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Suppose a locally compact group G (always second countable) has a Borel action on an analytic Borel space S so that each element of G transforms a given measure μ into an equivalent measure. If S_0 is the coset space for a closed subgroup H, then there is a natural action of G on S_0 which comes from translations of G on itself and there is such a quasi-invariant measure. Thus it is reasonable to think of such a space (S, μ), for some purposes, as a generalized sort of subgroup, or a virtual subgroup of G. In fact, the set $S \times G$ can be given algebraic and measure-theoretic structure so that many of the procedures used with subgroups can be carried over to this general setting. There is a general notion of virtual group, not necessarily "contained in" a group, which can be derived from this, and it turns out to include equivalence relations with suitable measures as a special case. These virtual groups appear in studying group representations, operator algebras, foliations, etc. Since there is a general setting for virtual groups, it seems desirable to see whether the intuitive idea of an action of a group as representing a subobject fits into this framework in a compatible way. The purpose of this paper is to show that "images" under homomorphisms, "kernels", etc. do fit together properly.

In this introduction we seek to summarize some of the motivation for the theory and give further explanation of the reasons for developing the results presented in the paper. Let G be a locally compact group, and let N be a closed normal subgroup. Let $\hat{N}$ (the *dual* of N) denote the space of equivalence classes of irreducible representations of N, with the Mackey Borel structure [3]. Suppose $\hat{N}$ is analytic, i.e., that N is a type I group [3]. This is the context of the paper of G.W. Mackey [12], in which he studied the problem of finding $\hat{G}$ in such a case. There is a natural action of G on representations of N: If L is a representation and $x \in G$, let $L^x(y) = (xyx^{-1})$ for $y \in N$. This gives a (right) Borel action of G on $\hat{N}$. If U is an irreducible representation of G, $U|N$ is a direct integral relative to an ergodic quasi-invariant measure on $\hat{N}$. Mackey confined his attention to the case in which for every ergodic quasi-invariant measure λ there is a conull orbit (one whose complement has measure 0). In this case we say the action is *essentially transitive* relative to λ. Mackey takes an arbitrary point in that orbit and his constructions are done with the closed subgroup of G

which stabilizes that point. Another point in the orbit will lead to a conjugate subgroup, and the results turn out to depend only on the orbit. There are many examples of pairs G, N for which there are ergodic quasi-invariant measures λ on $\hat{N}$ for which every G-orbit has measure zero [14, 20]. Then the class $[\lambda]$ of measures equivalent to λ is said to be a nontransitive quasi-orbit. This generalizes the notion of orbit in the same way that measure classes in general generalize the notion of subset of a set. For a nontransitive quasi-orbit, there is no subgroup which can be used to make the desired constructions. However, by introducing an algebraic structure in $\hat{N} \times G$, Mackey reformulated the essentially transitive case in a way which is meaningful in the general case.

Suppose S is any (right) G-space. Then for (s_1, x_1) and (s_2, x_2) in $S \times G$, the product is defined exactly when $s_1x_1 = s_2$, and then the result is (s_1, x_1x_2). Thus, only some pairs have a product, and the action determines which ones, while the group product from G shows itself in the formula for the product. With this product, $S \times G$ is a groupoid, i.e., a small category with inverses. There are several ways to formulate the definition of "groupoid", and we have chosen the following one for its intuitive content, preferring to think of the elements of a groupoid as abstractions of isomorphisms, i.e., mappings between objects of some type.

DEFINITION. A groupoid is a set F with a subset $F^{(0)}$ (of units), a pair of functions $d, r: F \to F^{(0)}$ (domain and range) and a product xy defined for pairs (x, y) in $F^{(2)} = \{(a, b) \in F \times F: d(a) = r(b)\}$. These must satisfy the following:

(a) (associativity) $d(xy) = d(y)$ and $r(xy) = r(x)$, and if $d(x) = r(y)$ and $d(y) = r(z)$, then $(xy)z = x(yz)$.

(b) (units) If $u \in F^{(0)}$ then $u = d(u) = r(u)$. If $u = d(x)$ then $xu = x$, while if $v = r(x)$ then $vx = x$.

(c) (inverses) For each $x \in F$ there is a y with $xy = r(x)$, $yx = d(x)$.

Notice that a groupoid for which there is only one unit is a group. The y of part (c) is unique and denoted x^{-1}. In a concrete small category with inverses, i.e., a groupoid of isomorphisms, the units are the identity mappings of the various objects. For $F = S \times G$, $r(s, x) = (s, e)$, $d(s, x) = (sx, e)$, and $(s, x)^{-1} = (sx, x^{-1})$.

If we identity S with $S \times \{e\}$, we can regard r and d as maps of $S \times G$ into S. Then points s_1, s_2 in S are in the same orbit under G iff there is an x in G with $s_1x = s_2$ iff there is an element z in F with $r(z) = s_1$ and $d(z) = s_2 (z = (s_1, x))$. In general, we call units u, v *equivalent* if there is an x with $r(x) = u, d(x) = v$. A set of units is called *saturated* if it is a union of equivalence classes. For a set A of units, its saturation $[A]$ is $r(d^{-1}(A)) = d(r^{-1}(A))$.

Besides $S \times G$ there are many ways to construct groupoids (we give a few):

Example 1. Let S be a set of groups and let F be the set of isomorphisms with domain and range elements of S. If we want the multiplication to be function composition, the other parts of the structure follow naturally, and the equivalence classes are isomorphism classes.

EXAMPLE 2. Let S be a partition of a set A, i.e., a collection of nonempty disjoint sets, and let F be the set of bijections between elements of S.

EXAMPLE 3. Let F be an equivalence relation on a set S (say a foliation on a manifold S). Define $F^{(0)}$ = diagonal, $r(x, y) = (x, x)$, $d(x, y) = (y, y)$, $(x, y)(y, z) = (x, z)$. Then $(x, y)^{-1} = (y, x)$.

Return now to the case of $S \times G$ and suppose S is the space of right cosets of a closed subgroup H, with s_0 the identity coset. There is a Borel $\gamma: S \to G$ so that $\gamma(s) \in S$ for each $s \in S$, and $\gamma(s_0) = e$. Define $\psi(s, x) = \gamma(s)x\gamma(sx)^{-1}$ for $(s, x) \in S \times G$ and $\varphi(h) = (s_0, h)$ for $h \in H$. Then $\psi: S \times G \to H$ and $\varphi: H \to S \times G$ are groupoid homomorphisms. Hence, if L is a representation of H then $L \circ \psi$ is a representation of $S \times G$, and if R is a representation of $S \times G$, then $R \circ \varphi$ is a representation of H. The pair (φ, ψ) establishes a kind of equivalence between H and $S \times G$ of which one consequence is the passing back and forth of representations. This equivalence is called *similarity* [15], which is defined as follows. Let F_1, F_2 be groupoids. A function $\varphi: F_1 \to F_2$ is a homomorphism if when xy is defined so is $\varphi(x)\varphi(y)$ and it equals $\varphi(xy)$. If $\varphi_1, \varphi_2: F_1 \to F_2$ are homomorphisms, a *similarity* of φ_1 and φ_2 is a function $\theta: F_1^{(0)} \to F_2$ such that for each $x \in F_1$, $\theta(r(x))\varphi_1(x)$ and $\varphi_2(x)\theta(d(x))$ are defined and equal. We write $\varphi_1 \approx \varphi_2$, and $[\varphi_1]$ is the similarity class of φ_1. (If we think of F_1 and F_2 as categories, a homomorphism is a functor and a similarity of homomorphisms is a natural equivalence of functors.) If F_1 and F_2 are groups, homomorphisms are the same, and since F_1 has only one unit, for similarity we simply have an element $a = \theta(e)$ in F_2 such that $\varphi_2(x) = a\varphi_1(x)a^{-1}$. F_1 and F_2 are *similar* if there are homomorphisms $\varphi_1: F_1 \to F_2$ and $\varphi_2: F_2 \to F$ such that $\varphi_1 \circ \varphi_2 \approx i_{F_1}$ and $\varphi_2 \circ \varphi_1 \approx i_{F_2}$.

In the example above, (φ, ψ) is a similarity of H with $S \times G$. In fact $\psi \circ \varphi = i_H$ = identity on H, but $\varphi \circ \psi(s, x) = (s_0, \gamma(s)x\gamma(sx)^{-1})$. If we define $\theta(s) = (s_0, \gamma(s))$, then $\varphi \circ \psi(s, x)\theta(sx) = \theta(s)(s, x)$. (Thus θ is a natural equivalence of $\varphi \circ \psi$ with the identity "functor" on $S \times G$.)

A consequence of this similarity is that the function taking L to $L\circ\psi$ induces a one-one map of equivalence classes of representations of H onto the same for $S\times G$. (Two representations R_1, R_2 of $S\times G$ are *equivalent* if there is a unitary operator valued function V on S such that $V(s)R_1(s, x) = R_2(s, x)V(sx)$ always.) This is important for the theory of virtual groups, because it is part of the basic pattern of using "virtual subgroup" (next paragraph) to extend the subgroup concept. We want the results to be consistent with the subgroup results in case the G-space is transitive.

