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The real connective K-theory spectrum, bo, has been shown to be a useful spectrum in homotopy theory. In particular, the bo-homology Adams Spectral Sequence, based on the cofiber sequence

$$\text{(A)}\qquad \begin{array}{ccccccccccc} S^0 & \longleftarrow & \bar{bo} & & \longleftarrow & & \bar{bo}\wedge\bar{bo} & & \longleftarrow & & \bar{bo}\wedge\bar{bo}\wedge\bar{bo} \\ & \searrow\nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\ & bo & & & bo\wedge\bar{bo} & & & & bo\wedge\bar{bo}\wedge\bar{bo} & & \end{array}$$

has been used extensively by Mahowald in his work on the image of the J-homomorphism. One of the problems encountered with the bo-spectrum is that, unlike the mod 2 Eilenberg-Maclane spectrum, $bo\wedge bo$ does not split as a wedge of suspensions of bo itself. However, Mahowald and Milgram have obtained a splitting

$$\text{(B)}\qquad bo\wedge bo \simeq X \vee G$$

where X is a wedge of spaces intimately related with bo itself, and G is a wedge of mod 2 Eilenberg-MacLane spectra. In this paper, we determine the structure of G, i.e., we calculate the number of Eilenberg-Maclane summands occuring in each dimension.

This should moreover permit the complete analysis of the iterated smash products $bo \wedge \bar{bo} \wedge \cdots \wedge \bar{bo}$, which occur in (A).

A second consequence is obtained using the results of [3], namely that the mod 2 cohomology Adams Spectral Sequence converging to $[bo, bo]_*$ collapses. This means, in view of the change of rings arguments in [3] and [4], that we have in fact obtained a basis for the vector space $[bo, bo]_*/I$, where I denotes the ideal of self-maps of bo which lie in Adams filtration higher than 0. Since I is well understood, this is a significant improvement in the understanding of the ring of operations in bo-theory. It should be pointed out, though, that we only give a basis, without discussion of the multiplicative structure, which seems more difficult.

The method of calculation can be summarized as follows: Mahowald has obtained a splitting of $H^*(bo, Z/2)$ as an $\mathscr{A}_1$-module ($\mathscr{A}_1 = Z/2(Sq^1, Sq^2)$)

$$H^*(bo, Z/2) \cong \bigoplus_i M_i \oplus F$$

where M is a direct sum of indecomposable $\mathscr{A}_1$-modules, and F is a free $\mathscr{A}_1$-module. Since $H^*(bo, Z/2) \cong \mathscr{A}(2)/\mathscr{A}(2)\{Sq^1, Sq^2\}$, this gives a splitting

$$\begin{aligned} H^*(bo \wedge bo, Z/2) &\cong H^*(bo, Z/2) \otimes H^*(bo; Z/2) \\ &\cong \mathscr{A}(2) \bigotimes_{\mathscr{A}_1} H^*(bo, Z/2) \\ &\cong \mathscr{A}(2) \bigotimes_{\mathscr{A}_1} (\bigoplus_i M_i) \oplus \mathscr{A}(2) \bigotimes_{\mathscr{A}_1} F \end{aligned}$$

of $\mathscr{A}(2)$-modules, which Mahowald and Milgram showed, using Adams operations in bo-theory, corresponds to the splitting of spectra $bo \wedge bo \simeq X \vee G$. The first step in our calculation of the structure of G is the calculation of

$$Z/2 \bigotimes_{\mathscr{A}(2)} H^*(bo \wedge bo, Z/2) = Z/2 \bigotimes_{\mathscr{A}_1} H^*(bo, Z/2) .$$

It turns out that it is more convenient to study the dual situation, and the main steps (Theorems III. 8 and III. 10) describe

$$\begin{aligned} Y &= \{x \in H_*(bo, Z/2) \mid Sq^1x = Sq^2x = 0\} \\ &= (Z/2 \bigotimes_{\mathscr{A}_1} H^*(bo, Z/2))^* \end{aligned}$$

as a graded $Z/2$-vector space, where Sq^1 and Sq^2 are dual Steenrod operations. To solve for $Z/2 \otimes_{\mathscr{A}_1} F$, it will be sufficient to identify the image of $\bigoplus_i M_i$ in $Z/2 \otimes_{\mathscr{A}_1} H^*(bo, Z/2)$, since $Z/2 \otimes_{\mathscr{A}_1} F$ can then be identified with the quotient of $Z/2 \otimes_{\mathscr{A}_1} H^*(bo, Z/2)$ by that image. Finally $Z/2 \otimes_{\mathscr{A}_1} F$ determines F, since F is free.

The paper is organized as follows: § I consists of preliminary material on the Steenrod algebra $\mathscr{A}(2)$ and its dual. § II contains a description of $\mathscr{A}(2)/\mathscr{A}(2)Sq^1 = H^*(K(Z_{(2)}, 0), Z/2)$ as a Sq^1-module, which will be needed in § III. ($K(Z_{(2)}, 0)$) denotes the Eilenberg-Maclane spectrum for $Z_{(2)}$, the integers localized at 2). § III calculates the graded $Z/2$-vector space Y described above. The main theorems are III. 8 and III. 10. § IV is a brief section which states the result describing the image of $\bigoplus_i M_i$ in $Z/2 \otimes_{\mathscr{A}_1} H^*(bo, Z/2)$, which by the above discussion gives F. IV. 2 states the algebraic result, and IV. 3 and IV. 4 are the obvious interpretations in terms of the geometric splitting of $bo \wedge bo$ and cohomology operations in bo-theory.

I. Preliminaries.

Let $\mathscr{A}(2)$ denote the mod 2 Steenrod algebra. It is a Hopf algebra with comultiplication given by the Cartan formula

$$\Delta(Sq^i) = \sum_{k=0}^{i} Sq^{i-k} \otimes Sq^k .$$

Milnor [5] proves that as an algebra, the dual Hopf algebra to $\mathscr{A}(2)$, $\mathscr{A}(2)^*$, is given by

$$\mathscr{A}(2)^* \cong P(\xi_1, \xi_2, \cdots) ,$$

the $Z/2$-polynomial algebra on $2^i - 1$ dimensional generators ξ_i. (Henceforth, the symbol P will denote the $Z/2$-polynomial algebra on stated generators.) The Steenrod algebra admits a canonical anti-automorphism χ, which identifies it with its opposite algebra. According to Milnor, the comultiplication in $\mathscr{A}(2)^*$ is given by

$$\Delta(\xi_i) = \sum_{j=0}^{i} \xi_j^{2^{i-j}} \otimes \xi_{i-j} .$$

Since $\mathscr{A}(2)$ is isomorphic to its opposite algebra, we may instead use the "reversed" diagonal

$$\Delta(\xi_i) = \sum_{j=0}^{i} \xi_{i-j} \otimes \xi_j^{2^{i-j}} .$$

Since $\mathscr{A}(2)$ is acted on both on the right and on the left by the operations Sq^1 and Sq^2, increasing degree, $\mathscr{A}(2)^*$ is also acted on by Sq^1 and Sq^2, lowering degree. The action is determined by

