

Pacific Journal of Mathematics

**ON SOME FIXED POINT THEOREMS FOR MULTIVALUED
MAPPINGS**

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We give some fixed point theorems for multivalued non-expansive mappings or generalized contractions with non-compact domains in Banach spaces. First, we give a fixed point theorem for nonexpansive mappings that generalizes the results of Lami-Dozo, Assad-Kirk and Ko. Furthermore we give similar theorems for nonexpansive mappings or generalized contractions with nonconvex domains.

In 1976, Caristi [4] obtained fixed point theorems for weakly inward singlevalued mappings. The essential part of his proof is based on the following useful existence theorem.

THEOREM (*Browder* [2], *Caristi-Kirk* [3], *Caristi* [4], *Kirk* [9], *Siegel* [18] and *Wong* [19]). *Let X be a complete metric space and $f: X \rightarrow X$ an arbitrary mapping. Suppose there exists a lower semi-continuous mapping ψ of X into the nonnegative real numbers such that for each $x \in X$,*

$$d(x, f(x)) \leq \psi(x) - \psi(f(x)).$$

Then f has a fixed point in X .

Fixed point theorems for multivalued nonexpansive mappings are obtained by Assad-Kirk [1], Downing-Kirk [5], Itoh-Takahashi [8], Ko [10], Lami-Dozo [11], Lim [12, 13], Reich [15, 16, 17] and the other. Recently Downing-Kirk and Reich obtained some existence theorems containing the results of Lim by using the above theorem essentially. In this paper we shall give extensions of results of Lami-Dozo, Assad-Kirk and Ko by using similar method to Downing-Kirk and Reich. Furthermore we shall obtain similar results in the case of nonconvex domain. Now we shall introduce some necessary notations and definitions. Let X be a Banach space and K be a nonempty convex subset of X . If $x \in K$, we define the inward set of x relative to K , denoted $I_K(x)$ as follows:

$$I_K(x) = \{x + \alpha(y - x) \mid y \in K, \alpha \geq 1\}.$$

We say that a mapping $f: K \rightarrow X$ is *weakly inward* if $f(x)$ belongs to the closure of $I_K(x)$ for each $x \in K$. We denote by $\mathcal{CB}(X)$ the family of nonempty bounded closed subsets of X and denote by $\mathcal{K}(X)$ the family of nonempty compact subsets of X . For $A \in$

$\mathcal{CB}(X)$, we define $d(x, A) = \inf \{\|x - y\| \mid y \in A\}$. If $K \subset X$, $\text{cl}(K)$, $\text{int}(K)$ and ∂K will stand for the closure, interior and boundary of K , respectively. We write $x_n \rightharpoonup x$ to indicate that the sequence of vectors $\{x_n\}$ converges weakly to x ; as usual $x_n \rightarrow x$ will symbolize (strong) convergence.

DEFINITION 1. Let D be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the norm of X and let $K \in \mathcal{CB}(X)$. $T: K \in \mathcal{CB}(X)$ is said to be *nonexpansive* if $D(T(x), T(y)) \leq \|x - y\|$ for every $x, y \in K$. $T: K \rightarrow \mathcal{CB}(X)$ is said to be a *contraction* if for every $x, y \in K$, $D(T(x), T(y)) \leq k\|x - y\|$, where $0 \leq k < 1$. $T: K \rightarrow \mathcal{CB}(X)$ is said to be a *generalized contraction* if for each $x \in K$ there is a number $\alpha(x) < 1$ such that $D(T(x), T(y)) \leq \alpha(x)\|x - y\|$ for each $y \in K$.

