \{e\}_s^{S_s}(y)]\}\ .$$

The main obstacle in achieving $W_a \subset_s X_{a,e} \subset_\infty N$ is that unlike Theorem 3.1 where we controlled B, here W_a is being enumerated by the "opponent" [4] and so after we enumerate an element $x \in T_{a,e,i}$, the opponent may enumerate x in W_a before x appears in V_i thereby jeopardizing the final clause of (4.2). To overcome this obstacle and meet requirement $R_{a,e}$ in case $W_a = \{e\}^S$ and $\bar{W}_a$ is infinite we wait for some $x < l(a, e, s)$, such that $x \notin W_a^s \cup X_{a,e}^s$ and we assign a certain marker $\Lambda_{a,e,n}$ to x with the intention that the final positions $\{\Lambda_{a\ e,n}^\omega\}_{n\in\omega}$ of the markers will constitute $\bar{X}_{a,e}$ thereby ensuring $X_{a,e} \subset_\infty N$. Now if the opponent enumerates x in W_a, say at stage $t + 1 > s$, while $S_t[u] = S_s[u]$, where $u = u(e, x, s)$ then

$$(4.3)\qquad W_a^{t+1}(x) = 1 \text{ and } \{e\}_t^{S_t}(x) = 0\ .$$

We then preserve $S_t[u]$ with priority $R_{a,e}$ thereby preserving (4.3) and ensuring that $W_a \neq \{e\}^S$. This negative restraint for $R_{a,e}$ can be injured only by a positive requirement P_j such that $j < \langle a, e\rangle$. (Let $\Lambda_{a,e,n}^s$ denote the position of marker $\Lambda_{a,e,n}$ at the end of stage s.) Hence, for almost every $x = \Lambda_{a,e,n}^s$ we can safely assume that x will remain in $\bar{W}_a$ until we enumerate in S some $y \leq u(e, x, s)$.

Corresponding to each negative requirement $N_{a,e,i}$ we have a "gate" $G_{a,e,i}$ as in Lerman's pinball machine model [6]. The gates are arranged in ascending order according to index so $G_{a,e,i}$ lies below $G_{a',e',i'}$ just if $\langle a, e, i\rangle < \langle a', e', i'\rangle$.

Now suppose that requirement P_j wishes at stage $s + 1$ to enumerate some element y into S. We first consider all x such that $y \leq u(e, x, s)$, since such x may enter W_a when we put y into S. If some such $x = \Lambda_{a,e,n}^s$ for some $\langle a, e, n\rangle \leq j$, then we do nothing since P_j does not have enough priority to move marker $\Lambda_{a,e,n}$. If there is no such $\langle a, e, n\rangle$ (and a few other conditions below are satisfied), then we appoint y as a follower of P_j and we attempt to let y pass through all gates of higher priority than P_j, namely gates $G_{\langle a,e,i\rangle}$, $\langle a, e, i\rangle \leq j$.

If the follower y eventually reaches gate $G_{a,e,i}$ at some stage $t+1 \geqq s+1$ we enumerate in $X_{a,e}$ all elements $x \in E(a, e, i, y, j)$, which is a certain set defined at stage $t+1$ and consisting of most elements $x \in T^{t+1}_{a,e,i} - W^t_a$ but excluding $\{A^t_{a,e,n}: \langle a, e, n\rangle \leqq j\}$. Follower y is later *released* by $G_{a,e,i}$ at some stage $v+1 \geqq t+1$ if $E(a, e, i, y, j) \subseteqq V^v_i$, whereupon y passes to the next lower gate. The point is that no $x \in E(a, e, i, y, j)$ can violate the last clause of (4.2) if y is later enumerated in S because $x \in V_i$ already.

Now follower y is eventually either released by all gates and enters S at some stage $w \geqq v+1$, or y is cancelled, or y is a permanent resident of some gate $G_{a,e,i}$. In the latter case $V_i \not\supseteq T_{a,e,i} \cap (X_{a,e} - W_a)$, $T_{a,e,i}$ is finite and so there are finitely many permanent residents of $G_{a,e,i}$.