Mackey used this connection between H and $S\times G$ even to derive a definition of homomorphism. If H_1 is a subgroup of G_1 with coset space S_1, and H_2, G_2, S_2 are another such triple, then we have γ_1, φ_1, ψ_1 and γ_2, φ_2, ψ_2. If $\varphi\colon H_1\to H_2$ is a homomorphism then $\varphi_2\circ\varphi\circ\psi_1$ should be a homomorphism and if $\varphi'\colon S_1\times G_1\to S_2\times G_2$ is a homomorphism then $\psi_2\circ\varphi'\circ\varphi_1$ should be a homomorphism. The result is the one we used above. Now we want to use this to get the "virtual subgroup" idea. If $i(h) = h$ for $h\in H$, then $i\circ\psi = \psi$. Thus ψ is related to the inclusion of H into G. If we define $j_S(s, x) = x$, then for $(s, x)\in S\times G$ we have $\psi(s, x)\gamma(sx) = \gamma(s)j_S(s, x)$. Thus ψ and j_S are similar. Now j_S makes sense in general, although ψ does not, and the notion of similarity of homomorphism allows us to think of j_S, or rather $[j_S]$, as an inclusion in general, and $S\times G$ as a *virtual subgroup* of G. To carry this one more step, suppose subgroups H_1 and H_2 have coset spaces S_1 and S_2. If $H_1\subseteq H_2$, $p(H_1a) = H_2a$ defines a G-equivariant map of S_1 onto S_2, and $j(s, x) = (p(s), x)$ corresponds to the inclusion of H_1 into H_2. Thus we define $S_1\times G$ to be "contained" in $S_2\times G$ if there is an equivariant map p of S_1 onto S_2. In section 5 of this paper we consider another way to define "subobject", also derived from group theory, and one purpose of the paper is to show the two ways agree. So far, we have arrived at a category of groupoids in which the maps are similarity classes of homomorphisms, so that $[j_S]$ is an "inclusion".

For a coset space S, $S\times G$ also has topological and measure theoretic structures. In this paper we are mainly concerned with the latter, and recall here some of the facts. It is known that on a coset space S there is exactly one quasi-invariant (σ-finite) measure, up to equivalence. If ν is a probability measure in the class of Haar measure on G, and s_0 is the identity coset, then we can define $\mu(A) = \nu(\{x\in G\colon s_0x\in A\})$ to get a quasi-invariant μ. Then $\mu\times\nu$ is quasi-invariant under $(s, x)\to(s, x)^{-1} = (sx, x^{-1})$ (Fubini). Now $\mu\times\nu = \int\varepsilon_s\times\nu\,d\mu(s)$, where ε_s denotes a unit point mass at s, and $\mu = r_*(\mu\times\nu)$. Thus we have $\mu\times\nu$ decomposed relative to r over

$r_*(\mu \times \nu)$. Since ν is quasi-invariant under left translation, for any (s, x), the map taking (sx, y) to $(s, xy) = (s, x)(sx, y)$ carries $\varepsilon_{sx} \times \nu$ to a measure equivalent to $\varepsilon_s \times \nu$. These properties suggest the measure theoretic structure we will use in the general case. If we take $F = S \times G$ and $\lambda = \mu \times \nu$, then (F, λ) is a measured groupoid in the sense of the definition below.

It is convenient to denote the equivalence class of a measure μ by $[\mu]$. Then a measure μ is quasi-invariant iff $[\mu]$ is invariant as a set. Suppose F is an analytic Borel groupoid, i.e., it is analytic as a Borel space and $r, d, (\ )^{-1}$ and multiplication are Borel functions. Let λ be a probability measure on F, and denote by $(\lambda)^{-1}$ the measure whose value at a Borel set A is $\lambda(\{x^{-1}: x \in A\})$. We say that λ is *quasi-symmetric* if $(\lambda)^{-1} \sim \lambda$; this is true iff $[\lambda]$ is symmetric, and iff there is a symmetric $\lambda_1 \sim \lambda$ (take $\lambda_1 = 1/2(\lambda + (\lambda)^{-1})$). Now let $\tilde{\lambda} = r_*(\lambda)$ be the image of λ in $F^{(0)}$ via r, and decompose λ over $\tilde{\lambda}$ relative to r [15, 18], $\lambda = \int \lambda^u d\tilde{\lambda}(u)$. For $x \in F$ define $x\lambda^{d(x)}$ to be the measure whose value at a set A is $\lambda^{d(x)}(\{y: r(y) = d(x)$ and $xy \in A\}) = \lambda^{d(x)}(x^{-1}A)$. We say the decomposition is *left quasi-invariant* if there is a $\tilde{\lambda}$-conull set $U \subseteq F^{(0)}$ such that $x\lambda^{d(x)} \sim \lambda^{r(x)}$ when x is in the set $r^{-1}(U) \cap d^{-1}(U)$, which is denoted $F|U$ and called the contraction or reduction of F to U. When U is conull this is an *inessential contraction* (i.c.). An i.c. is a conull set in F, but also a subgroupoid, and it is important to use an i.c. in the definition of quasi-invariant decomposition. If the set U can be taken to be $F^{(0)}$ we say the decomposition is *strictly quasi-invariant*. If λ has a (strictly) quasi-invariant decomposition and $\lambda_1 \sim \lambda$ let g be a strictly positive and finite Radon-Nikodym derivative $d\lambda_1/d\lambda$. If $\tilde{\lambda}_1 = r_*(\lambda_1)$, then $\tilde{\lambda}_1 \sim \tilde{\lambda}$, so there is a strictly positive and finite Radon-Nikodym derivative $f = d\tilde{\lambda}/d\tilde{\lambda}_1$. Then

$$\lambda_1 = \int g\lambda^u d\tilde{\lambda}(u) = \int f(u) g\lambda^u d\tilde{\lambda}_1(u) ,$$

so we can get a (strictly) left quasi-invariant decomposition of λ_1 by taking $\lambda_1^u = f(u)g\lambda^u$. Thus the existence of a (strictly) left quasi-invariant decomposition depends only on the measure class. The same holds for right quasi-invariance, defined using d. In particular, for $F = S \times G$, since $\mu \times \nu$ has a strictly left quasi-invariant decomposition and is quasi-symmetric, it also has a strictly right quasi-invariant decomposition, although a direct construction of it is less obvious.

With this background, we define a *measured groupoid* to be a pair (F, λ) or $(F, [\lambda])$ where F is an analytic Borel groupoid and λ is a probability measure on F which is quasi-symmetric and has a left (or equivalently, right) quasi-invariant decomposition. When convenient, we may take λ to be symmetric.

Now we want to define homomorphisms for measured groupoids. There are at least two possibilities. We have given an example in [18, p. 282] showing why we choose the null set condition we use. There the definition was given for virtual groups (defined below), and here we extend it to measured groupoids in general. Suppose $\varphi: F_1 \to F_2$ is a Borel function. For φ to be a *homomorphism* of measured groupoids (F_1, λ_1), (F_2, λ_2) we require two conditions:

(a) There is an i.c. of F_1 on which φ is algebraically a homomorphism.

(b) For saturated analytic sets $A \subseteq F_2^{(0)}$,

$$\varphi_*(\tilde{\lambda}_1)(A) = 0 \text{ iff } \tilde{\lambda}_2(A) = 0 .$$

A measured groupoid is called *ergodic*, or a *virtual group* iff every saturated Borel set is null or conull. Then the same is true of saturated analytic sets, and one can show that (b) follows from a weaker condition:

(b′) $\tilde{\lambda}_2(A) = 0$ and A saturated implies $\varphi_*(\tilde{\lambda}_1)(A) = 0$.

If the i.c. in condition (a) can be taken to be F_1, we say φ is *strict* homomorphism.

We have given a brief explanation of how to derive the measure theoretic definition of measured groupoid. For homomorphisms, the allowance for an i.c. and condition (b) are more complicated to motivate. One reason for (b) is given in [18]. The use of the i.c. arises because there are necessary constructions which only produce that amount of good algebraic bebavior. The author has been able to sharpen this under additional hypotheses (unpublished), but only to improve the type of i.c. By further study of the connection between $S \times G$ and H for coset spaces S, Mackey has extended several other group theoretic notions to groupoids. A primary example is that of induced representation. Suppose μ is a quasi-invariant measure on S, and let $\rho(s, x) = (d\mu(sx)/d\mu(s))^{1/2}$. If R is a unitary representation of $S \times G$ on a Hilbert space K and $\mathscr{H} = L^2(\mu' K)$, we can induce R to get a representation U of G on $\mathscr{H}$: $(U_1 f)(s) = \rho(s, x)R(s, x)f(sx)$. If $R = L \circ \psi$, this is one of the standard forms for inducing from H to G, but it is meaningful in general.

Another example of a notion extended from groups is that of the closure of the range of a homomorphism into a group [15, 16, 18]. Suppose H_1 is a closed subgroup of G_1 with coset space S_1. Take $\gamma_1: S_1 \to G_1$ and $\psi_1: S_1 \times G_1 \to H_1$ as before. If $\varphi: H_1 \to G_2$ is a homomorphism, so is $\varphi_1 = \varphi \circ \psi_1$. Now $S_1 \times G_1$ acts on $G_2 \times S_1$ as follows: $(x_2, s_1)(s_1, x_1) = (x_2\varphi_1(s_1, x_1), s_1x_1)$, and G_2 acts via $(x_2, s_1)y_2 =$

$(y_2^{-1}x_2, s_1)$. Then orbits under $S_1 \times G_1$ partition $G_2 \times S_1$ and are permuted by G_2. The quotient space of $G_2 \times S_1$ is analytic iff $\varphi(H_1)$ is closed, but in general there is a G_2 equivariant map f of $G_2 \times S_1$ onto the coset space S_2 of $\varphi(H_1)^-$ such that for any Borel equivariant $g: G_2 \times S_1 \to S$ there is a Borel h with $g = h \circ f$. By our earlier definition of "containment" this makes $S_2 \times G_2$ "the smallest subobject containing $\varphi(S_1 \times G_1)$". This construction can be carried out for groupoids in general. It generalizes the construction of a "flow built under a function" [16]. Details of one approach to this can be found in [18], and another approach is spelled out in the Appendix.