DEFINITION 2. A Banach space X is said to satisfy *Opial's condition* if the following holds: If a sequence $\{x_n\}$ is weakly convergent to x in X and $x \neq y$, then

$$(*) \quad \liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|.$$

A Banach space X is said to satisfy *weak Opial's condition* if the following holds: If a sequence $\{x_n\}$ is weakly convergent to x in X , then for every y in X ,

$$(**) \quad \liminf_{n \rightarrow \infty} \|x_n - x\| \leq \liminf_{n \rightarrow \infty} \|x_n - y\|.$$

We remark that $(*)$ and $(**)$ are equivalent to $(*)'$ and $(**)'$, respectively (cf. [11]):

$$(*)' \quad \limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|,$$

$$(**)' \quad \limsup_{n \rightarrow \infty} \|x_n - x\| \leq \limsup_{n \rightarrow \infty} \|x_n - y\|.$$

Hilbert spaces and l^p ($1 \leq p < \infty$) satisfy Opial's condition and Banach spaces with weakly continuous duality mappings satisfy weak Opial's condition (cf. [14]).

DEFINITION 3. Let K be a convex set in X . $T: K \rightarrow \mathcal{CB}(X)$ is said to be *demiclosed* on K if $x_n \rightharpoonup x$, $y_n \rightarrow y$ and $y_n \in T(x_n)$ imply $y \in T(x)$. $T: K \rightarrow \mathcal{CB}(X)$ is said to be *semiconvex* on K if for any $x, y \in K$, $z = \lambda x + (1 - \lambda)y$, where $0 \leq \lambda \leq 1$, and any $x_1 \in T(x)$, $y_1 \in T(y)$, there exists $z_1 \in T(z)$ such that $\|z_1\| \leq \max \{\|x_1\|, \|y_1\|\}$.

PROPOSITION 1 (K_0 [10]). *Let K be a convex set in X and let $T: K \rightarrow \mathcal{CB}(X)$. If $I - T$ is semiconvex on K , then for any $x, y \in K$*

and $z = \lambda x + (1 - \lambda)y$, where $0 \leq \lambda \leq 1$, we have $d(z, T(z)) \leq \max \{d(x, T(x)), d(y, T(y))\}$.

PROPOSITION 2 (Ko [10], Downing-Kirk [5]). *Let K be a set in X . If $T: K \rightarrow \mathcal{CB}(X)$ is upper semicontinuous, then $d(x, T(x))$ is a lower semicontinuous mapping of K into the nonnegative real numbers.*

Before we obtain main theorems, we shall state the following result related to multivalued contractions.

PROPOSITION 3 (Downing-Kirk [5], Reich [17]). *Let K be a nonempty closed convex subset of X and let $T: K \rightarrow \mathcal{K}(X)$ be a contraction. If $T(x) \subset \text{cl}(I_K(x))$ for each $x \in K$, then T has a fixed point.*

We shall obtain the first theorem.

THEOREM 1. *Let K be a nonempty weakly compact convex subset of a Banach space X and let $T: K \rightarrow \mathcal{K}(X)$ be nonexpansive such that $T(x) \subset \text{cl}(I_K(x))$ for each $x \in K$. If $I - T$ is demiclosed or semiconvex on K , then T has a fixed point.*

Proof. Choose a point x_0 in K and a sequence $\{k_n\}$, $0 < k_n < 1$, that converges to 0. By Proposition 3, the mapping $T_n: K \rightarrow \mathcal{K}(X)$ defined by $T_n(x) = k_n x_0 + (1 - k_n)T(x)$ for all $x \in K$ has a fixed point x_n . Consequently there exists $y_n \in T(x_n)$ such that $x_n = k_n x_0 + (1 - k_n)y_n$. Suppose $I - T$ is demiclosed on K . Since K is weakly compact, there is a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in K$. Also

$$\|x_{n_i} - y_{n_i}\| = \frac{k_{n_i}}{1 - k_{n_i}} \|x_0 - x_{n_i}\| \longrightarrow 0.$$

Therefore $0 \in (I - T)(z)$, i.e., $z \in T(z)$. Suppose $I - T$ is semiconvex on K . We have $\inf \{d(x, T(x)) \mid x \in K\} = 0$ because

$$d(x_n, T(x_n)) \leq \|x_n - y_n\| = \frac{k_n}{1 - k_n} \|x_0 - x_n\| \longrightarrow 0.$$