(i) $Sq^1(\xi_i) = \xi_{i-1}^2 \forall i$

$$(\xi_i)Sq^1 = 0 \text{ unless } i = 1, \xi_1 Sq^1 = 1 .$$

(ii) $Sq^1(xy) = (Sq^1x)y + xSq^1y$

$$(xy)Sq^1 = (xSq^1)y + x(ySq^1) .$$

(iii) $Sq^2(\xi_k) = 0 \forall i,\ Sq^2(\xi_i^2) = \xi_{i-1}^4 \forall i.$

$$(\xi_1)Sq^2 = \xi_1,\ \xi_1^2 Sq^2 = 1,\ \xi_i Sq^2 = 0 \forall i \neq 2 ,$$
$$(\xi_i^2)Sq^2 = 0 \forall i \neq 1 .$$

(iv) $Sq^2(xy) = (Sq^2x)y + (Sq^1x)(Sq^1y) + x(Sq^2y)$

$$(xy)Sq^2 = (xSq^2)y + (xSq^1)(ySq^1) + x(ySq^2) .$$

Define a map σ; $P(\xi_1, \xi_2, \cdots) \to P(\xi_1, \xi_2, \cdots)$ by

$$\sigma(\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}) = \xi_2^{\alpha_1} \cdots \xi_{n+1}^{\alpha_n}$$
$$\sigma(1) = 1 .$$

This is a *nongraded* vector space endomorphism. Let $A = \mathscr{A}(2)/\mathscr{A}(2)Sq^1$, $B = \mathscr{A}(2)/\mathscr{A}(2)\{Sq^1, Sq^2\}$. A and B are left $\mathscr{A}(2)$-modules, hence their duals are left sub-comodules of $P(\xi_1, \xi_2, \cdots)$. Let $A^* = V$, $B^* = W$, and $\mathscr{A}(2)^* = U$. We quote from [2].

PROPOSITION 1.
(a) $V = P(\xi_1^2, \xi_2, \cdots)$.
(b) $W = P(\xi_1^4, \xi_2^2, \xi_3, \cdots)$.
Note that V and W are closed under the action of Sq^1 and Sq^2, and are therefore left $\mathscr{A}_1$-modules, where $\mathscr{A}_1$ is the subalgebra of $\mathscr{A}(2)$ generated by Sq^1 and Sq^2. The following lemma is immediate.

LEMMA 2.
(a) *$\sigma^j V$ is closed under the action of Sq^1.*
(b) *$\sigma^j W$ is closed under Sq^1 and Sq^2.*
(c) $Sq^1\sigma^j U \subseteq \sigma^{j-1}U$
$Sq^2\sigma^j U \subseteq \sigma^{j-1}U$.

Throughout this paper, we will be discussing graded vector spaces. All bases will be required to be *graded*, i.e., they should respect the grading. Consequently, the bases will be "graded sets", i.e., sets X together with a function d from X to the nonnegative integers. Of course, the isomorphism type of a basis as a graded set determines the isomorphism type of the graded vector space. Also, define the *suspension* of a graded vector space V, ΣV, to be V as a vector space, with the grading of all elements increased by one.

We recall from [6] that $H^*(bo, Z/2) \cong \mathscr{A}(2)/\mathscr{A}(2)\mathscr{A}_1$ and

$$H^*(K(Z(2), 0), Z/2) \cong \mathscr{A}(2)/\mathscr{A}(2)Sq^1 ,$$

so

$$H_*(bo, Z/2) = W,\ H_*(K(Z_{(2)}, 0), Z/2) = V .$$

II. Sq^1-calculations. By the results of § I, V is isomorphic as a left $\mathscr{A}_1$-module to $P(\xi_1^2, \xi_2, \cdots)$.

PROPOSITION 1. *Let $X = \{x \in V \mid Sq^1 x = 0\}$. Then a basis for X is given by the elements of the form*

$$\sigma_{j,k}(P) = \xi_j^{2k}P + \xi_j^{2k-2}\xi_{j+1}Sq^1P ,$$

where P is a monomial in $\sigma^j V = P(\xi_{j+1}^2, \xi_{j+2}, \xi_{j+3}, \cdots)$.

Proof. It is clear that $\sigma_{j,k}(P) \in X$, since $Sq^1(\sigma_{j,k}(P)) = Sq^1(\xi_j^{2k}P + \xi_j^{2k-2}\xi_{j+1}Sq^1P) = \xi_j^{2k}Sq^1P = 0$. Also, the $\sigma_{j,k}$'s form an independent set, since each involves only one monomial in $\sigma^j V$, and all these monomials are distinct. It remains to show that every element of X may be written as a linear combination of the $\sigma_{j,k}$'s.

Claim. If $\varphi \in \sigma^{j-1}U = P(\xi_j, \xi_{j+1}, \cdots)$, and $Sq^1\varphi = 0$, then $\varphi \in$

$\sigma^{j-1}V \subseteqq \sigma^{j-1}U$. For, $\varphi = \sum_s \xi_j^s\varphi_s$, $\varphi_s \in \sigma^j U$, and $Sq^1\varphi = \sum_s s\xi_{j-1}^2\xi_j^{s-1}\varphi_s + \xi_j^s Sq^1\varphi_s = \xi_{j-1}^2(\sum_s s\xi_j^{s-1}\varphi_s) + \sum_s \xi_j^s Sq^1\varphi_s$, and $Sq^1\varphi_s \in \sigma^{j-1}U$, hence $\varphi_s = 0$ for s odd.