These definitions are obtained by use of the similarity between H and $(G/H) \times G$, and we arrive at a category whose objects are groupoids and whose maps are similarity classes of homomorphisms. In this category we define relationships and constructions (e.g., subobject and range closure) by extension from the groups. Another approach would be to apply the standard definitions of category theory. An earlier version of this paper nearly ignored the category theory approach, but in this one we explain some of the relationships between the two approaches. Our primary purpose for the theory is to have a workable extension of group and subgroup methods to the context of ergodic group actions. If the definitions are workable, we are not committed to agreement with category theory. However it may be of interest to compare the two approaches.

As one example, we point out that already the category of groups with similarity classes of homomorphisms is noticeably different. A group is a groupoid with only one unit, so homomorphisms φ_1, φ_2 from a group G to a group H are similar iff there is an element $a \in H$ such that for all x in G we have $\varphi_2(x) = a\varphi_1(x)a^{-1}$. Thus an inner automorphism is identified with the identity function. This reflects the fact that stabilizers of different points in a transitive G-space are conjugate subgroups. Now suppose N is a normal subgroup of G for which there is an inner automorphism α of G such that $\alpha | N$ is outer (these are easy to find). Let φ be the identity homomorphism of N and let $\psi = \alpha | N$. If i is the inclusion of N into G, $i \circ \varphi$ and $i \circ \psi$ are similar homomorphisms of N into G, but φ and ψ are not similar homomorphisms of N into N. Thus in our category the map which is the similarity class of i is not left-cancellable, even though we surely want to regard it as an imbedding.

The outline of the rest of the paper is as follows. In the first four sections we give definitions and statements of results which are needed later. Some of these are generalizations to groupoids of published results on groups. The methods are not always the same, but we have relegated most of the proofs to an appendix, in order to get the reader more quickly to § 5. In § 1 we define measured groupoids and actions and state a few results about them. Section 2 is about ergodic decompositions. The existence proof, in § 2 of the Appendix, depends on a simple characterization of ergodicity for groupoids and hence for group actions. Peter Hahn has given an independent proof, using other methods [7, Theorem 6.1]. The basic technical result in Section 3 is that if two groupoids have commuting actions on a given space, then each will have an action on the space of ergodic parts for the other. This provides a way to construct "range closures" of homomorphism into groupoids in the manner suggested by Mackey in [15] and close to that of K. Lange in [10]. It is also similar to the reasoning used by C. C. Moore in pages 112-117 of [1]. The Boolean G-space approach used in [18] seems harder to implement when G is no longer a group. The present method applies, for example to construct the "range closure" of a homomorphism into a virtual subgroup $(S \times G, [\mu \times \nu])$ of a group G, without referring directly to G itself. The result in section four is that the assignment of G-spaces to homomorphisms of groupoids into G is functorial.

We have mentioned above one group theoretic motivation for thinking of $S \times G$ as a subobject of G when a group G acts on a space S. In section five we develop another approach to this and related questions. The general problem is to find measure theoretic equivalences to topological and algebraic notions. For example, let H be a subgroup of a locally compact group G. Then H is closed iff the coset space is countably separated, and hence analytic, in the quotient Borel structure [11]. If F and G are groups and $\varphi: F \rightarrow G$ is a continuous homomorphism, then F acts on G: $g \cdot f = g\varphi(f)$, and $\varphi(F)$ is closed iff the orbit space in G for the action of F is analytic. Such equivalences allow us to define "closed range", "imbedding", etc., and we show that these properties are invariant under similarity of homomorphisms. In § 6 we show that the relation of being a subobject is transitive and is consistent with Mackey's definition of virtual subgroup of a group. We also show that a composition of two homomorphisms with dense range has dense range, and that the composition of an irreducible representation with a homomorphism having dense range is an irreducible representation. What is different here is that these notions are defined measure theoretically rather than topologically. In § 7, we discuss trivial homomorphisms, imbeddings, surjections, etc., in connection with "containment of subobjects" and various notions of category theory. For instance, we show that a homomorphism which has dense range is an epimorphism in the category sense.

For most terminology and notation we refer the reader to [18, 19, 20]. We point out that measures are assumed to be finite unless described otherwise. If $\mathscr{H}$ is a Hilbert space then $\mathscr{L}(\mathscr{H})$ is the space of bounded operators on $\mathscr{H}$ and $\mathscr{U}(\mathscr{H})$ is the (Polish) group of unitary operators on $\mathscr{H}$ and $(:)$ is the usual notation for the inner product. $(\ )^{-1}$ is used for the function taking x to x^{-1}, and if μ is a measure $(\mu)^{-1}$ may be used for $(\ )_*^{-1}(\mu)$. In decomposing a measure μ relative to f the measures may be $\mu(f, t)$, μ_t or μ^t. We will use $*$ instead of $\times$ for relative products of sets or measures.

The author is indebted to Caroline Series for the opportunity to see her Harvard thesis [21] and for results and ideas in it, and to Raymond Fabec for pointing out an error in an earlier proof of Lemma A.1.7. I also thank Alain Connes, Peter Hahn and Calvin Moore for suggesting ways to improve the paper.

1. Actions of groupoids and equivariant maps. In this section we discuss an algebraic aspect of groupoid actions and revise the terminology of [18] to agree with that of [5]. We also discuss various notations of action and equivariant map when measures are involved. We give some results relating these notions among themselves, and finally consider a 'universal G-space' construction for groupoids [13]. These are technicalities, intended to make things run more smoothly later.

Thinking only algebraically for the moment, let G be a groupoid with a right action on a set S, and set $F = S^*G = \{(s, x) \in S \times G$: sx is defined$\}$. We want to make a groupoid of F, in precisely the same way as when G is a group. Thus we want $(s, x)(t, y)$ to be defined iff $sx = t$, and then the product is (s, xy). For this to define a product in F, xy and $s(xy)$ must be defined whenever sx and $(sx)y$ are defined. In other words, to make $S*G$ a groupoid by the definition used when G is a group, the action must be true [18, p. 258]. Therefore we will adopt the following definition of action, in agreement with [5].

DEFINITION 1.1. If G is a groupoid and S is a set, an action of G on S (on the right) is a pair (p, a) where p is a function from S onto $G^{(0)}$ and a is a function from $S*G = \{(s, x) \in S \times G$: $p(s) = r(x)\}$ to S such that whenever $(s, x) \in S*G$ and $(x, y) \in G^{(2)}$, then $p(a(s, x)) = d(x)$ and $a(s, xy) = a(a(s, x), y)$. If G and S are Borel, we say the action is Borel iff p and a are Borel functions. We also will refer to (S, p, a) as a (Borel) G-space if (p, a) is an action (a Borel action) of G on S. If G_1 is a contraction of G and

$B \subseteqq S$, we say B is G_1-invariant iff $s \in B$, $x \in G$ and $(s, x) \in S*G$ imply $a(s, x) \in B$. To give a weak action of G on S, we give for each $x \in G$ sets $D(x)$ and $R(x) \subseteqq S$ and a bijection $\psi(x)\colon D(x) \to R(x)$. If $\psi(x)(s)$ is denoted sx, we require

(i) $S = \cup \{D(x)\colon x \in G\}$

(ii) $u \in G^{(0)}$ and $s \in D(u)$ imply $su = s$

(iii) $(x, y) \in G^{(2)}$ and $s \in D(x)$ imply $sx \in D(y)$, $s \in D(xy)$ and $(sx)y = s(xy)$.

This is Borel if F and $(s, x) \mapsto (sx, x^{-1})$ are Borel.

REMARKS. (1) $(a(s, x), y) \in S^*G$ because $p(a(s, x)) = d(x)$.

(2) We will ordinarily write sx for $a(s, x)$ and refer to the G-space (S, p). We may even let the function p be implicit and refer to the G-space S.

(3) The associative law holds under this definition, i.e., if either of $s(xy)$ and $(sx)y$ is defined, then the other is also defined and they are equal. We leave it for the reader to verify that $S*G$ is in fact a groupoid.

(4) $j_S(s, x) = x$ defines a homomorphism of $S*G$ into G called the inclusion.

(5) B is G-invariant iff $\{(s, p(s))\colon s \in B\}$ is saturated in $S*G$, and B is G_1-invariant iff $B \cap p^{-1}(G_1^{(0)})$ is G_1-invariant.

(6) Define $s_1 \sim s_2$ iff there is an x with $s_1x = s_2$. Then $\sim$ is an equivalence relation on S.

DEFINITION 1.1 is suitable when no measures are involved, but when we deal with measured groupoids, there may be null sets which we want to discard. This needs to be considered in making the definitions. For homomorphisms of measured groupoids, we found it convenient to have the most used term include the possibility of some null sets on which there is imprecise behavior. This avoids repetitions of such phrases as "there is an i.c. G_0 on which φ is a homomorphism." We simply say, "φ is a homomorphism." For the same kind of reason, we want to allow for a carefully controlled amount of algebraic imprecision in the definitions for $(G, [\mu])$-spaces and $(G, [\mu])$-equivariant functions. This is one way to simplify the statements of theorems.