Let $r > 0$, define $H_r = \{x \in K \mid d(x, T(x)) \leq r\}$. Since Proposition 1 and Proposition 2 imply that H_r are closed convex, H_r are weakly closed for every $r > 0$. The family $\{H_r \mid r > 0\}$ has the finite intersection property. Therefore, by the weak compactness of K , we have $\bigcap \{H_r \mid r > 0\} \neq \emptyset$. It is clear that any point in $\bigcap \{H_r \mid r > 0\}$ is a fixed point of T . \square

We obtain the following

COROLLARY 1. *Let K be a nonempty weakly compact convex subset of a Banach space X which satisfies Opial's condition (or weak Opial's condition). If $T: K \rightarrow \mathcal{K}(X)$ is nonexpansive (or a generalized contraction) such that $T(x) \subset \text{cl}(I_X(x))$ for each $x \in K$, then T has a fixed point.*

Proof. If X satisfies Opial's condition and T is nonexpansive, then $I - T$ is demiclosed on K by the result of Lami-Dozo. Therefore we show that $I - T$ is demiclosed on K if X satisfies weak Opial's condition and T is a generalized contraction. Suppose that $x_n \rightarrow x$, $y_n \rightarrow y$ and $y_n \in (I - T)(x_n)$. Hence there exists $u_n \in T(x_n)$ such that $y_n = x_n - u_n$. Since $T(x)$ is compact, there exists $v_n \in T(x)$ such that

$$\|v_n - u_n\| \leq D(T(x), T(x_n)) \leq \alpha(x)\|x - x_n\|.$$

Also there is a sequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i} \rightarrow v \in T(x)$. We have the following relation,

$$\begin{aligned} \alpha(x) \limsup_{i \rightarrow \infty} \|x_{n_i} - x\| &\geq \limsup_{i \rightarrow \infty} \|u_{n_i} - v_{n_i}\| \\ &= \limsup_{i \rightarrow \infty} \|x_{n_i} - y_{n_i} - v_{n_i}\| \\ &= \limsup_{i \rightarrow \infty} \|x_{n_i} - y - v + y - y_{n_i} + v - v_{n_i}\| \\ &\geq \limsup_{i \rightarrow \infty} (\|x_{n_i} - y - v\| - \|y_{n_i} - y\| - \|v_{n_i} - v\|) \\ &\geq \limsup_{i \rightarrow \infty} \|x_{n_i} - y - v\| - \limsup_{i \rightarrow \infty} \|y_{n_i} - y\| - \limsup_{i \rightarrow \infty} \|v_{n_i} - v\| \\ &= \limsup_{i \rightarrow \infty} \|x_{n_i} - y - v\|. \end{aligned}$$

Since $x_{n_i} \rightarrow x$ and X satisfies weak Opial's condition, we have $\limsup_{i \rightarrow \infty} \|x_{n_i} - x\| = 0$. Hence $x_{n_i} \rightarrow x$ and $x_{n_i} \rightarrow y + v$. Therefore $y = x - v \in (I - T)(x)$. \square

If K is compact in Theorem 1, we obtain the following

COROLLARY 2. *Let K be a nonempty compact convex subset of a Banach space X and let $T: K \rightarrow \mathcal{K}(X)$ be nonexpansive such that $T(x) \subset \text{cl}(I_K(x))$ for each $x \in K$. Then T has a fixed point.*

We shall obtain fixed point theorems for nonexpansive mappings or generalized contractions on starshaped subsets of Banach spaces.

DEFINITION 4. A subset K of a Banach space is called *starshaped* if there exists an element $x_0 \in K$ such that for $x \in K$ and $k(0 < k < 1)$, $kx_0 + (1 - k)x \in K$.