The proof of the proposition will now be by induction. We will show that for $\varphi \in \sigma^{j-1}V$, with $Sq^1\varphi = 0$, there are polynomials $P_k \in \sigma^j U$ for which $\varphi + \sum_k \sigma_{j,k}(P_k) \in \sigma^j U$. By the claim, $\varphi + \sum_k \sigma_{j,k}(P_k) \in \sigma^j V$, so we may iterate the procedure, eventually obtaining an expression for φ in terms of elements $\sigma_{j,k}(P_k)$. We now prove the inductive step. φ may be written uniquely as

$$\varphi = \sum_{k=0}^{N} \xi_j^{2k}\varphi_k, \quad \varphi_k \in \sigma^j U ,$$

so

$$Sq^1\varphi = \sum_{k=0}^{N} \xi_j^{2k} Sq^1\varphi_k .$$

We claim $\varphi_N \in \sigma^j V$. For note that the power of ξ_j occurring in all the terms $\xi_j^{2k}Sq^1\varphi_k$, $k < N$, is less than or equal to $2N$. Let

$$\varphi_N = \sum_s \xi_{j+1}^s\psi_s, \quad \psi_s \in \sigma^{j+1}U ,$$

so

$$Sq^1\varphi_N = \xi_j^2(\sum_s s\xi_{j+1}^{s-1}\psi_s) + \sum_s \xi_{j+1}^s Sq^1\psi_s .$$

$Sq^1\psi_s \in \sigma^j U$, so the term in $Sq^1\varphi$ involving monomials in which the power of ξ_j occurring is $2N+2$ is precisely $\xi_j^{2N+2}(\sum_s s\xi_{j+1}^s\psi_s)$.

Since we assume $Sq^1\varphi = 0$, this term must be zero, so $\psi_s = 0$ for s odd, showing that $\varphi_N \in \sigma^j U$. Now consider $\tilde{\varphi} = \varphi + \sigma_{j,N}(\varphi_N) = \varphi + \xi_j^{2N}\varphi_N + \xi_j^{2N-2}\xi_{j+1}Sq^1\varphi_N$. $Sq^1\tilde{\varphi} = 0$, and $\tilde{\varphi}$ may be expressed as $\varphi = \sum_{k=0}^{N-1}\xi_j^{2k}\tilde{\varphi}_k$. After iterating this step $N-1$ times, we may write φ as $\alpha + \beta$, where $\alpha \in \sigma^j U$ and $\beta = \sum_s \sigma_{j,s}(P_s)$.

We finally observe that if φ involved only $\{\xi_j \mid j \leqq l\}$, then α and β could be chosen so that they also only involve only $\{\xi_j \mid j \leqq l\}$. Therefore, this procedure terminates, and we have proven the result (*).

We interpret this proposition as a description of the structure of X as a graded $Z/2$-vector space. Note that $\{\sigma^j X\}_{j=1}^{\infty}$ provides a filtration of X, and that each $\sigma^j X$ is graded compatibly with the grading of X. The inductive step in the proof of 1 showed that

$$im(X \longrightarrow X/\sigma X)$$

is isomorphic to $\bigoplus_{j=1}^{\infty} \xi_1^{2j}\sigma V$. Since it is clear that the associated graded version of X is isomorphic to X as a graded $Z/2$-vector space, we obtain

$$X \cong \sigma X \oplus \bigoplus_{j=1}^{\infty} \xi_1^{2j}\sigma V .$$

Since

$$\bigcap_{i=0}^{\infty} \sigma^i X = Z/2(1) ,$$

$$X \cong \bigoplus_{k=1}^{\infty} \bigoplus_{j=1}^{\infty} \xi_k^{2j}\sigma^k V \oplus Z/2(1) , \quad \text{or}$$

COROLLARY 2. *As a graded $Z/2$-vector space, X is isomorphic to the subalgebra of V consisting of all monomials $\prod_{i=0}^{t} \xi_{s+i}^{\alpha_i}$, such that α_0 and α_1 are multiples of 2, where α_0 is the first nonzero exponent, and 1.*

III. $\mathscr{A}(2)/\mathscr{A}(2)\bar{\mathscr{A}_1}$. In this section we will extend the techniques of § II to obtain the structure of

$$Y = \{x \in W \mid Sq^1 x = Sq^2 x = 0\}$$

as a graded $Z/2$-vector space.

We first note that there is a splitting of $Z/2$-vector spaces $W = \bigoplus_i W_i$, where

$$W_i = \xi_1^{4i}\cdot\sigma V .$$

Let $\Gamma_j = \bigoplus_{i=0}^{j} W_i$, so $\{\Gamma_j\}$ provides a filtration of W, with

$$\Gamma_j/\Gamma_{j-1} \cong \xi_1^{4j}\cdot\sigma V .$$

Define an operator

$$\phi\colon \sigma V \longrightarrow \sigma V$$

on monomials by $\phi(\xi_2^{2k}Q) = k\cdot\xi_2^{2k-2}Q$, $Q \in \sigma^2 U$, and extend by linearity.

LEMMA 1.

(a) $Sq^1 W_i \subseteq W_i$.

(b) $Sq^2\Gamma_j \subseteq \Gamma_{j+1}$, *and if $x \in \Gamma_j$, say $x = \sum_{i=0}^{j} \xi_1^{4i}P_i$, $P_i \in \sigma V$, then the projection of $Sq^2 x$ in Γ_{j+1}/Γ_j is $\xi_1^{4j+4}\phi(P_j)$.*

Proof.

(a) is clear since V is closed under the action of Sq^1 by Lemma I.2.a, and $Sq^1\xi_1^{4j} = 0$.

(b) We first calculate the action of Sq^2 on σV. Let $y \in \sigma V$,

$$y = \sum_s \xi_2^{2s}\psi_s, \quad \psi_s \in \sigma^2 U .$$

$$Sq^2 y = \sum_s s\xi_1^4\xi_2^{2s-2}\psi_s + \sum_s \xi_2^{2s} Sq^2\psi_s .$$

By LEMMA I.2.c, $Sq^2\psi_s \in \sigma V$, so we find that $Sq^2 y = \xi_1^4\phi(y) + \alpha$, where $\alpha \in \sigma V$. Now, if $x = \sum_{i=0}^{j} \xi_1^{4i} P_i$, $P_i \in \sigma V$, $Sq^2 x = \xi_1^{4j+4}\phi(P_j) + \beta$, where $\beta \in \Gamma_j$, and $\xi_1^{4j+4}\phi(P_j) \in \Gamma_{j+1}$, which proves the result, (*).

COROLLARY 2. *Let $x \in W$ be written uniquely as $x = \sum_{i=0}^{j} \xi_1^{4i} P_i$, $P_i \in \sigma V$, and suppose $Sq^1 x = Sq^2 x = 0$. Then*

(a) $Sq^1 P_i = 0$.

(b) $P_j \in \sigma W$.

Proof.

(a) is again clear since the splitting $W = \bigoplus_i W_i$ is preserved under Sq^1.