Suppose (S, p, a) is a G-space and G_1 is a contraction of G, and set $S_1 = p^{-1}(G_1^{(0)})$. Then S_1 is G_1-invariant. If $p(S_1) = G_1^{(0)}$ and S_1 is G_1-invariant, let $p_1 = p | S_1$ and $a_1 = a | S_1*G_1$; then (S_1, p_1, a_1) is a G_1-space. Also notice that S_1*G_1 is the contraction of $S*G$ to $\{(s, p(s))\colon s \in S_1\}$. For $S_1 \subseteqq S$, the contraction to $\{(s, p(s))\colon s \in S_1\}$ is S_1*G_1 iff S_1 is invariant under $G_1 = G | p(S_1)$.

DEFINITION 1.2. Let $(G, [\mu])$ be a measured groupoid, let S be an analytic Borel space, let p be Borel from S onto $G^{(0)}$ and let $a: S \times G \to S$.

(a) (S, p, a) is a $(G, [\mu])$-space if there is an i.c. G_1 of G such that $S_1 = p^{-1}(G_1^{(0)})$ is a G_1-space under $p|S_1$ and $a|S_1{*}G$. A measure λ on S is then called quasi-invariant iff $p_*(\lambda) \sim \tilde{\mu}$ and λ has a decomposition $\lambda = \int \lambda_u d\tilde{\mu}(u)$ such that $(\lambda_{r(x)})x \sim \lambda_{d(x)}$ for almost all x in G. In this case we call (S, λ, p, a) or (S, λ) or even $(S, [\lambda])$ a $(G, [\mu])$-space.

(b) If we can take $G_1 = G$ we call S a strict G-space, and if $(\lambda_{r(x)})x \sim \lambda_{d(x)}$ for every x, we say λ, or its decomposition, is strictly quasi-invariant.

Let (S, p) be a strict $(G, [\mu])$-space and let λ be a finite Borel measure on S with $p_*(\lambda) \sim \tilde{\mu}$. Decompose λ as $\int \lambda_u d\tilde{\mu}(u)$ relative to p. By Theorem 2.9 of [16], λ is quasi-invariant iff $\lambda{*}\mu$ is quasi-invariant under $\tau(s, x) = (sx, x^{-1})$, the inverse map in $S{*}G$. Suppose λ is quasi-invariant and let

$$\nu = \lambda{*}\mu = \int \lambda_u \times \mu^u d\tilde{\mu}(u) = \int \lambda_{r(x)} \times \varepsilon_x d\mu(x) = \int \varepsilon_s \times \mu^{p(s)} d\lambda(s)$$

[16, pages 63, 64]. We have $r(s, x) = (s, r(x))$ and $d(s, x) = (sx, d(x))$, and $(S{*}G)^{(0)}$ is just the graph of p, which is isomorphic to S via the coordinate projection onto S. Hence $r_*(\nu) = \int \lambda_u \times \varepsilon_u d\tilde{\mu}(u)$, by Lemma 1.2 of [19]. This is just the image of λ in $(S{*}G)^{(0)}$, so the last formula for ν above is its decomposition relative to r, i.e., $\nu(r, (s, p(s))) = \varepsilon_s \times \mu^{p(s)}$. Let G_1 be an i.c. of G such that $x \in G_1$ implies $x\mu^{d(x)} \sim \mu^{r(x)}$ for $x \in G_1$ [19, Lemma 6.2], and let $S_1 = p^{-1}(G_1^{(0)})$. Then $S_1{*}G_1$ is an i.c. and for $(s, x) \in S_1{*}G_1$ we have $(s, x)[\varepsilon_{sx} \times \mu^{d(x)}] = \varepsilon_s \times (x\mu^{d(x)}) \sim \varepsilon_s \times \mu^{r(x)}$. Hence $(S{*}G, [\nu])$ is a measured groupoid. Thus the process of forming $S{*}G$ does not give a new kind of object when applied the second time.

Here are some examples of G-spaces.

EXAMPLE 1. Any G-space for a group G is a strict G-space. Any quasi-invariant measure on it is strictly quasi-invariant.

EXAMPLE 2. Let G be a groupoid, $S = G^{(0)}$, $p =$ the identity function. Then $S{*}G = \{(r(x), x): x \in G\}$. Define $a(r(x), x) = d(x)$. If $(G, [\mu])$ is a measured groupoid, $\tilde{\mu}$ is strictly quasi-invariant. The orbit of $u \in G^{(0)}$ is its equivalence class. Thus every equivalence relation is induced by an action.

EXAMPLE 3. Suppose $U \subseteq G^{(0)}$ meets each equivalence class in

$G^{(0)}$ and let $S = r^{-1}(U)$. Let $p = d|S$. Then $S*G = S \times G \cap G^{(2)}$. If $(s, x) \in S*G$, let $a(s, x) = sx$. The orbit of $s \in S$ is then $r^{-1}(r(s))$. In the proof of Theorem 3.5, we show how to get some quasi-invariant measures.

EXAMPLE 4. Let φ be a homomorphism from G to a groupoid H. Let $T(\varphi) = \{(\xi, u) \in H \times G^{(0)}: d(\xi) = \varphi(u)\}$. Define $p(\xi, u) = u$. Then $T(\varphi)^*G = \{((\xi, r(x)), x): \xi \in H, x \in G$ and $d(\xi) = r \circ \varphi(x)\}$. Define $a((\xi, r(x)), x) = (\xi\varphi(x), d(x))$. This generalizes correctly the action of one group on another via a homomorphism. We use this space to construct the "closure of the range" of φ, in section three.

The term for a function between spaces on which a group acts, which preserves the group action, is equivariant. Next we want to define this word in the context of groupoid actions.

DEFINITION 1.3. Let $(G[\mu])$ be a measured groupoid.

(a) If (S_1, λ_1) and (S_2, λ_2) are $(G, [\mu])$-spaces and $f: S_1 \to S_2$ is Borel, we say f is $(G, [\mu])$-equivariant if

(i) there are an i.c. G_0 of G and conull G_0-invariant analytic sets $S_3 \subseteq S_1$ and $S_4 \subseteq S_2$ such that when $(s, x) \in S_3*G_0$ then $(f(s), x) \in S_4*G_0$ and $f(sx) = f(s)x$, and

(ii) for saturated analytic sets $A \subseteq S_2$, $\lambda_1(f^{-1}(A)) = 0$ iff $\lambda_2(A) = 0$.

(b) If we can take G_0, S_3 and S_4 so that (a) holds and f takes S_3 one-one onto S_4, we call f an isomorphism.

(c) If we can take $G_0 = G$, $S_3 = S_1$ and $S_4 = S_2$, we say f is strictly equivariant or a strict isomorphism.

(d) If S_2 has no measure, we delete the requirement that S_4 be conull, as well as condition (ii) in (a).

(e) We say f is almost equivariant if $\{(s, x) \in S_1*G: f(s)x$ is defined and equal to $f(sx)\}$ is conull [21].

It may be of interest to note that for an equivariant map f, $f_*(\lambda_1) \sim \lambda_2$. This means they are what C. Series called *normalized* [21]. This is Lemma A1.4 in the Appendix. Another useful fact is the following regularization result for almost equivariant maps. It is a little stronger than we can get by applying the homomorphism regularization lemma to f^*i, and its proof is also in the Appendix, as Lemma A1.1.

LEMMA 1.4. *Let $(G, [\mu])$ be a measured groupoid, let (S, λ, p) be an analytic Borel $(G, [\mu])$-space and let T be a strict analytic Borel $(G, [\mu])$-space. If $f_1: S \to T$ is almost $(G, [\mu])$-equivariant, then there is an equivariant function $f: S \to T$ which agrees with f_1* a.e. *Furthermore, $f_{1*}(\lambda) = f_*(\lambda)$ and is quasi-invariant. The function f exists even if T is a weak G-space.*

We also need a notion of similarity of equivariant functions. Suppose $f, g: (S_1, \lambda_1) \to (S_2, \lambda_2)$ are strictly $(G, [\mu])$-equivariant and let $\theta: S_1 \to S_1 * G$ give a strict similarity of $f*i$ and $g*i$, i.e., suppose $\theta(s)(f(s), x) = (g(s), x)\theta(sx)$ for $(s, x) \in S_1 * G$. Let $\theta(s) = (\alpha(s), \beta(s))$ where $\alpha: S_1 \to S_2$, $\beta: S_1 \to G$. Then the similarity equation is equivalent to these: $\alpha = g$, $g(s)\beta(s) = f(s)$ for $s \in S_1$ and $\beta(s)x = x\beta(sx)$ for $(s, x) \in S_1 * G$. This motivates our definition.

DEFINITION 1.5. (a) Let $f, g: (S_1, \lambda_1) \to (S_2, \lambda_2)$ be strictly $(G, [\mu])$-equivariant. They are strictly similar iff there is a Borel function $\beta: S_1 \to G$ such that $g(s)\beta(s) = f(s)$ for $s \in S_1$ and $\beta(s)x = x\beta(sx)$ for $(s, x) \in S_1 * G$.

(b) Let $f, g: (S_1, \lambda_1) \to (S_2, \lambda_2)$ be $(G, [\mu])$-equivariant. They are similar if there are an i.c. G_1 and conull strict $(G_1, [\mu])$-spaces $S_3 \subseteq S_1$ and $S_4 \subseteq S_2$ such that $f|S_3$ and $g|S_3$ are strict and strictly similar, from S_3 to S_4.