DEFINITION 5. For a subset K of a Banach space X and a bounded sequence $\{x_n\}$ in X , we define

$$AR(K, \{x_n\}) = \inf \left\{ \limsup_{n \rightarrow \infty} \|y - x_n\| \mid y \in K \right\}$$

and

$$A(K, \{x_n\}) = \left\{ z \in K \mid \limsup_{n \rightarrow \infty} \|z - x_n\| = AR(K, \{x_n\}) \right\}.$$

The set $A(K, \{x_n\})$ and the number $AR(K, \{x_n\})$ are called, respectively, the *asymptotic center* and the *asymptotic radius* of $\{x_n\}$ relative to K .

PROPOSITION 4. *The following hold:*

- (1) *If K is convex, then $A(K, \{x_n\})$ is convex;*
- (2) *if K is closed, then $A(K, \{x_n\})$ is closed;*
- (3) *if K is weakly compact, then $A(K, \{x_n\})$ is nonempty;*
- (4) *if X is uniformly convex and K is bounded closed convex, then $A(K, \{x_n\})$ consists of exactly one point;*
- (5) $A(K, \{x_n\}) \subset \partial K \cup A(X, \{x_n\})$;
- (6) *There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $AR(K, \{x_{n_i}\}) = AR(K, \{x_n\})$ and $A(K, \{x_{n_i}\}) \subset A(K, \{x_{n_j}\})$ for any subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$.*

Proof. (1), (2), (3) and (4) are clear (cf. [6]). We prove at first (5). Suppose that $A(K, \{x_n\}) \not\subset \partial K \cup A(X, \{x_n\})$. Then there exists $x \in \text{int}(K)$ such that $x \in A(K, \{x_n\})$ and $x \notin A(X, \{x_n\})$. We have

$$\begin{aligned} \inf \left\{ \limsup_{n \rightarrow \infty} \|y - x_n\| \mid y \in X \right\} &< \limsup_{n \rightarrow \infty} \|x - x_n\| \\ &= \inf \left\{ \limsup_{n \rightarrow \infty} \|y - x_n\| \mid y \in K \right\}. \end{aligned}$$

Hence there is $v \in X$ such that

$$\limsup_{n \rightarrow \infty} \|v - x_n\| < \inf \left\{ \limsup_{n \rightarrow \infty} \|y - x_n\| \mid y \in K \right\}.$$

Since $x \in \text{int}(K)$, there exists $\lambda \in (0, 1)$ such that $\lambda x + (1 - \lambda)v \in K$. Hence

$$\begin{aligned} \inf \left\{ \limsup_{n \rightarrow \infty} \|y - x_n\| \mid y \in K \right\} &\leq \limsup_{n \rightarrow \infty} \|\lambda x + (1 - \lambda)v - x_n\| \\ &\leq \lambda \limsup_{n \rightarrow \infty} \|x - x_n\| + (1 - \lambda) \limsup_{n \rightarrow \infty} \|v - x_n\|. \end{aligned}$$

Therefore $\limsup_{n \rightarrow \infty} \|x - x_n\| \leq \limsup_{n \rightarrow \infty} \|v - x_n\|$. This is a con-

tradiction. Next we show (6). By Lim [13, Proposition 1], there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $AR(K, \{x_{n_{i,j}}\}) = AR(K, \{x_{n_i}\})$ for any subsequence $\{x_{n_{i,j}}\}$ of $\{x_{n_i}\}$. Let $x \in A(K, \{x_{n_i}\})$. For any subsequence $\{x_{n_{i,j}}\}$ of $\{x_{n_i}\}$,

$$\begin{aligned} \limsup_{j \rightarrow \infty} \|x_{n_{i,j}} - x\| &\leq \limsup_{i \rightarrow \infty} \|x_{n_i} - x\| = AR(K, \{x_{n_i}\}) \\ &= AR(K, \{x_{n_{i,j}}\}) \leq \limsup_{j \rightarrow \infty} \|x_{n_{i,j}} - x\|. \end{aligned}$$

Hence $\limsup_{j \rightarrow \infty} \|x_{n_i} - x_j\| = AR(K, \{x_{n_{i,j}}\})$. Therefore $x \in A(K, \{x_{n_{i,j}}\})$. \square

We shall obtain the following theorem for nonexpansive mappings.