(b) x has been assumed to lie in Γ_j. Since $Sq^2 x = 0$, we must in particular have that the projection of $Sq^2 x$ in Γ_{j+1}/Γ_j is zero, so $\phi(P_j) = 0$. But $\phi(P_j) = 0 \Leftrightarrow P_j \in P(\xi_2^4, \xi_3, \xi_4, \cdots)$. We must show that $P_j \in \sigma W = P(\xi_2^4, \xi_3^2, \xi_4, \cdots)$. So, expand P_j as

$$P_j = \sum_k \xi_2^{4k} Q_k,\ Q_k \in \sigma^2 U .$$

Part (a) gives that $Sq^1 P_j = 0$, which implies $Sq^1 Q_k = 0 \forall k$. By the claim in the proof of Proposition II. 1, $Q_k \in \sigma^2 V = P(\xi_3^2, \xi_4, \cdots)$, proving (b)$\cdot$(*).

PROPOSITION 3. *For any $x \in \sigma W$, with $Sq^1 x = 0$, and any $j \geqq 2$, there is an element $\tilde{x} \in \Gamma_j$ with $Sq^1\tilde{x} = Sq^2\tilde{x} = 0$, and the projection of $\tilde{x}$ in Γ_j/Γ_{j-1} equal to $\xi_1^{4j}x$.*

Proof. Since $Sq^1Sq^1 = 0$, we may compute the homology of W under this differential. In [2], it is shown that

$$H_*(W; Sq^1) \cong P(\xi_1^4) .$$

By Lemma I.2.b, σW and W are isomorphic as $\mathscr{A}_1$-modules (although the isomorphism does not preserve grading). Thus

$$H_*(\sigma W; Sq^1) \cong P(\xi_2^4) .$$

For any Sq^1-homology generator, say $x = \xi_2^{4s}$, $\tilde{x} = \xi_1^{4j}\xi_2^{4s}$ satisfies $Sq^1\tilde{x} = Sq^2\tilde{x} = 0$, so we may suppose that x is a Sq^1-boundary, $x = Sq^1 y$. Now let

$$\begin{aligned} x = \xi_1^{4j} Sq^1 y &+ \xi_1^{4j-4}\xi_2^2 Sq^2 Sq^1 y + \xi_1^{4j-4}\xi_3 Sq^1 Sq^2 Sq^1 y \\ &+ \xi_1^{4j-8}\xi_2^2\xi_3 Sq^1 Sq^2 Sq^1 Sq^2 y + \xi_1^{4j-8}\xi_2^4 Sq^2 Sq^1 Sq^2 y . \end{aligned}$$

It is easy to check that $Sq^1\tilde{x} = Sq^2\tilde{x} = 0$, and the projection of $\tilde{x}$ in Γ_j/Γ_{j-1} is $\xi_1^{4j} Sq^1 y = \xi_1^{4j} x$. (*).

We must now examine the case $j = 1$.

PROPOSITION 4. *Let $x \in \sigma W$, with $Sq^1 x = 0$. Then there is $\tilde{x} \in \Gamma_1$ with $Sq^1\tilde{x} = Sq^2\tilde{x} = 0$, and the projection of $\tilde{x}$ in Γ_1/Γ_0 equal to $\xi_1^4 x$ if and only if $Sq^2Sq^1Sq^2x = 0$.*

Proof. Notice that if $Sq^2Sq^1Sq^2x = 0$, then the expression

$$\tilde{x} = \xi_1^4 x + \xi_2^2 Sq^2 x + \xi_3 Sq^1Sq^2 x$$

satisfies the conditions on $\tilde{x}$.

Conversely, suppose $\tilde{x}$ exists. Thus $x = \xi_1^4 x + \omega_0$, $\omega_0 \in \Gamma_0$, with $\omega_0 = \nu_0 + \xi_2^2\nu_1 + \xi_3\nu_2 + \xi_2^2\xi_3\nu_3$, where $\nu_j \in \sigma W$, and $Sq^1\omega_0 = 0$, $Sq^2\omega_0 = \xi_1^4 Sq^2 x$. But,

$$\begin{aligned} Sq^2\omega_0 = Sq^2\nu_0 &+ \xi_1^4\nu_1 + \xi_2^2 Sq^2\nu_1 + \xi_2^2 Sq^1\nu_2 \\ &+ \xi_3 Sq^2\nu_2 + \xi_1^4\xi_3\nu_3 + \xi_2^4 Sq^1\nu_3 + \xi_2^2\xi_3 Sq^2\nu_3 , \end{aligned}$$

so $\nu_1 = Sq^2 x$. Secondly,

$$\begin{aligned} 0 = Sq^1\omega_0 = Sq^1\nu_0 &+ \xi_2^2 Sq^1\nu_1 + \xi_2^2\nu_2 \\ &+ \xi_3 Sq^1\nu_2 + \xi_2^4\nu_3 + \xi_2^2\xi_3 Sq^1\nu_3, \text{ so } \nu_2 = Sq^1\nu_1 , \end{aligned}$$

and we have

$$\omega_0 = \nu_0 + \xi_2^2 Sq^2 x + \xi_3 Sq^1Sq^2 x + \xi_2^2\xi_3\nu_3 .$$

Using this reduction, we again calculate

$$Sq^2\omega_0 = Sq^2\nu_0 + \xi_2^4 Sq^2 x + \xi_3 Sq^2Sq^1Sq^2 x + \xi_1^4\xi_3\nu_3 + \xi_2^4 Sq^1\nu_3 + \xi_2^2\xi_3 Sq^2\nu_3 .$$
$$(Sq^1 x = 0, \text{ so } Sq^2Sq^2 x = Sq^1Sq^2Sq^1 x = 0.)$$

Thus, $\nu_3 = 0$, $Sq^1\nu_0 = Sq^2\nu_0 = 0$, and we must have

$$\xi_3 Sq^2Sq^1Sq^2 x = 0 \Longrightarrow Sq^2Sq^1Sq^2 x = 0. \quad (*) .$$

We will now construct various subspaces of W. Let

$$\begin{aligned} W_1 &= \{w \in W \mid Sq^1 w = Sq^2Sq^1Sq^2 w = 0\} \\ W_2 &= \{w \in W \mid Sq^1Sq^2 w = 0\} \\ W_3 &= \{w \in W \mid Sq^2 w = 0\} . \end{aligned}$$

Let $\pi_j\colon \Gamma_j \to \Gamma_j/\Gamma_{j-1}$ denote the projection.