Let $T = \{t \in G: r(t) = d(t)\}$, which is the "union of the stabilizers" if G comes from a group action. Let $p = d|T$ and define $a(t, x) = x^{-1}tx$ for $(t, x) \in T * G$. Then the equation $\beta(s)x = x\beta(sx)$ just says that β is strictly equivariant from S_1 to T. The next lemma is proved as Lemma A1.10.

LEMMA 1.6. *Let $(G, [\mu])$ be a measurable groupoid and let $(S_1, [\lambda_1])$ and $(S_2, [\lambda_2])$ be analytic $(G, [\mu])$-spaces. Suppose $f: S_1 \to S_2$ and $g: S_2 \to S_1$ are equivariant maps with $f \circ g$ similar to the identity on S_2 and $g \circ f$ similar to the identity on S_1. Then $(S_1, [\lambda_1])$ and $(S_2, [\lambda_2])$ are isomorphic.*

Now let us turn to the construction of a 'universal G-space'. For groups the locally square-integrable functions make a good space, but we have no topology and hence no compact sets. However, we work with finite measures, so any bounded function is in L^2. For each unit $u \in G^{(0)}$ and each Borel $f: G \to [0, 1]$ we can define $[f]_u = \{g: g$ is Borel from G to C and $g = f$ a.e. $d\mu(r, u)\}$. Then let $\mathscr{F}(u) = \{[f]_u: f$ is Borel from G to $[0, 1]\}$. Now $\mathscr{F}(u)$ may be regarded as a subset of $L^2(\mu(r, u))$ and as such it is a weakly closed norm-bounded set and hence is weakly compact. We now form a bundle over $G^{(0)}$ as one does with Hilbert bundles. Let $G^{(0)} * \mathscr{F} = \cup \{\{u\} \times \mathscr{F}(u): u \in G^{(0)}\}$, and give $G^{(0)} * \mathscr{F}$ the Borel structure it inherits as a subset of $G^{(0)} * \mathscr{H} = \cup \{\{u\} \times L^2(\mu(r, u)): u \in G^{(0)}\}$, which is a Hilbert bundle [18, 20]. This is the smallest Borel structure for which the projection onto $G^{(0)}$ is Borel along with all the functions ψ_g, for bounded Borel functions g where

$$\psi_g(u, [f]_u) = \int fgd(\mu(r, u)) .$$

If $\mathscr{A}$ is a countable algebra generating the Borel sets then $G^{(0)}*\mathscr{F} = \{(u, [f]_u) \in G^{(0)}*\mathscr{H}: A \in \mathscr{A} \text{ implies } 0 \leqq \psi_{\phi_A}(u, [f]_u) \leqq 1\}$. Hence $G^{(0)}*\mathscr{F}$ is a Borel subset of $G^{(0)}*\mathscr{H}$ and must be analytic. Now G acts on $G^{(0)}*\mathscr{F}$ as follows: $(r(x), [f]_{r(x)})x = (d(x), [g]_{d(x)})$ where $g(y) = f(xy)$ for $y \in r^{-1}(d(x))$ and $g(y) = 0$ otherwise. This is well defined if μ has a left quasi-invariant decomposition. The next lemma is proved as Lemma A1.11.

LEMMA 1.7. *$G^{(0)}*\mathscr{F}$ is an analytic G-space, provided the given decomposition of μ relative to r is quasi-invariant.*

2. Ergodic decompositions of measurable groupoids. John von Neumann proved that a measure preserving flow can be decomposed into ergodic flows [17]. This decomposition into ergodic parts has also been done for other groups of transformations [2, 9]. We shall need to decompose groupoids of transformations into ergodic parts. This follows from a decomposition of measurable groupoids, since we can simply form the new groupoid $S*G$. It will be convenient to begin with a Hilbert bundle characterization of ergodic groupoids. It is possible to work with measure algebra bundles, but Hilbert bundles are more familiar, so we shall use them instead.

Let (G, C) be a measurable groupoid and suppose $\lambda \in C$ is a symmetric probability measure and has a quasi-invariant decomposition $\lambda = \int \lambda_u d\tilde{\lambda}(u)$ relative to d. Define $\mathscr{H}_r = \{f \circ r: f \in L^2(\tilde{\lambda})\}$, $\mathscr{H}_d = \{f \circ d: f \in L^2(\tilde{\lambda})\}$. Since almost every fiber measure is a probability measure, $f \to f \circ r$ and $f \to f \circ d$ are isometric imbeddings of $L^2(\tilde{\lambda})$ into $L^2(\lambda)$. If $(Jf)(x) = f(x^{-1})$, J is a unitary operator on $L^2(\lambda)$ with $J^2 = I$, and $J(\mathscr{H}_r) = \mathscr{H}_d$. Notice that $\mathscr{H}_r \cap \mathscr{H}_d$ contains the constant functions.

LEMMA 2.1. *The measurable groupoid (G, C) is ergodic iff $\mathscr{H}_r \cap \mathscr{H}_d$ is one-dimensional.*

DEFINITION 2.2. Let $(G, [\lambda])$ be a measurable groupoid. A strict ergodic decomposition of $(G, [\lambda])$ is a mapping q of $G^{(0)}$ into an analytic Borel space T such that if $\nu = q_*(\tilde{\lambda})$ and $\tilde{\lambda} = \int \tilde{\lambda}(p, t)d\nu(t)$ is a decomposition of $\tilde{\lambda}$ relative to q, then for ν-almost all t, $q^{-1}(t)$ is saturated and $(G|q^{-1}(t), [\lambda^t])$ is an ergodic groupoid, where

$$\lambda^t = \int \lambda_u d(\tilde{\lambda}(p, t))(u) .$$

An ergodic decomposition of $(G, [\lambda])$ is a Borel mapping q of $G^{(0)}$ into an analytic space T such that for some conull Borel set $U \subseteq G^{(0)}$, $q|U$ is a strict ergodic decomposition of $(G|U, [\lambda])$.

If U is conull in $G^{(0)}$, then it is conull for almost every $\tilde{\lambda}(q, t)$. Thus $G|(q|U)^{-1}(t)$ is almost always an i.c. of $G|q^{-1}(t)$, so the basic difference between strict and nonstrict decompositions is that in the strict case the sets $q^{-1}(t)$ are almost all saturated, whereas in the nonstrict case there is a conull set U such that the sets $q^{-1}(t) \cap U$ are almost all saturated relative to $G|U$.

There is a property which characterizes ergodic decompositions and which is more useful than the definition in most cases. This property is stated in terms of factoring of functions. This is a measure theoretic version of a familiar procedure in elementary algebra: If f maps X onto Y and g maps X to Z and is constant on level sets of f then there is an $h: Y \to Z$ with $g = h \circ f$. After this lemma we state first the uniqueness and then the existence of ergodic decompositions.

LEMMA 2.3. *Let $(G, [\mu])$ be a measured groupoid, and let a Borel function q from $G^{(0)}$ to an analytic space T be an ergodic decomposition. If a Borel function g from $G^{(0)}$ to an analytic space Z is constant on equivalence classes, then there is a Borel $h: T \to A$ such that $h \circ q = g$* a.e. *Such an h is determind* a.e. *relative to $\mu = q_*(\tilde{\lambda})$.*

THEOREM 2.4. (*Uniqueness of Ergodic Decompositions*). *Let $q_1: G^{(0)} \to T_1$ and $q_2: G^{(0)} \to T_2$ be ergodic decompositions of the measured groupoid $(G, [\lambda])$. Then there are a conull Borel set $U \subseteq G^{(0)}$ and a Borel isomorphism $f: q_1(U) \to q_2(U)$ such that $q_2 = f \circ q_1$ on U. Also, q_1 and q_2 have the same level sets in U. If q_1 and q_2 are strict decompositions, U may be taken to be saturated.*

THEOREM 2.5. *If $(G, [\lambda])$ is a measured groupoid, then $(G, [\lambda])$ has an ergodic decomposition. If λ has a (right or left) quasi-invariant decomposition, then $(G, [\lambda])$ has a strict ergodic decomposition.*

DEFINITION 2.6. Let $(G, [\mu])$ be a measurable groupoid and let (S, λ) be an analytic Borel G-space with q.i. measure. The measure λ is ergodic iff $(S*G, [\lambda*\mu])$ is an ergodic groupoid. An ergodic decomposition of (S, λ) relative to G is a Borel mapping q of S into an analytic Borel space T such that if $\lambda = \int \lambda_t dq_*(\lambda)(t)$ is a decomposition of λ relative to q then for $q_*(\lambda)$-almost all t in T the set

$q^{-1}(t)$ is invariant and the measure λ_t is concentrated on $q^{-1}(t)$ and is q.i. and ergodic.

COROLLARY 2.7. *If (S, λ) is an analytic G-space with a quasi-invariant measure for a measurable groupoid (G, C) and C has an element with a left quasi-invariant decomposition then S has a decomposition into ergodic parts, which is essentially unique.*

LEMMA 2.8. *The converse of Lemma 2.3 is true.*

3. Commuting groupoid actions and closing of ranges of homomorphisms. In constructing the closure of the range of a homomorphism $\varphi: F \to G$, the idea is to make a G-space out of the space of ergodic parts for the action of F on $G*F^{(0)}$[16, 18]. The reason this should work is that F and G have actions on $G*F^{(0)}$ which commute in the sense of Definition 3.1 below. Theorem 3.2 is a precise formulation of a theorem needed for working with such pairs of actions, and we apply it in Theorem 3.5 to construct range closures. Parts of the proof seem easier than when done as in [18].