To see that this strategy succeeds in meeting $N_{a,e,i}$ we need to know that no new $x \notin E(a, e, i, y, j)$ is enumerated in $T_{a,e,i}$ between stages $v+1$ and w. This requires not just a single set $T_{a,e,i}$ but an infinite list of candidates $\{T_{a,e,i,p}\}_{p\in\omega}$, such that $T_{a,e,i,p}$ will be the true $T_{a,e,i}$ satisfying (4.2) just if p is the canonical index of the finite set of permanent residents at gates $G_{a',e',i'}$, $\langle a', e', i'\rangle \leqq \langle a, e, i\rangle$. (As in Rogers [13, p. 70] let D_p denote the finite set whose canonical index is p.)

If y and y' are followers of P_j and $P_{j'}$ respectively we say that y' has *lower order* than y if $j < j'$ or if $j = j'$ and y' was appointed after y was appointed. If y and y' are followers at stage s we shall arrange that $y < y'$ iff y' has lower order than y. A follower y of P_j once cancelled can later be appointed to follow $P_{j'}$ only if $j' < j$.

CONSTRUCTION.

Stage $s = 0$. Do nothing. Set $Z^0 = \phi$ for all sets Z, $M^0 = -1$ for all movable markers M, and $r(a, e, 0) = -1$ for all a and e.

Stage $s+1$. Perform in order the following steps.

Step 1. For all $\langle a, e\rangle \leqq s$, if either $r(a, e, s) = -1$ or there is a $y \in S^s - S^{s-1}$ such that $y < r(a, e, s)$, set $r'(a, e, s) = -1$. Otherwise, set $r'(a, e, s) = r(a, e, s)$ (whether or not $\langle a, e\rangle \leqq s$). If $r'(a, e, s) = -1$, and there exists $x \in W^s_a$ such that $\{e\}^{S_s}_s(x) = 0$, choose x minimal for a and e and set $r(a, e, s+1) = u(e, x, s)$. Otherwise, set $r(a, e, s+1) = r'(a, e, s)$. If $r(a, e, s+1) \neq r'(a, e, s+1) = -1$, then cancel every follower of every requirement P_j, $j \geqq \langle a, e\rangle$.

Step 2. If $r(a, e, s+1) > -1$ or $\langle a, e\rangle > s$, let $L^{s+1}_{a,e} = \phi$ and

$\Lambda^s_{a,e,n} = -1$ for all n. If $r(a, e, s+1) = -1$ let $\Lambda^{s+1}_{a,e,n}$ be the nth element of the set

$$L^{s+1}_{a,e} = \{x: x \in W^s_a \cup X^s_{a,e} \ \& \ x < l(a, e, s)\}$$

if such element exists and $\Lambda^{s+1}_{a\ e\ n} = -1$ otherwise. Define

$$F(j, s+1) = \{x: x = \Lambda^{s+1}_{a,e\ n} \ \& \ \langle a, e, n\rangle \leqq j\} .$$

If either $F(j, s+1) \neq F(j, s)$ or $u(e, x, s) \neq u(e, x, s \dot{-} 1)$ for some $x \in \bigcup_{t \leqq s+1} F(j, t)$ then cancel every follower of P_{j}, for all $j' \geqq j$.

Step 3. If follower y of requirement P_j is at gate $G_{a,e,i}$ and $u(e, x, s) \neq u(e, x, s \dot{-} 1)$ for some $x \in E(a, e, i, y, j)$ then cancel all followers $z \geqq y$.

Step 4. Define

$$\Gamma^{s+1}_{a\ e\ i} = \begin{cases} (\mu x)[x \in (X^s_{a,e} - W^s_a) \cap (T^s_{a,e\ i} - V^s_i)] & \text{if } x \text{ exists,} \\ s & \text{otherwise.} \end{cases}$$

Define

$$H(a, e, i, s+1) = \{y: y \text{ is a follower now at a gate } G_{a',e',i'} \text{ for some } \langle a', e', i'\rangle \leqq \langle a, e, i\rangle\} .$$

Choose p such that $D_p = H(a, e, i, s+1)$. Let

$$x = (\mu z)[z \leqq \Gamma^{s+1}_{a,e,i} \ \& \ z \in (U^s_i \cap L^{s+1}_{a\ e}) - T^s_{a,e,i,p}] .$$

Enumerate x in $T_{a,e,i,p}$. If x is not defined do nothing. Let $T^{s+1}_{a,e,i} = \bigcup_p T^{s+1}_{a,e,i,p}$.