PROPOSITION 5. *Let $j \geqq 1$, and let $x \in \sigma V$, so x has a unique expression as*

$$x = \nu_0 + \xi_2^2\nu_1 + \xi_3\nu_2 + \xi_2^2\xi_3\nu_3$$

with $\nu_i \in \sigma W$. *Then*

(a) $\exists \tilde{x} \in W_1 \cap \Gamma_j$ *with* $\pi_j(\tilde{x}) = \xi_1^{4j} x \Leftrightarrow \nu_3 = 0$ *and* $Sq^1 x = 0$.

(b) $\exists \tilde{x} \in W_2 \cap \Gamma_j$ *with* $\pi_j(\tilde{x}) = \xi_1^{4j} x \Leftrightarrow \nu_3 = 0$ *and* $Sq^1 \nu_1 = 0$.

(c) $\exists \tilde{x} \in W_3 \cap \Gamma_j$ *with* $\pi_j(\tilde{x}) = \xi_1^{4j} x \Leftrightarrow \nu_1, \nu_3 = 0$ *and* $Sq^1 \nu_2 = 0$.

Proof.

(a) First, observe that $Sq^2Sq^1Sq^2\Gamma_j \subseteqq \Gamma_{j+2}$, since $Sq^1\Gamma_j \subseteqq \Gamma_j$, $Sq^2\Gamma_j \subseteqq \Gamma_{j+1}$. Secondly, expanding $Sq^2Sq^1Sq^2(\xi_1^{4j}x)$ gives $Sq^2Sq^1Sq^2(\xi_1^{4j}x) = \xi_1^{4j+8}\nu_3 + \omega$, $\omega \in \Gamma_{j+1}$, implying that $\nu_3 = 0$. For the converse, suppose that $x = \nu_0 + \xi_2^2\nu_1 + \xi_3\nu_2$, $Sq^1x = 0$. Since $Sq^1x = 0$, we obtain $Sq^1\nu_0 = 0$, $\nu_2 = Sq^1\nu_1$. If ν_0 is a Sq^1-homology generator, ξ_2^{4s}, then $\xi_1^{4j}\nu_0 = \xi_1^{4j}\xi_2^{4s} \in W_1$. Thus, we may assume $\nu_0 = Sq^1y$. On the other hand, $\xi_2^2\nu_1 + \xi_3Sq^1\nu_1 = Sq^1(\xi_3\nu_1)$, so $x = Sq^1z$, $z = y + \xi_3\nu_1$.

Now let $\tilde{x} = \xi_1^{4j}x + \xi_1^{4j-4}\xi_2^2Sq^1Sq^2z$.

It is easily verified that $\tilde{x} \in W_1$.

(b) Observe that $Sq^1Sq^2\Gamma_j \subseteqq \Gamma_{j+1}$. We obtain

$$Sq^1Sq^2\xi_1^{4j}x = \xi_1^{4j+4}(Sq^1\nu_1 + \xi_2^2\nu_3 + \xi_3Sq^1\nu_3) + \omega ,$$

where $\omega \in \Gamma_j$, so $Sq^1\nu_1 = 0 = \nu_3$. Conversely, suppose $Sq^1\nu_1 = 0 = \nu_3$. If ν_1 is a Sq^1-homology generator, ξ_2^{4s}, then $\xi_1^{4j}\xi_2^2\xi_2^{4s} \in W_2$, so we may assume that ν_1 is a Sq^1-boundary, say $\nu_1 = Sq^1y$, hence $x = \nu_0 + \xi_2Sq^1y + \xi_3\nu_2$. Then if $\eta = \xi_1^{4j-4}(\xi_2^2Sq^2\nu_0 + \xi_2^2\xi_3Sq^1Sq^2Sq^1y + \xi_2^4Sq^1Sq^2y + \xi_2^2\xi_3Sq^2\nu_2 + \xi_2^4Sq^1\nu_2)$, $\xi_1^{4j}x + \eta \in W_2$, and $\pi_j(\xi_1^{4j}x + \eta) = \xi_1^{4j}x$.

(c) $Sq^2\Gamma_j \subseteqq \Gamma_{j+1}$, and $Sq^2(\xi_1^{4j}x) = \xi_1^{4j+4}(\nu_1 + \xi_3\nu_3) + \omega$, $\omega \in \Gamma_j$, so $\nu_1 = \nu_3 = 0$. If $\exists \tilde{x} \in \Gamma_j \cap W_3$, with $\pi_j(\tilde{x}) = \xi_1^{4j}x$, then there is an element

$$y = \mu_0 + \xi_2^2\mu_1 + \xi_3\mu_2 + \xi_2^2\xi_3\mu_3 ,$$

with $\mu_i \in \sigma W$, so that $\pi_j(Sq^2y) = \pi_j(\xi_1^4Sq^2x)$. Now,

$$Sq^2x = Sq^2\nu_0 + \xi_3Sq^2\nu_2 + \xi_2^2Sq^1\nu_2$$

and

$$\begin{aligned} Sq^2y = Sq^2\mu_0 + \xi_1^4\mu_1 + \xi_2^2Sq^2\mu_1 \\ + \xi_2^2Sq^1\mu_2 + \xi_3Sq^2\mu_2 + \xi_1^4\xi_3\mu_3 + \xi_2^4Sq^1\mu_3 + \xi_2^2\xi_3Sq^2\mu_3 . \end{aligned}$$

Sq^2y thus contains no coefficient of $\xi_1^4\xi_2^2$, hence $Sq^1\nu_2 = 0$. As usual, if ν_2 is a Sq^1-homology generator, ξ_2^{4s}, then $\xi_1^{4j}\xi_3\xi_2^{4s} \in W_3$, so we may assume $\nu_2 = Sq^1y$, and $x = \nu_0 + \xi_3Sq^1y$. Now if $\lambda = \xi_3Sq^2Sq^1\nu_0 + \xi_2^4Sq^2y + \xi_2^2Sq^2\nu_0 + \xi_2^2\xi_3Sq^2Sq^1y$, one may check that $\xi_1^{4j}x + \xi_1^{4j-4}\lambda \in W_3$, proving the proposition. (*).

PROPOSITION 6. *Let* $x \in \sigma V = \Gamma_0$, *with the* ν_j's *as in Proposition* 5.

(a) $x \in W_1 \Leftrightarrow \nu_3 = 0$, $Sq^1Sq^2\nu_1 = 0$, $\nu_2 = Sq^1\nu_1$, $\nu_0 \in \sigma W_1$.
(b) $x \in W_3 \Leftrightarrow \nu_1, \nu_3 = 0$, $Sq^1\nu_2 = Sq^2\nu_2 = 0$, *and* $\nu_0 \in \sigma W_3$.