DEFINITION 3.1. If S is an F-space and a G-space, we say the actions commute iff for $s \in S$, $\xi \in F$ and $x \in G$, if sx and $s\xi$ are defined then so are $(sx)\xi$ and $(s\xi)x$ and they are equal.

THEOREM 3.2. *Let $(F, [\mu])$ and $(G, [\nu])$ be measured groupoids and let (S, λ, p) and (S, λ, q) be strict $(F, [\mu])$- and $(G, [\nu])$-spaces respectively. Suppose these actions commute. Then there is a strictly G-equivariant function $f: S \to G^{(0)}*\mathscr{F}$ which is an ergodic decomposition of $S*F$. If S' is an analytic $(G, [\nu])$-space and $f': S \to S'$ is a $(G, [\nu])$-equivariant ergodic decomposition of $S*F$, then $(G^{(0)}*\mathscr{F}, f_*(\lambda))$ and $(S', f'_*(\lambda))$ are isomorphic $(G, [\nu])$-spaces.*

In the process of constructing the closure of the range of a homomorphism, it will be necessary to construct some quasi-invariant measures. The next lemma gives one of the basic ingredients. First some preparation is needed.

Let $(G, [\nu])$ be a measured groupoid and let E be the equivalence relation on $G^{(0)}$ induced by $G: E = (r, d)(G) \subseteq G^{(0)} \times G^{(0)}$. We take $\nu' = (r, d)_*(v)$ and are interested in a special kind of decomposition of ν relative to ν'. The important thing about ν is that one of these decompositions exist.

DEFINITION 3.3. We shall say that ν is (r, d)-quasi-invariant if it has decompositions $\nu = \int \nu_u d\tilde{\nu}(u)$ and $\nu = \int \nu_{v,u} d\nu'(v, u)$ such that

(a) for $(v, u) \in E$, $\nu_{v,u}$ is concentrated on $r^{-1}(v) \cap d^{-1}(u)$,

(b) for $(v, u) \in E$, $(\nu_{v,u})^{-1} \sim \nu_{u,v}$.

(c) if $r(x) \sim u$, then $\nu_{u,r(x)} \cdot x \sim \nu_{u,d(x)}$ and $x \cdot \nu_{d(x),u} \sim \nu_{r(x),u}$, and

(d) for $u \in G^{(0)}$, $\nu_u = \int \nu_{v,u} d(r_*(\nu_u))(v)$.

If we assume ν is (r, d)-quasi-invariant, we mean that such decompositions should be used. By Lemma 6.8 of [19] there is a measure $\nu^* \sim \nu$ and an i.c. G_0 of G such that $\nu^* | G_0$ is (r, d)-quasi-invariant. Now take ρ to be an everywhere positive and finite version of $d\nu/d\nu^*$, ρ' the same for $d\nu^{*\prime}/d\nu'$ and define $\nu_{v,u} = \rho'(v, u)\rho\nu^*_{v,u}$. If $G_0 = G | U_0$ and $E_0 = E | U_0$, then (a) (b) and (c) hold for E_0 and G_0. Hence $\nu_u = \int \nu_{v,u} d(r_*(\nu_u))(v)$ for almost all u, by uniqueness of decompositions. By removing another null set, we see that we have an i.c. G_1 on which ν is (r, d)-quasi-invariant. Thus in matters where we can safely pass to an i.c., we may assume that ν is (r, d)-quasi-invariant for technical convenience. Of course in concrete situations one would expect this to hold globally anyway.

LEMMA 3.4. *Let $(G, [\nu])$ be a measured groupoid and suppose ν is (r, d)-quasi-invariant. Let λ be a finite measure on $G^{(0)}$ such that $\lambda(A) = 0$ iff $\tilde{\nu}(A) = 0$ for saturated analytic sets $A \subseteq G^{(0)}$. Let $\nu_1 = \int \nu_u d\lambda(u)$, and let $y \in G$ act on $x \in G$ by $x*y = y^{-1}x$ provided $r(x) = r(y)$. Then ν_1 is quasi-invariant.*

THEOREM 3.5. *Let $(F, [\mu])$ be a measured grupoid, let $(G, [\nu])$ be a measured groupoid for which ν is (r, d)-quasi-invariant and let $\varphi: F \to G$ be a homomorphism. Then there are* i.c.'s *F_0 and G_0 of F and G, a strict $(G_0, [\nu])$-space (S_ϕ, λ) and a strict homomorphism $\varphi': F_0 \to S_\phi * G_0$ such that $\varphi | F_0 = j \circ \varphi'$, where $j: S_\phi * G_0 \to G_0$ is the inclusion (coordinate projection).*

DEFINITION 3.6. We call $(S_\phi * G, [\lambda * \nu])$ the closure of the range of φ, and will denote j by j_ϕ when necessary to identify its connection with φ.

Notice here that $S_\phi * G = S_\phi * G_0$, and that the proof is by a construction. The very statement of the theorem allows some ambiguity in the choice of S_ϕ, because F_0 and G_0 are not unique. The construction, given in the Appendix, produces S_ϕ as an ergodic decomposition space of $T(\varphi) = \{(x, u) \in G \times F^{(0)}: d(x) = \varphi(u)\}$ for a certain action of F on $T(\varphi)$ and a natural measure on $T(\varphi)$ (see Lemma 3.7). As such, it is determined up to isomorphism modulo null sets, which is sufficient. This also depends on ν being (r, d)-

quasi-invariant, but we know that $(G, [\nu])$ always has an i.c. G_0 on which ν is (r, d) quasi-invariant, and φ is similar to a homomorphism φ_0 taking values in G_0. We need to see that S_{ϕ_0} does not really depend on the choice of φ_0, as the following lemma shows.

Lemma 3.7. *Let $(G, [\nu])$ be a measured groupoid in which ν is (r, d)-quasi-invariant and let φ_1, φ_2 be similar homomorphisms of a measurable groupoid $(F, [\mu])$ into $(G, [\nu])$. Let $T_1 = T(\varphi_1) = \{(x, u) \in G \times F^{(0)}: d(x) = \varphi_1(u)\}$ and take the measure $\nu_1 = \int \nu_u d(\varphi_1 * (\tilde{\mu}))(u)$ on $d^{-1}(\varphi_1(F^{(0)}))$ and $\nu_1 * \tilde{\mu}$ on T_1. Similarly form $T_2 = T(\varphi_2)$, ν_2 and $\nu_2 * \tilde{\mu}$. Then there are* i.c.'s *F_0 and G_0 of F and G and F_0 and G_0-invariant conull analytic sets $T_1^* \subseteq T_1$ and $T_2^* \subseteq T_2$ which are strictly isomorphic as F_0 and G_0-spaces under a measure-class-preserving function f. Hence $(S_{\varphi_1}, \lambda_1)$ and $(S_{\varphi_2}, \lambda_2)$ have strictly isomorphic analytic conull G_0-invariant subspaces.*

Starting with an arbitrary φ, if we choose a G_0 on which ν is (r, d)-quasi-invariant and a φ_0 similar to φ taking values in G_0, we have i.c.'s F_1 and G_1 and φ_0': $F_1 \to S_{\varphi_0} * G$ such that $j \circ \varphi_0' = \varphi_0 | F_1$. Then $j \circ \varphi_0' \sim \varphi | F_1$, but we do not have equality. In fact, there probably would not be a φ': $F_1 \to S_{\varphi_0} * G_1$ with $j \circ \varphi' = \varphi | F_1$, because φ may not carry F_1 into G_1. Thus we speak of $S_{\varphi_0} * G$ as "the" range closure of φ in the following sense: it is constructed from φ by way of a choice of G_0 and $\varphi_0 \sim \varphi$, but if we choose instead an i.c. G_1 on which ν is (r, d)-quasi-invariant and a $\varphi_1 \sim \varphi$ taking values in G_1, then there is a $\varphi_2 \sim \varphi$ taking values in $G_2 = G_0 \cap G_1$, and Lemma 3.7 says we have isomorphisms $S_{\varphi_0} \sim S_{\varphi_2}$ and $S_{\varphi_1} \sim S_{\varphi_2}$, so $S_{\varphi_0} \sim S_{\varphi_1}$. Since we could never have an S_φ determined more than within isomorphism, it is agreeable to take $S_\varphi = S_{\varphi_0}$. Also, we actually can choose $S_{\varphi_1} = S_{\varphi_2}$ whenever $\varphi_1 \sim \varphi_2$.

It seems natural to ask about the uniqueness of S_φ in the following way. Suppose φ: $F \to G$ and there exists a G-space S and a homomorphism φ': $F \to S * G$ such that $j \circ \varphi' \sim \varphi$. Is S determined up to isomorphism? According to Lemma 4.1, there is a map $M(\varphi')$: $S_{j \circ \varphi'} \to S_j$. We have $S_{j \circ \varphi'} \simeq S_\varphi$. By Lemma 6.3, $S_j \simeq S$ and by Theorems 6.7 and 6.11, $M(\varphi')$ is an isomorphism, so the answer is, yes.

4. Functorial properties of the range closure construction.

It seems worthwhile to extend some of the results of [10] to our situation. We restrict our attention to a few facts, but presumably the other results extend also.