Step 5. Requirement P_j *requires attention* if $W_{j,s} \cap S_s = \phi$, and one of the following two conditions holds.

Condition 1. A follower y of P_j now at some gate $G_{a,e,i}$ is *released* by $G_{a,e,i}$ namely

$$E(a, e, i, y, j) \subseteqq V^s_i \cup W^s_a .$$

Condition 2. All existing followers of P_j currently reside at gates, and there exists $y \in W^s_j$ such that $y > 2j$, $y >$ all previously appointed followers of requirements $P_{j'}$, $j' \leqq j$, and

$$(\forall x)[x \in \bigcup_{t \leqq s+1} F(j, t) \Longrightarrow u(e, x, s) < y], \tag{4.4}$$

and

(4.5) $$(\forall a)(\forall e)[\langle a, e\rangle \leqq j \Longrightarrow r(a, e, s+1) < y] .$$

If no P_j requires attention go to step 6. Otherwise, choose the least j such that P_j requires attention and the least y corresponding to P_j. Cancel all followers z of lower order than y, and adopt the first case below which holds.

Case 1. Condition 1 holds. If $\langle a, e, i\rangle = 0$ enumerate y in S. Otherwise move y to gate $G_{a',e',i'}$, the next gate below $G_{a,e,i}$. Let C be the set of all x such that $x \in E(a', e', i', y', j')$ for some follower $y' < y$ now residing at gate $G_{a',e',i'}$ and following some $P_{j'}$. Define

$$E(a', e', i', y, j) = \{x: x \in T^{s+1}_{a',e',i'} - \bigcup_{t \leqq s+1} F(j, t)\} - \{x: x \in C \ \& \ u(e', x, s) < y\} .$$

Enumerate in $X_{a',e'}$ all $x \in E(a', e', i', y, j)$.

Case 2. Condition 2 holds. Appoint y to follow P_j. Place y at gate $G_{a',e',i'}$, where $j = \langle a', e', i'\rangle$, and proceed as in Case 1.

Step 6. If $x \in W^s_a - W^{s-1}_a$ enumerate x in $X_{a,e}$.

This completes the construction at stage $s + 1$. Define $Z = \bigcup_s Z^s$ for each set Z mentioned above.

LEMMA 1. *For all e and x, $u(e, x) = \lim_s u(e, x, s)$ exists, and S is low* (i.e., $S' \equiv_T \phi'$).

Proof. This follows automatically from the restraint function r as in [18, Theorem 4.1 and Remark 4.5]. Fix e and x. Uniformly effectively in e and x we can choose a such that $W_a = \{0\}$ and b such that for every $\sigma \in \omega^{<\omega}$, $\{b\}^\sigma_s(0) = 0$ iff $\{e\}^\sigma_s(x)$ is defined. Some $y < r(a, b, s)$ can be put into S at stage $s+1$ by P_j only if $j < \langle a, b\rangle$, but each P_j contributes at most one such element to S, so $\lim_s r(a, b, s)$ exists. Now if $\{e\}^{S_s}_s(x)$ is defined for infinitely many s, then $\{b\}^{S_s}_s(0) = 0$ for infinitely many s and hence for cofinitely many s by step 1 and the definition of r. Thus, $\lim_s r(a, b, s) > -1$ and $\{e\}^S(x)$ is defined. Hence, $u(e, x) = \lim_s u(e, x, s)$ exists and is recursive in ϕ' by the Limit Lemma [15, p. 29]. Finally, S is low since $e \in S'$ iff $u(e, e) > -1$.