THEOREM 2. *Let K be a nonempty weakly compact starshaped subset of a uniformly convex Banach space X and let $T: K \rightarrow \mathcal{K}(X)$ be nonexpansive. If for each $x \in \partial K$, $T(x) \subset K$ and $\lambda x + (1 - \lambda)T(x) \subset K$ for some $\lambda \in (0, 1)$ or $T(x) \subset \text{int}(K)$, then T has a fixed point.*

Proof. Let x_0 be a starcenter and choose a sequence $\{k_n\}$, $0 < k_n < 1$, that converges to 0. By Assad-Kirk [1], the mapping $T_n: K \rightarrow \mathcal{K}(X)$ defined by $T_n(x) = k_n x_0 + (1 - k_n) T(x)$ for all $x \in K$, has a fixed point x_n . Consequently there exists $y_n \in T(x_n)$ such that $x_n = k_n x_0 + (1 - k_n) y_n$. Since $\{x_n\}$ is bounded, we can take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ as (6) in Proposition 4. We rewrite $\{x_{n_i}\}$ to $\{x_n\}$. Let $z \in A(K, \{x_n\})$. Since $T(z)$ is compact, there exists $z_n \in T(z)$ such that $\|z_n - y_n\| \leq D(T(z), T(x_n)) \leq \|z - x_n\|$, and there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightarrow \bar{z} \in T(z)$. By (6) in Proposition 4, $A(K, \{x_n\}) \subset A(K, \{x_{n_i}\})$. Hence $z \in A(K, \{x_{n_i}\})$. Since

$$\|x_{n_i} - y_{n_i}\| = \frac{k_{n_i}}{1 - k_{n_i}} \|x_0 - x_{n_i}\| \longrightarrow 0,$$

we have

$$\begin{aligned} \limsup_{i \rightarrow \infty} \|\bar{z} - x_{n_i}\| &\leq \limsup_{i \rightarrow \infty} \|\bar{z} - z_{n_i}\| + \limsup_{i \rightarrow \infty} \|z_{n_i} - y_{n_i}\| + \limsup_{i \rightarrow \infty} \|y_{n_i} - x_{n_i}\| \\ &= \limsup_{i \rightarrow \infty} \|z_{n_i} - y_{n_i}\| \\ &\leq \limsup_{i \rightarrow \infty} \|z - x_{n_i}\| = \inf \left\{ \limsup_{i \rightarrow \infty} \|y - x_{n_i}\| \mid y \in K \right\}. \end{aligned}$$

If $z \in \partial K$, then $w = \lambda z + (1 - \lambda)\bar{z} \in K$ for some $\lambda \in (0, 1)$ by hypothesis. Suppose that $z \neq \bar{z}$. By uniform convexity of X , we have for some $\delta \in (0, 1)$,

$$\limsup_{i \rightarrow \infty} \|w - x_{n_i}\| \leq (1 - \delta) \inf \left\{ \limsup_{i \rightarrow \infty} \|y - x_{n_i}\| \mid y \in K \right\}.$$

This contradicts the choice of w . If $z \in A(X, \{x_{n_i}\})$, we have

$$\begin{aligned} AR(X, \{x_{n_i}\}) &\leq \limsup_{i \rightarrow \infty} \|\bar{z} - x_{n_i}\| \\ &\leq \limsup_{i \rightarrow \infty} \|\bar{z} - z_{n_i}\| + \limsup_{i \rightarrow \infty} \|z_{n_i} - y_{n_i}\| + \limsup_{i \rightarrow \infty} \|y_{n_i} - x_{n_i}\| \\ &= \limsup_{i \rightarrow \infty} \|z_{n_i} - y_{n_i}\| \leq \limsup_{i \rightarrow \infty} \|z - x_{n_i}\| = AR(X, \{x_{n_i}\}). \end{aligned}$$

Hence $\bar{z} \in A(X, \{x_{n_i}\})$. By uniform convexity of X , we obtain $z = \bar{z} \in T(z)$. □

The following theorem for generalized contractions is obtained.