Recall from [18] that if $(F, [\lambda])$ and $(G, [\mu])$ are measurable groupoids and ψ: $(F, [\mu]) \to (G, [\mu])$ is a homomorphism then $[\psi, F]$ or

$[\psi]$ denotes the set of homomorphisms similar to ψ. If $\psi: (F, [\lambda]) \to (G, [\mu])$ and $\varphi: (G, [\mu]) \to (H, [\nu])$ then there are $\psi_1 \sim \psi$ and an i.c. G_1 so that φ is strict on G_1 and $\psi_1(F) \subseteqq G_1$, i.e., (φ, ψ_1) is composable [18, Definition 6.7]. Then $[\varphi \circ \psi_1]$ depends only on $[\psi]$ and $[\varphi]$ and is denoted $[\varphi]\circ[\psi]$. This operation is associative [18, Lemma 6.13].

If $(G, [\mu])$ is a measured groupoid, let $\mathscr{M}(G)$ denote the class of pairs $((F, [\lambda]), \varphi)$ where $(F, [\lambda])$ is a measurable groupoid and $\varphi: (F, [\lambda]) \to (G, [\mu])$ is a homomorphism. If we insist that $F \subseteqq [0, 1]$ as a Borel space, then $\mathscr{M}(G)$ becomes a set. For $\mathscr{F}_1 = ((F_1, [\lambda_1]), \varphi_1)$ and $\mathscr{F}_2 = ((F_2, [\lambda_2]), \varphi_2)$ in $\mathscr{M}(G)$, a homomorphism $\psi: \mathscr{F}_1 \to \mathscr{F}_2$ is a homomorphism $\psi: (F_1, [\lambda_1]) \to (F_2, [\lambda_2])$ such that $[\varphi_2]\circ[\psi] = [\varphi_1]$. We denote by $M((F, [\lambda]), \varphi)$ the G-space (S_φ, ν) for which the groupoid $(S_\varphi * G, [\nu * \mu])$ is the closure of the range of φ, and we want to define $M[\psi]$ so as to make a functor out of M. We have a series of lemmas generalizing those of [10, § 2]. The proofs are clearly related to those of [10], but are not identical, because we have a groupoid for G and because we have a different construction for S_φ. Since we start with homomorphisms which need not be strict, we will expect to product G-space maps which are not strictly equivariant. In fact, we may need to restrict to a conull analytic set which is invariant for some i.c. G_0 in order to get strictness. Thus if we take some i.c.'s in the process nothing will be lost, and we can work with strict homomorphisms when necessary.

LEMMA 4.1. *Suppose* $\mathscr{F}_1 = ((F_1, [\lambda_1]), \varphi_1)$ *and* $\mathscr{F}_2 = ((F_2, [\lambda_2]), \varphi_2)$ *are in* $\mathscr{M}(G)$, φ_2 *is strict,* ψ *is a homomorphism of* $\mathscr{F}_1$ *to* $\mathscr{F}_2$ *and* $\theta: F_1^{(0)} \to G$ *is a Borel function for which* $\theta \circ r(\xi)\varphi_2 \circ \psi(\xi) = \varphi_1(\xi)\theta \circ d(\xi)$ *for almost all* ξ. *Then there is a G-equivariant normalized* $h = M(\psi, \theta): S_{\varphi_1} \to S_{\varphi_2}$ *obtained as the essential quotient of the function* f^θ *from* $T_1 = G * F_1^{(0)}$ *to* $T_2 = G * F_2^{(0)}$ *defined by* $f^\theta(x, u) = (x\theta(u), \psi(u))$.

LEMMA 4.2. *Under the hypotheses of Lemma* 4.1, *if* δ *is another similarity of* $\varphi_2 \circ \psi$ *with* φ_1 *and* φ_2 *is strict, then* $M(\psi, \delta)$ *is similar to* $M(\psi, \theta)$.

DEFINITION 4.3. Call this class of maps $[M(\psi)]$.

LEMMA 4.4. *If* μ *is* (r, d)*-quasi-invariant on* G *and* $\psi_1: \mathscr{F}_1 \to \mathscr{F}_2$ *is a homomorphism, where* $\mathscr{F}_2 = ((F_2[\lambda_2]), \varphi_2)$ *with* φ_2 *strict, and* $\psi_2: (F_1, [\lambda_1]) \to (F_2, [\lambda_2])$ *is a homomorphism with* $[\psi_2] = [\psi_1]$ *then* $\psi_2: \mathscr{F}_1 \to \mathscr{F}_2$ *is a homomorphism and* $[M(\psi_1)] = [M(\psi_2)]$.

Now we can define $M[\psi]$ for any $\psi: \mathscr{F}_1 \to \mathscr{F}_2$ by $M[\psi] = [M(\psi_1)]$ where (φ_2, ψ_1) is composable and $\psi_1 \sim \psi$. Indeed, in such circumstances we may pass to an i.c. F_4 of F_2 on which φ_2 is strict and an i.c. F_3 of F_1 such that $\psi_1(F_3) \subseteqq F_4$, and the construction of $M(\psi_1)$ is valid. If also $\psi_2 \sim \psi$ and (φ_2, ψ_2) is composable, there are i.c.'s F_5 of F_1 and F_6 of F_2 such that $\psi_2(F_4) \subseteqq F_6$ and $\varphi_2 | F_6$ is strict. Hence $F_4 \cap F_6$ is an i.c. on which φ_2 is strict, and there is a $\psi_3: F_1 \to F_2$ such that $\psi_3(F_1) \subseteqq F_4 \cap F_6$ and $\psi_3 \sim \psi$. Then $\psi_3 \sim \psi_1$ and by Lemma 4.4 we have $[M(\psi_3)] = [M(\psi_1)]$, similarly $[M(\psi_3)] = [M(\psi_2)]$. Thus $M[\psi]$ is well defined.

Finally, we can remove the restriction that μ be (r, d)-quasi-invariant on G, as follows. If $\mathscr{F}_1 = ((F_1, [\lambda_1]), \varphi_1)$ and $\mathscr{F}_2 = ((F_2, [\lambda_2]), \varphi_2)$ are in $\mathscr{M}(G)$, there is an i.c. G_0 on which μ is (r, d)-quasi-invariant and then there are $\varphi_3 \sim \varphi_1$ and $\varphi_4 \sim \varphi_2$ taking values in G_0. To construct a space called S_{φ_1} in § 3, we used S_{φ_3}, and also $S_{\varphi_2} = S_{\varphi_4}$. If $\psi: (F_1, [\lambda_1]) \to (F_2, [\lambda_2])$ then $[\varphi_2]\circ[\psi] = [\varphi_4]\circ[\psi]$ and $[\varphi_3] = [\varphi_1]$, so ψ is an $\mathscr{M}(G)$-homomorphism of $\mathscr{F}_1$ to $\mathscr{F}_2$ iff it is such from $((F_1, [\lambda_1]), \varphi_3)$ to $((F_2, [\lambda_2]), \varphi_4)$. To get a class of maps $M[\psi]$ from S_{φ_1} to S_{φ_2}, we may use the ones we constructed from ψ using φ_3 and φ_4. Suppose now that we choose instead $\varphi_5 \sim \varphi_1$ and $\varphi_6 \sim \varphi_2$. We want to see that $M[\psi]$ is invariant. We may assume we have $\theta_1: F_1^{(0)} \to G$ and $\theta_2: F_2^{(0)} \to G$ so that $\theta_1\circ r(\xi)\varphi_5(\xi) = \varphi_3(\xi)\theta_1\circ d(\xi)$ for $\xi \in F_1$ and $\theta_2\circ r(\eta)\varphi_6(\eta) = \varphi_4(\eta)\theta_2\circ d(\eta)$ for $\eta \in F_2$. We start with a $\theta: F_1^{(0)} \to G$ so that $\theta\circ r(\xi)\varphi_4\circ\psi(\xi) = \varphi_3(\xi)\theta\circ d(\xi)$ for $\xi \in F_1$, and define $\theta'(u) = \theta_1(u)^{-1}\theta(u)\theta_2\circ\tilde{\psi}(u)$ for $u \in F_1^{(0)}$. Then $\theta'\circ r(\xi)\varphi_6\circ\psi(\xi) = \varphi_4(\xi)\theta'\circ d(\xi)$ for $\xi \in F_1$. There are isomorphisms $f_1: T(\varphi_3) \to T(\varphi_5)$ and $f_2: T(\varphi_4) \to T(\varphi_6)$ given by $f_1(x, u) = (x\theta_1(u), u)$ and $f_2(x, u) = (x\theta_2(u), u)$ (proof of Lemma A3.7). These satisfy $f^{\theta'}\circ f_1 = f_2\circ f^{\theta}$ and induce isomorphisms $S_{\varphi_3} \to S_{\varphi_5}$ and $S_{\varphi_4} \to S_{\varphi_6}$. Hence $M(\psi, \theta)$ is equivalent to $M(\psi, \theta')$ under these isomorphisms, so the class $M[\psi]$ transfers from maps of S_{φ_3} to S_{φ_4} to maps of S_{φ_5} to S_{φ_6} in a consistent way.

LEMMA 4.5. *If $\psi_1: \mathscr{F}_1 \to \mathscr{F}_2$ and $\psi_2: \mathscr{F}_2 \to \mathscr{F}_3$ are homomorphisms, for $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$ in $\mathscr{M}(G)$, then $M([\psi_2]\circ[\psi_1]) = M[\psi_2]\circ M[\psi_3]$.*

5. Special properties of groupoid homomorphisms.

Here we give definitions of several properties a homomorphism of measured groupoids may have. In keeping with the viewpoint expressed in the introduction, we begin with an interpretation of certain properties of continuous group homomorphisms in ways which apply to groupoids. Suppose F and G are locally compact groups and $\varphi: F \to G$ is a continuous homomorphism. Then F acts on G via φ by $x\cdot\xi = x\varphi(\xi)$, and we have the following equivalences, by which we learn how to define the terms for groupoids:

(1) φ is one-one iff F acts freely on G iff the groupoid $G \times F$ (thinking of G as an F-space) is principal.