LEMMA 2. *For each gate $G_{a,e,i}$ there are at most finitely many followers y which reside permanently at $G_{a,e,i}$.*

Proof. Suppose follower y of P_j resides at $G_{a,e,i}$ at all stages

$\geqq s_0$. Then $E(a, e, i, y, j) \nsubseteq V_i \cup W_a$ so $(X_{a,e} - W_a) \cap T_{a,e,i} \nsubseteq V_i$, and hence $T_{a,e,i}$ is finite because of step 4. Choose $s_1 \geqq s_0$ such that $T^s_{a,e,i} = T_{a,e,i}$ and $u(e, x, s) = u(e, x)$ for all $x \in T_{a,e,i}$ and $s \geqq s_1$. Let $z_1 = \max\{u(e, x): x \in T_{a,e,i}\}$.

Now suppose for a contradiction that $G_{a,e,i}$ has infinitely many permanent residents. For each $m > 1$ let $y_m > z_1$ be a permanent resident of $G_{a,e,i}$ which follows some P_{j_m} and arrives at $G_{a,e,i}$ at some stage $s_m + 1 > s_1 + 1$. Let C_m be defined for y_m as in Case 1 of step 5 (with a, e, i in place of a', e', i'). Now since y_m is a permanent resident of $G_{a,e,i}$, $E(a, e, i, y_m, j_m) \neq \phi$, and so must contain an element $v_m \in T_{a,e,i} - C_m$. But the definition of C_m implies that $v_{m_1} \neq v_{m_2}$ for $m_1 \neq m_2$, so there can be finitely many such elements y_m because $T_{a,e,i}$ is finite.

LEMMA 3. *S is simple.*

Proof. First $\bar{S}$ is infinite since by step 5 Case 2 if y is appointed to follow P_j then $y > 2j$. It remains to show that for all j requirement P_j is met. Fix j and assume that for all $j' < j$, $P_{j'}$ is met and receives attention at most finitely often. Choose s_0 such that no $P_{j'}$, $j' < j$, requires attention after stage s_0. Now we can choose $s_1 > s_0$ such that $r(a, e, s) = r(a, e, s_1)$ for all $s \geqq s_1$ and all $\langle a, e\rangle \leqq j$, because $P_{j'}$ can contribute to S an element $y \leqq r(a, e, s)$ only if $j' < \langle a, e\rangle$.

Next we choose $s_2 > s_1$ such that $F(j, s_2) = F(j, s)$ for all $s \geqq s_2$. To see that this is possible fix $\langle a, e, n\rangle \leqq j$ and assume that for all $n' < n$, marker $\Lambda_{a,e,n'}$ does not move after stage $v > s_1$. Suppose $\Lambda^s_{a,e,n} = x > -1$ for some $s > v$, where x is minimal for all $s > v$. Then $r(a, e, t) = -1$ for all $t \geqq s_1$. Hence, by the choice of s_0 and cancellation of step 2 and (4.4), $u(e, x, t) = u(e, x, s)$ for all $t \geqq s$, and $\{e\}^{S_t}_t(x) = 0$ for all $t \geqq s$. Now x cannot be enumerated in W_a after stage s else step 1 later applies to $\langle a, e\rangle$ contrary to the choice of s_1. But x cannot be enumerated in $X_{a,e}$ at a stage $t+1 > s$ else step 5 applies to some $P_{j'}$, $j' < \langle a, e, n\rangle$, contrary to choice of s_0. Thus, $\Lambda^t_{a,e,n} = x$ for all $t \geqq s$, and s_2 exists. Set $F(j) = \bigcup_s F(j, s) = \bigcup_{s \leqq s_2} F(j, s)$.

Choose $s_3 \geqq s_2$ such that $u(e, x) = u(e, x, s)$ for all $s \geqq s_3$, $e \leqq j$, and $x \in F(j)$. Now after stage s_3 the cancellation of steps 1 and 2 cannot apply to P_j. By Lemma 2 there can be at most finitely many permanent residents $y_1, \cdots, y_m$ at gates $G_{a,e,i}$, $\langle a, e, i\rangle \leqq j$. Choose $z_1 > u(e, x)$ for every $x \in \bigcup\{E(a, e, i, y_k, j_k: 1 \leqq k \leqq m)\}$, where y_k permanently follows P_{j_k}. Choose $s_4 \geqq s_3$ such that $u(e, x, s) = u(e, x)$ for all $s \geqq s_4$ and $x \leqq z_1$, and such that each y_k, $k \leqq m$, has reached its final gate position by stage s_4.