Proof.
(a) The proof of Proposition 5.a shows that $\nu_3 = 0$, so

$$x = \nu_0 + \xi_2^2\nu_1 + \xi_3\nu_2 ,$$

and

$$Sq^1x = Sq^1\nu_0 + \xi_2^2Sq^1\nu_1 + \xi_2^2\nu_2 + \xi_3Sq^1\nu_2 .$$

Thus, $Sq^1\nu_0 = 0$, $Sq^1\nu_1 = \nu_2$. Now,

$$Sq^2Sq^1Sq^2x = Sq^2Sq^1Sq^2\nu_0 + \xi_1^4Sq^1Sq^2\nu_0 + \xi_3Sq^1Sq^2Sq^1Sq^2\nu_1$$

so $Sq^2Sq^1Sq^2\nu_0 = 0$, $Sq^1Sq^2\nu_1 = 0$. That these conditions imply $x \in W_1$ is clear.

(b) Expanding Sq^2x, the coefficients of ξ_1^4 and $\xi_1^4\xi_3$ are ν_1 and ν_3 respectively, so $\nu_1 = \nu_3 = 0$, and

$$x = \nu_0 + \xi_3\nu_2$$
$$Sq^2x = Sq^2\nu_0 + \xi_2^2Sq^1\nu_2 + \xi_3Sq^2\nu_2 ,$$

so

$$Sq^2\nu_0 = 0,\ Sq^1\nu_2 = 0,\ Sq^2\nu_2 = 0 .$$

Again, the converse is clear. (*).

LEMMA 7. *Define a subspace B of $\sigma V = \Gamma_0 = \sigma W + \xi_2^2\sigma W + \xi_3\sigma W + \xi_2^2\xi_3\sigma W$ by $B = \xi_3\sigma W + \xi_2^2\xi_3\sigma W$, so $\sigma V/B \cong \sigma W + \xi_2^2\sigma W$. Let $\pi: \sigma V \to \sigma V/B$ be the projection. Then $\nu_0 + \xi_2^2\nu_1 \in \pi(\sigma V \cap W_2) \Leftrightarrow \nu_0 \in \sigma W_2$, $Sq^1\nu_1 = 0$. Secondly, $\xi_3\nu_2 + \xi_2^2\xi_3\nu_3 \in W_2 \Leftrightarrow \nu_3 = 0$, $Sq^2\nu_2 = 0$.*

Proof. Let $x \in \sigma V$, $x = \nu_0 + \xi_2^2\nu_1 + \xi_3\nu_2 + \xi_2^2\xi_3\nu_3$. Then

$$\begin{aligned} Sq^1Sq^2x = {} & Sq^1Sq^2\nu_0 + \xi_1^4Sq^1\nu_1 \\ & + \xi_2^2Sq^1Sq^2\nu_1 + \xi_2^2Sq^2\nu_2 + \xi_3Sq^1Sq^2\nu_2 + \xi_1^4\xi_2^2\nu_3 \\ & + \xi_1^4\xi_3Sq^1\nu_3 + \xi_2^4Sq^2\nu_3 + \xi_2^2\xi_3Sq^1Sq^2\nu_3 . \end{aligned}$$

Thus, $Sq^1\nu_1 = 0$, $\nu_3 = 0$, $Sq^1Sq^2\nu_0 = 0$. Suppose $Sq^1Sq^2\nu_0 = 0$, $Sq^1\nu_1 = 0$. If ν_1 is a Sq^1-homology generator, ξ_2^{4s}, then $\xi_2^2\xi_3^{4s} \in W_2$, so we assume ν_1 to be a Sq^1-boundary, $\nu_1 = Sq^1z$. Now $\lambda = \nu_0+\xi_2^2\nu_1+\xi_3Sq^2z$ satisfies $\lambda \in W_2 \cap \Gamma_0$, $\pi(\lambda) = \nu_0 + \xi_2^2\nu_1$. For the second part, we have already observed that ν_3 is necessarily zero.

$$Sq^1Sq^2(\xi_3\nu_2) = \xi_2^2Sq^2\nu_2 + \xi_3Sq^1Sq^2\nu_2 ,$$

so $Sq^2\nu_2 = 0$. The converse is clear. (*)

We now interpret Propositions 3, 4, 5, and 6 as statements about the structure of the various graded vector spaces we have defined. Let $T = \{w \in W \mid Sq^1 w = 0\}$, so

$$T \cong \bigoplus_{i=0}^{\infty} \xi_1^{4i}\sigma X ,$$

where X is defined in § II.

As in § II, Propositions 3 and 4 give

$$\text{(a)} \qquad Y \cong \bigoplus_{j=2}^{\infty} \xi_1^{4j}\sigma T + \xi_1^4\sigma W_1 + \sigma Y$$

and Propositions 5, 6 and Lemma 7 give

$$\text{(b)} \qquad W_1 \cong \bigoplus_{j=1}^{\infty} \xi_1^{4j}(\sigma T + \xi_2^2\sigma W) + \xi_2^2\sigma W_2 + \sigma W_1 .$$

(For $Sq^1(x) = 0 \Longrightarrow \nu_2 = Sq^1\nu_1$.)

$$\text{(c)} \qquad W_2 \cong \bigoplus_{j=1}^{\infty} \xi_1^{4j}(\sigma W + \xi_2^2\sigma T + \xi_3\sigma W) + \xi_2^2\sigma T + \xi_3\sigma W_3 + \sigma W_2$$

$$\text{(d)} \qquad W_3 \cong \bigoplus_{j=1}^{\infty} \xi_1^{4j}(\sigma W + \xi_3\sigma T) + \xi_3\sigma Y + \sigma W_3 .$$