THEOREM 3. *Let K be a nonempty weakly compact starshaped subset of a Banach space X and $T: K \rightarrow \mathcal{K}(X)$ be a generalized contraction. If for each $x \in \partial K$, $T(x) \subset K$, then T has a fixed point.*

Proof. As in Theorem 2, we obtain $x_n \in K$ such that $x_n \in T_n(x_n)$. Consequently, there exists $y_n \in T(x_n)$ such that $x_n = k_n x_0 + (1 - k_n)y_n$. Since $\{x_n\}$ is bounded, we can take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ as (6) in Proposition 4. We rewrite $\{x_{n_i}\}$ to $\{x_n\}$. Let $z \in A(K, \{x_n\})$. Since $T(z)$ is compact, there exists $z_n \in T(z)$ such that

$$\|z_n - y_n\| \leq D(T(z), T(x_n)) \leq \alpha(z) \|z - x_n\|,$$

and there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightarrow \bar{z} \in T(z)$. Since $A(K, \{x_n\}) \subset A(K, \{x_{n_i}\})$, $z \in A(K, \{x_{n_i}\})$. Also

$$\|x_{n_i} - y_{n_i}\| = \frac{k_{n_i}}{1 - k_{n_i}} \|x_0 - x_{n_i}\| \longrightarrow 0.$$

If $z \in \partial K$, then $\bar{z} \in K$ by hypothesis. Hence

$$\begin{aligned} AR(K, \{x_{n_i}\}) &\leq \limsup_{i \rightarrow \infty} \|\bar{z} - x_{n_i}\| \\ &\leq \limsup_{i \rightarrow \infty} \|\bar{z} - z_{n_i}\| + \limsup_{i \rightarrow \infty} \|z_{n_i} - y_{n_i}\| + \limsup_{i \rightarrow \infty} \|y_{n_i} - x_{n_i}\| \\ &= \limsup_{i \rightarrow \infty} \|z_{n_i} - y_{n_i}\| \leq \limsup_{i \rightarrow \infty} \alpha(z) \|z - x_{n_i}\| \\ &= \alpha(z) AR(K, \{x_{n_i}\}). \end{aligned}$$

Since $1 - \alpha(z) > 0$, $AR(K, \{x_{n_i}\}) = 0$, which implies that $x_{n_i} \rightarrow \bar{z}$ and $x_{n_i} \rightarrow z$. Therefore $z = \bar{z} \in T(z)$. If $z \in A(X, \{x_{n_i}\})$, we have

$$\begin{aligned} AR(X, \{x_{n_i}\}) &\leq \limsup_{i \rightarrow \infty} \|\bar{z} - x_{n_i}\| \\ &\leq \limsup_{i \rightarrow \infty} \|\bar{z} - z_{n_i}\| + \limsup_{i \rightarrow \infty} \|z_{n_i} - y_{n_i}\| + \limsup_{i \rightarrow \infty} \|y_{n_i} - x_{n_i}\| \end{aligned}$$

$$\begin{aligned}
 &= \limsup_{i \rightarrow \infty} \|z_{n_i} - y_{n_i}\| \leq \limsup_{i \rightarrow \infty} \alpha(z) \|z - x_{n_i}\| \\
 &= \alpha(z) AR(X, \{x_{n_i}\}) .
 \end{aligned}$$

Since $1 - \alpha(z) > 0$, $AR(X, \{x_{n_i}\}) = 0$, which implies that $x_{n_i} \rightarrow \bar{z}$ and $x_{n_i} \rightarrow z$. Therefore $z = \bar{z} \in T(z)$. \square

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Received October 18, 1978.

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