Now suppose that W_j is infinite and $W_j \cap S = \phi$. Choose a stage $s_5 + 1 > s_4 + 1$ at which some follower y of P_j is appointed and such that no follower $y' < y$ ever receives attention after stage $s_5 + 1$. Now y cannot be cancelled at step 3 or step 5 by choice of s_4 and s_5 respectively. Note that when y reaches gate $G_{a,e,i}$, all residents of $G_{a,e,i}$ must be permanent. Hence, y can never be cancelled. But y is not a permanent resident of any gate $G_{a,e,i}$, $\langle a, e, i\rangle \leqq j$ so y eventually enters S. Hence, requirement P_j is met and receives attention at most finitely often.

LEMMA 4. *For all a and e requirement $R_{a,e}$ is met.*

Proof. Assume $W_a = \{e\}^S$ and $\bar{W}_a$ is infinite. Hence, by the proof of Lemma 3, for all a, e, and n, $\Lambda^{\omega}_{a,e,n} = \lim_s \Lambda^s_{a,e,n}$ exists and $\Lambda^{\omega}_{a,e,n} > -1$. Hence, $X_{a,e} \subset_{\infty} N$, and $W_a \subset X_{a,e}$ by step 6. To prove $W_a \subset_s X_{a,e}$ we must verify that requirement $N_{a,e,i}$ is met for all i.

Fix $\langle a, e, i\rangle$. Now assume the hypothesis of (4.1), namely $V_i \supset U_i \cap (X_{a,e} - W_a)$. Then $V_i \supset T_{a,e,i} \cap (X_{a,e} - W_a)$, and hence $\lim_s \Gamma^s_{a,e,i} = \infty$. By Lemma 2 choose p such that

$$D_p = \{y\colon y \text{ is a permanent resident of some gate } G_{a',e',i'}, \langle a', e', i'\rangle < \langle a, e, i\rangle\}\,.$$

We shall show that $T_{a,e,i,p}$ satisfies (4.2).

We call stage $s + 1$ a *nondeficiency* stage of the construction if some requirement P_j receives attention or is cancelled at stage $s+1$ and for all $j' \leqq j$ no follower y of $P_{j'}$ receives attention or is cancelled at any stage $t > s + 1$. Note that there are infinitely many nondeficiency stages, and at all sufficiently large nondeficiency stages s, $H(a, e, i, s) = D_p$ because whenever a follower is cancelled, all followers of lower order are cancelled, so any follower existing at stage s remains in existence and in its current position at all later stages. Therefore, each $x \in U_i - X_{a,e}$ will be eventually enumerated in $T_{a,e,i,p}$ at step 4 of some nondeficiency stage.

Thus, $T_{a,e,i,p}$ satisfies the first two clauses of (4.2). To see that it satisfies the final clause also, choose s_0 such that all followers $y \in D_p$ are in their final gate positions by stage s_0, no requirement P_j, $j < \langle a, e, i\rangle$, receives attention at any stage $s \geqq s_0$, and $r(a, e, s) = -1$ for all $s \geqq s_0$.

Now suppose $x \in T^{s+1}_{a,e,i,p} - T^s_{a,e,i,p}$ for some $s > s_0$. Then $x \in L^{s+1}_{a,e}$ and $\{e\}^{S_s}_s(x) = 0$. Now if $x \in W^{t+1}_a - W^t_a$ for some $t + 1 > s + 1$ then $y \in S^{v+1} - S^v$ for some $y \leqq u = u(e, x, v)$, and v such that $s \leqq v \leqq t$ else $S_t[u] = S_s[u]$, $\{e\}^{S_t}_t(x) = 0$, and step 1 applies to $\langle a, e\rangle$ after stage t contrary to the choice of s_0. Let y be the first such follower so that v is minimal. Then $u(e, x, s) = u(e, x, v)$. Suppose y follows P_j. Then $j > \langle a, e, i\rangle$ by choice of s_0. Now y cannot have been at a gate $G_{a',e',i'}$ for $\langle a', e', i'\rangle \leqq \langle a, e, i\rangle$ at stage $s+1$ because $y \notin D_p$. Therefore, y enters $G_{a,e,i}$ at some stage $s_1 + 1$, such that $s \leqq s_1 \leqq v$.