Solving these equations inductively, noting that

$$\bigcap_{k=0}^{\infty} \sigma^k Y = (1) ,$$

we obtain

$$\text{(a)} \qquad Y \cong \bigoplus_{k=1}^{\infty}\bigoplus_{j=2}^{\infty} \xi_k^{4j}\sigma^k T + \bigoplus_{k=1}^{\infty} \xi_k^4\sigma^k W_1 + Z/2\ (1)$$

$$\text{(b)} \qquad W_1 \cong \bigoplus_{k=1}^{\infty}\bigoplus_{j=1}^{\infty} \xi_k^{4j}(\sigma^k T + \xi_{k+1}^2\sigma^k W) + \bigoplus_{k=1}^{\infty} \xi_{k+1}^2\sigma^k W_2 + Z/2\ (1)$$

$$\text{(c)} \qquad W_2 \cong \bigoplus_{k=1}^{\infty}\bigoplus_{j=1}^{\infty} \xi_k^{4j}(\sigma^k W + \xi_{k+1}^2\sigma^k T + \xi_{k+2}\sigma^k W) + \bigoplus_{k=1}^{\infty} \xi_{k+1}^2\sigma^k T + \bigoplus_{k=1}^{\infty} \xi_{k+2}\sigma^k W_3 + Z/2\ (1)$$

$$\text{(d)} \qquad W_3 \cong \bigoplus_{k=1}^{\infty}\bigoplus_{j=1}^{\infty} \xi_k^{4j}(\sigma^k W + \xi_{j+2}\sigma^k T) + \bigoplus_{k=1}^{\infty} \xi_{k+2}\sigma^k Y + Z/2\ (1) .$$

By now substituting (d) in (c), (c) in (b), and (b) in (a), we obtain

$$Y \cong \bigoplus_{k=1}^{\infty}\bigoplus_{j=2}^{\infty} \xi_k^{4j}\sigma^k T + \bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\bigoplus_{j=1}^{\infty} \xi_k^4\xi_{l+k}^{4j}(\sigma^{l+k} T + \xi_{l+k+1}^2\sigma^{l+k} W)$$

$$\bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\bigoplus_{m=1}^{\infty}\bigoplus_{j=1}^{\infty}\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k}^{4j}(\sigma^{m+l+k}W+\xi_{m+l+k+1}^2\sigma^{m+l+k}T$$

$$+\xi_{m+l+k+2}\sigma^{m+l+k}W)+\bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\bigoplus_{m=1}^{\infty}\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k+1}^2\sigma^{m+l+k}T$$

$$+\bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\bigoplus_{m=1}^{\infty}\bigoplus_{n=1}^{\infty}\bigoplus_{j=1}^{\infty}\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k+2}\xi_{n+m+l+k}^{4j}(\sigma^{n+m+l+k}W$$

$$+\xi_{n+m+l+k+2}\sigma^{n+m+l+k}T)+Z/2(1)+\bigoplus_{k=1}^{\infty}\xi_k^4(Z/2(1))$$

$$+\bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\xi_k^4\xi_{l+k+1}^2(Z/2(1))+\bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\bigoplus_{m=1}^{\infty}\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k+2}(Z/2(1))$$

$$+\bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\bigoplus_{m=1}^{\infty}\bigoplus_{n=1}^{\infty}\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k+2}\xi_{n+m+l+k+2}\sigma^{n+m+l+k}Y\,.$$

In § II, we showed that as graded $Z/2$-vector spaces,

$$X\cong\bigoplus_{k=1}^{\infty}\xi_1^{2k}\sigma V+\sigma X\,,$$

so

$$X\cong\bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\xi_l^{2k}\sigma^l V+Z/2(1)\,.$$

This shows that the graded set

$$B=\{1\}\cup\{\xi_k^{2a}\xi_{k+1}^{2j}\sigma^{k+1}(\mu)\}_{\substack{a\geq 1\\ j\geq 0}}\,,$$

μ a monomial in U, is isomorphic to a basis for X. Since $T\cong\bigoplus_{j=0}^{\infty}\xi_1^{4j}\sigma X$, we obtain a basic C for T, namely

$$C=\{\xi_1^{4j}\sigma\beta\}_{\substack{j\geq 0\\ \beta\in B}}\,.$$

Let

$$Z=Z/2(1)+\bigoplus_{k=1}^{\infty}\xi_k^4(Z/2(1))+\bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\xi_k^4\xi_{l+k+1}^2(Z/2(1))$$

$$+\bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\bigoplus_{m=1}^{\infty}\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k+2}(Z/2(1))+\bigoplus_{k=1}^{\infty}\bigoplus_{j=2}^{\infty}\xi_k^{4j}\sigma^kT$$

$$+\bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\bigoplus_{j=1}^{\infty}\xi_k^4\xi_{l+k}^{4j}(\sigma^{l+k}T+\xi_{l+k+1}^2\sigma^{l+k}W)$$

$$+\bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\bigoplus_{m=1}^{\infty}\bigoplus_{j=1}^{\infty}\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k}^{4j}(\sigma^{m+l+k}W$$

$$+\xi_{m+l+k+1}^2\sigma^{m+l+k}T+\xi_{m+l+k+2}\sigma^{m+l+k}W)$$

$$+\bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\bigoplus_{m=1}^{\infty}\bigoplus_{n=1}^{\infty}\bigoplus_{j=1}^{\infty}\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k+2}\xi_{n+m+l+k}^{4j}$$

$$\cdot(\sigma^{n+m+l+k}W+\xi_{n+m+l+k+2}\sigma^{n+m+l+k}T)\,.$$

Using the basis C obtained for T above, and the monomial basis for W, we obtain

THEOREM 8. *A basis for Z is, as a graded set, isomorphic to the collection of all monomials of the following types:*

(i) $\underline{1,\ \xi_k^4,\ \xi_k^4\xi_{l+k+1}^2,\ \xi_k^4\xi_{l+k+1}^2\xi_{m+l+k+2},\ \xi_k^{4a+4}\xi_{k+1}^{4u}}$,
$\underline{\xi_k^4\xi_{l+k}^{4a}\xi_{l+k+1}^{4u},\ \xi_k^4\xi_{l+k+1}^2\xi_{m+l+k}^{4u}\xi_{m+l+k+1}^{4v+2}}$,
$\underline{\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k+2}\xi_{n+m+l+k}^{4a}\xi_{n+m+l+k+1}^{4u}\xi_{n+m+l+k+2}}$.