By step 5 Case 1 x is put into $E(a, e, i, y, j)$ at stage $s_1 + 1$ unless either: (1) $x \in \bigcup \{F(j, k): k \leqq s_1 + 1\}$; or (2) $u(e, x, s_1) < y$ and $x \in E(a, e, i, y', j')$ for some follower $y' < y$ which rests at $G_{a,e,i}$ at stage $s_1 + 1$ and follows some $P_{j'}$. But (1) cannot hold else $u(e, x, s_1) < y$ and hence $u(e, x, v) < y$ by (4.4) and because otherwise y is cancelled before stage $v + 1$ according to step 2. (Notice that (4.4) also rules out y being appointed too late.) Likewise, (2) cannot hold else $u(e, x, s_1) < y$ and hence $u(e, x, v) < y$ since otherwise y is cancelled before stage $v + 1$ according to step 3.

Therefore $x \in E(a, e, i, y, j)$ and x must have been released by $G_{a,e,i}$ at some stage $s_2 + 1$ such that $s_1 \leqq s_2 \leqq v$ at which time $x \in V_i^{s_a}$. Therefore $(T_{a,e,i,p} - V_i) \cap A = {}^*\phi$ and $T_{a,e,i,p}$ satisfies (4.2) so requirement $N_{a,e,i}$ is met.

5. Final remarks and open questions. In view of the close resemblance between non-d-simple sets and small sets we would like to know whether every non-d-simple set is small. The obvious attack that A non-d-simple via X implies $A \subset_s X$ fails. If these classes fail to coincide (as seems likely) is it at least true that $\boldsymbol{D} = \boldsymbol{N}$, the degrees containing non-small sets? The construction and proof in Theorem 4.1 strongly used the fact that S is low. Can this be extended to non-low degrees? In particular, is it true that $\boldsymbol{D} \supset \bar{\boldsymbol{L}}_1$? Is there any elegant description of $\boldsymbol{D}$ analogous to the definitions of $\boldsymbol{L}_n$ and $\boldsymbol{H}_n$?

A major open question is to find all the invariant classes of degrees and in particular to determine whether $\boldsymbol{H}_n$ and $\bar{\boldsymbol{L}}_n$ are invariant for every n. In particular, is $\bar{\boldsymbol{L}}_1$ invariant? If so one should be able to find a condition analogous to "atomless" and carry out the procedure of Lachlan [4] and Shoenfield [16]. After repeated attempts no such condition has emerged. If $\bar{\boldsymbol{L}}_1$ is not invariant then one ought to be able to prove using automorphisms that for any invariant class $\boldsymbol{C}$ if $\bar{\boldsymbol{L}}_2 \subsetneqq \boldsymbol{C}$ then $\boldsymbol{C} \cap \boldsymbol{L}_1 \neq \phi$. To do this one would hope to show that for any coinfinite set A such that $\deg(A) \in \boldsymbol{L}_2$ there exists $\Phi \in \operatorname{Aut} \mathscr{E}$ such that $\deg(\Phi(A)) \in \boldsymbol{L}_1$. However, by [20] this would imply that $\mathscr{L}(A) \cong \mathscr{E}$ for every such A and we have been unable to push Lachlan's construction [2, Theorem 4] to verify this. We do not even know whether for such an A and for any $\exists\forall\exists$-Boolean algebra $\mathscr{B}$ [2, p. 21] there exists $B \supseteqq A$ such that $\mathscr{L}^*(B) \cong \mathscr{B}$. This has been verified however [1, Theorem 4.8] for coinfinite A satisfying a lowness property similar to $\deg(A) \in \mathbf{L}_2$.

Finally, we are interested in the role of d-simplicity and its stronger versions in classifying the automorphism types of members of $\mathscr{E}$. If A and B are d-simple and low is A automorphic to B? What are sufficient conditions on A and B for (1.5) to hold? Which other definable classes $\mathscr{C} \subseteq \mathscr{E}$ besides maximal sets and infinite, coinfinite sets constitute orbits under Aut $\mathscr{E}$?

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