(ii) $\underline{\xi_k^{4a+4}\xi_{k+1}^{4u}\xi_{k+l+1}^{2b}\xi_{k+l+2}^{2v}\sigma^{k+l+2}(\mu)}$

(iii) $\underline{\xi_k^4\xi_{l+k}^{4a}\xi_{l+k+m+1}^{4u}\xi_{l+k+m+1}^{2b}\xi_{l+k+m+2}^{2v}\sigma^{l+k+m+2}(\mu)}$

(iv) $\underline{\xi_k^4\xi_{l+k}^{4a}\xi_{l+k+1}^{4u+2}\xi_{l+k+2}^{2v}\sigma^{l+k+3}(\mu)}$

(v) $\underline{\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k}^{4a}\xi_{m+l+k+1}^{4u}\sigma^{m+l+k+1}(\mu)}$

(vi) $\underline{\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k}^{4u}\xi_{m+l+k+1}^{4v+2}\xi_{n+m+l+k+1}^{2a}\xi_{n+m+l+k+2}^{2u}\sigma^{n+m+l+k+2}(\mu)}$

(vii) $\underline{\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k+2}\xi_{n+m+l+k}^{4a}\xi_{n+m+l+k+1}^{4u}\xi_{n+m+l+k+2}^{2v}\sigma^{n+m+l+k+2}(\mu)}$

(viii) $\underline{\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k+2}\xi_{n+m+l+k}^{4a}\xi_{n+m+l+k+1}^{4u}\xi_{n+m+l+k+2}^{2v}\sigma^{n+m+l+k+2}(\mu)}$

where μ is any momomial in U, $a, b \geqq 1$, and $u, v \geqq 0$.

(i) *above asserts that*

(ii) $Y \cong Z \oplus \bigoplus_{k=1}^{\infty}\bigoplus_{l=1}^{\infty}\bigoplus_{m=1}^{\infty}\bigoplus_{n=1}^{\infty}\xi_k^4\xi_{l+k+1}^2\xi_{m+l+k+2}\xi_{n+m+l+k+2}\sigma^{n+m+l+k+k}Y.$

DEFINITION 9. *A λ-sequence will be a collection $\alpha = \{i_s, j_s, k_s, l_s\}_{s=1}^{m}$ of integers satisfying $2 < i_s < j_s < k_s < l_s < i_{s+1}$. Given a λ-sequence α, we define $q(\alpha) = \prod_{s=1}^{m} \xi_{i_s-2}^4\xi_{j_s-1}^2\xi_{k_s}\xi_{l_s}$, and let $r(\alpha) = l_{s-2}$.*

(ii) now gives

THEOREM 10. *As a graded set, a basis for Y is given by*

$$\bigcup_{\alpha,\delta}\{q(\alpha(\sigma)\sigma^{(\alpha)}(\delta))\}\ ,$$

as α ranges over all λ-sequences and δ ranges over all monomials in Theorem 8.

IV. Relations with the Mahowald-Milgram splitting. We recall from [4] that as an $\mathscr{A}_1$-module, $W \cong \bigoplus_i M_i \oplus F$, where F is free and M_i is a certain $\mathscr{A}_1$-module. In order to obtain F, we must know the image of $Y \cap M_i$ in terms of the basis we have constructed for Y. This calculation is entirely straightforward, and we only state the result.

PROPOSITION 1. *Let Z be as in* § III. *Then $Z \cap (\bigoplus_i M_i)$ may be identified with the subspace spanned by all monomials of type* (i) *in Theorem* III. 8. *Moreover, $Y \cap (\bigoplus_i M_i)$ may be identified with the subspace spanned by*

$$\bigcup_{\alpha,\beta}\{q(\alpha)\sigma^{r(\alpha)}(\delta)\}\ ,$$

as α ranges over all λ-sequences, and δ ranges over all monomials of type (i).

This immediately gives

THEOREM 2. *A basis for F as a free, graded $\mathscr{A}_1$-module is given by the set*

$$\bigcup_{\alpha,\delta}\{q(\alpha)\sigma^{r(\alpha)}(\delta)\}$$

where α ranges over all λ-sequences, and δ ranges over all monomials of types (ii)–(viii) *in Theorem* III. 8.

From § I, $H^*(bo, Z_2) \cong W^*$, so $H^*(bo \wedge bo, Z_2) \cong \mathscr{A}(2)/\mathscr{A}(2)\bar{\mathscr{A}_1} \otimes Z/2\, W^* = \mathscr{A}(2) \otimes_{\mathscr{A}_1} W^*$. Thus the splitting of W^* as $\mathscr{A}_1$-modules tensors to a splitting of $H^*(bo \wedge bo, Z/2)$ as $\mathscr{A}(2)$-modules. In [4], it is shown that this algebraic splitting is actually a geometric splitting, and we obtain

COROLLARY 3. *$bo \wedge bo \cong X \bigvee_{\gamma\in\Gamma} \Sigma^{d(\gamma)}K(Z/2, 0)$ where Γ is the set of all monomials in Theorem 2, and $d(\gamma)$ denotes the degree of γ, and X is the spectrum mentioned in* (B) *of the introduction.*

In [3], it was shown that the Adams Spectral Sequence with E_2-term

$$\mathrm{Ext}^{**}_{\mathscr{A}(2)}(H^*(bo), H^*(bo)) ,$$

and converging to $[bo, bo]_*$, collapses. Thus, if $\mathscr{B}$ denotes the ring of self-maps $bo \to bo$, and I denotes the ideal of all maps which vanish in mod 2 cohomology,

$$\mathscr{B}/I \cong \mathrm{Hom}_{\mathscr{A}(2)}(H^*(bo), H^*(bo)) .$$

A standard change of rings result gives that as a graded $Z/2$-vector space.

$$\mathscr{B}/I \cong \mathrm{Hom}_{\mathscr{A}_1}(Z/2, H^*(bo))$$

which in turn is isomorphic to $\{x \in \mathscr{A}(2)/\mathscr{A}(2)\bar{a}_1 \mid Sq^1x = Sq^2x = 0\}$. Since we have a splitting of $\mathscr{A}(2)/\mathscr{A}(2)\bar{\mathscr{A}_1}$, the calculation in Theorem 2 gives

COROLLARY 4. *As a graded $Z/2$-vector space,*

$$\mathscr{B}/I \cong \bigoplus_i \mathrm{Hom}_{\mathscr{A}_i}(Z/2, M_i) \oplus \textstyle\sum^6 F .$$

REFERENCES

1. D. Anderson, Thesis, Berkeley, 1964.

2. D. Anderson, E. Brown and F. P. Petersen, *The structure of the spin cobordism ring*, Ann. of Math., **86** (1967), 217-298.

3. G. Carlsson, *Operations in Connective K-theory and Associated Cohomology Theories*, Thesis, Stanford, (1976).

4. R. J. Milgram, *The Steenrod algebra and its dual for connective K-theory*, Proceedings of the Northwestern Conference on Homotopy Theory, August, 1974. Ed. D. Davis (published by the Mexican Mathematical Society).

5. J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math., **67** (1958),.

6. R. Stong, *Determination of* $H^*(BO[k, \cdots, \infty), Z/2)$ *and* $H^*(Bu[k, \cdots, \infty). Z/2)$, Trans. Amer. Math. Soc., **107** (1963), 526-544.

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