(2) $\varphi(F)$ is dense in G iff the natural homomorphism of $\varphi(F)^-$ into G is an isomorphism onto.

(3) $\varphi(F)$ is closed in G iff the space of orbits in G under the action of F, G/F, is analytic [11, Theorem 7.2].

(4) $\varphi(F) = G$ iff φ has dense, closed range iff G/F consists of one point up to a null set.

(5) φ is a topological embedding iff φ is an isomorphism of F onto $\varphi(F)^-$ iff $G \times F$ is principal and G/F is analytic.

DEFINITION 5.1. Let $(F, [\mu])$ and $(G, [\nu])$ be measurable groupoids and let $\varphi: (F, [\mu]) \to (G, [\nu])$ be a strict homomorphism, and suppose $\varphi^* \sim \varphi$ and φ^* takes values in an i.c. on which ν is (r, d)-quasi-invariant. Set $T = T(\varphi^*) = \{(x, u) \in G \times F^{(0)}: d(x) = \varphi^*(u)\}$,

$$\nu_1 = \int \nu_u d(\varphi^*_*(\tilde{\mu}))(u)$$

and $\lambda_1 = \nu_1 * \tilde{\mu}$. Form the measured groupoid $(T*F, [\lambda_1 * \mu])$. Let $(S_\varphi, \lambda) = (S_{\varphi^*}, \lambda)$ as in Theorem 3.5.

(a) φ is called strictly immersive iff $T*F$ is principal.

(a′) φ is called immersive iff $\varphi | F_1$ is strictly immersive for some i.c. F_1 of F.

(b) We say $\varphi(F)$ is dense or φ has dense range iff there is an i.c. G_0 of G and a conull strict G_0-space $S_0 \subseteqq S_\varphi$ such that $j | S_0 * G_0$ is an isomorphism onto G_0.

(c) We say $\varphi(F)$ is closed or φ has a strictly closed range iff the orbit space T/F is analytic.

(c′) We say φ has closed range iff $\varphi | F_1$ has strictly closed range for some i.c. F_1 of F.

(d) We say φ is surjective iff φ has a dense closed range.

(e) We say φ is a strict imbedding iff $T*F$ is principal and T/F is analytic.

(e′) We say φ is an imbedding iff $\varphi | F_1$ is a strict imbedding for some i.c. F_1 of F.

REMARKS. (1) There can always be sets of measure zero which are basically irrelevant, as when a null set of units is adjoined to a group, and the nonstrict forms of the definitions are to take account of such cases, even though they should be exceptional. The nonstrict definitions may also be much easier to verify in concrete cases, even when the strict definitions are satisfied. The extra freedom makes the machinery a little more tractable.

(2) We will see in Theorem 6.7 that for any homomorphism φ the φ' associated with it by Theorem 3.5 has dense range. (3) The definition of "dense range" is phrased so that it says the range

closure is isomorphic to G (up to null sets) under its natural imbedding. This sounds natural. However another formulation is more convenient for applications of the concept. The function p taking (x, u) to $r(x)$ is the projection of $T(\varphi)$ onto $G^{(0)}$ relative to which the action of G on $T(\varphi)$ is defined and it is constant on F-orbits. Thus it factors through the ergodic decomposition $f\colon T(\varphi) \to S_\varphi$ via the projection $q\colon S_\varphi \to G^{(0)}$ in the definition of the action of G on S_φ. The units of $S_\varphi * G$ are just the graph of q, and if $j \mid (S_\varphi * G)^{(0)}$ is one-one a.e., that means q is one-one a.e. Thus whenever φ has dense range the projection p is an ergodic decomposition. We use this in Theorems 7.16, 7.17 and 7.18.

(4) Let (S, μ) be an ergodic $\boldsymbol{Z}$-space and let $\varphi\colon S \times \boldsymbol{Z} \to \boldsymbol{R}$ be a homomorphism for which the function f defined by $f(s) = \varphi(s, 1)$ has constant sign, say $f > 0$ everywhere. Then the set $T_0 = \{(s, x) \in S \times \boldsymbol{R}\colon -f(s) < x \leqq 0\}$ meets each $\boldsymbol{Z}$-orbit exactly once. Hence φ has closed range. Furthermore, $\boldsymbol{Z}$ acts freely on almost all of S and hence on $S \times \boldsymbol{R}$, so φ is in fact an imbedding. (The set T_0 is the space for the flow built under f; see [16].)

Before proceeding to our main objective, we prove the following theorem, which asserts that a properly ergodic groupoid cannot be mapped *onto* a group. A consequence is that in Corollaries 2.1 and 3.3 of [22], "dense range" cannot be strengthened to "onto".

THEOREM 5.2. *If $(F, [\mu])$ is a measurable groupoid which has a homomorphism φ onto a locally compact group G, then $(F, [\mu])$ is similar to a group,* i.e., *is essentially transitive.*

Proof. The groupoid $(G \times F^{(0)}) * F$ has a homomorphism into F and the assumption that φ is onto implies that $(G \times F^{(0)}) * F$ is essentially transitive. It follows that F is essentially transitive.

Now we want to show that these definitions are similarity invariant in $\mathscr{M}(G)$. The first lemma is immediate from Lemma 3.7.

LEMMA 5.3. *Suppose $((F_1, [\mu_1]), \varphi_1)$ and $((F_2, [\mu_2]), \varphi_2)$ are similar elements of $\mathscr{M}(G)$. Then φ_1 has dense range iff φ_2 has dense range.*

LEMMA 5.4. *Suppose $((F_1, [\mu_1]), \varphi_1)$ and $((F_2, [\mu_2]), \varphi_2)$ are similar elements of $\mathscr{M}(G)$. Then φ_1 has closed range iff φ_2 has closed range.*

Proof. Because of the symmetry, we need only prove one implication. Let $\psi_1\colon F_1 \to F_2$ and $\psi_2\colon F_2 \to F_1$ be a similarity. These may be replaced by similar homomorphisms, if needed, so we may begin with (φ_1, ψ_2) composable. Then we may choose an i.c. F_4 of

F_2 such that if $\varphi_4 = \varphi_2|F_4$ and $\psi_4 = \psi_2|F_4$ then φ_4 and ψ_4 are strict, $\varphi_1\circ\psi_4$ is strictly similar to φ_4 and $T(\varphi_4)/F_4$ is analytic. Next, choose ψ_1 and an i.c. F_3 of F_1 so that $\psi_3 = \psi_1|F_3$ is strict, $\psi_3(F_3) \subseteqq F_4$, $\psi = \psi_4\circ\psi_3$ is strictly similar to the identity on F_3, and $\varphi_4\circ\psi_3$ is strictly similar to $\varphi_3 = \varphi_1|F_3$.

There exist strict similarities θ_1, θ_2 and θ:

$$\theta_1\circ r(\xi)\varphi_4\circ\psi_3(\xi) = \varphi_3(\xi)\theta_1\circ d(\xi) \text{ for } \xi\in F_3\ ,$$
$$\theta_2\circ r(\xi)\varphi_3\circ\psi_4(\xi) = \varphi_4(\xi)\theta_2\circ d(\xi) \text{ for } \xi\in F_4\ ,$$

and

$$\theta\circ r(\xi)\psi(\xi) = \xi\theta\circ d(\xi) \text{ for } \xi\in F_3\ .$$

Define $f^{\theta_1}(x, u) = (x\theta_1(u), \psi_1(u))$, for $(x, u)\in T(\varphi_3)$, $f^{\theta_2}(x, u) = (x\theta_2(u), \psi_2(u))$ for $(x, u)\in T(\varphi_2)$ and $f(x, u) = (x\varphi_3\circ\theta(u), u)$ for $(x, u)\in T(\varphi_3)$. Then $(x, u)\sim(y, v)$ in $T(\varphi_3)\Rightarrow f^{\theta_1}(x, u)\sim f^{\theta_1}(y, v)$ in $T(\varphi_2)$. If $f^{\theta_1}(x, u)\sim f^{\theta_1}(y, v)$ in $T(\varphi_2)$, then $f^{\theta_2}\circ f^{\theta_1}(x, u)$ is in $T(\varphi_3)$ because $\psi(u)\in F_3^{(0)}$, and so is $f^{\theta_2}\circ f^{\theta_1}(y, v)$, and these are equivalent under F_3 because they are equivalent under F_1 and both units are in $F_3(\psi(u)$ and $\psi(v))$. Thus $(x\theta_1(u)\theta_2\circ\psi_1(u), \psi(u)) \sim (y\theta_1(v)\theta_2\circ\psi_1(v), \psi(v))$.

Now $w\in F_3^{(0)} \Rightarrow d\circ\theta(w) = \psi(w)$ and $r\circ\theta(w) = w$, so we can operate on the points with $\theta(u)^{-1}$ and $\theta(v)^{-1}$, getting two points which are equivalent in $T(\varphi_3)$:

$$(x\theta_1(u)\theta_2\circ\psi_1(u)\varphi_3(\theta(u)^{-1}), u) \sim (y\theta_1(v)\theta_2\circ\psi_1(v)\varphi_3(\theta(v)^{-1}), v)\ .$$

Now $\varphi_3\circ\theta$ is a similarity (strict) of $\varphi_3\circ\psi$ with φ_3, so the function f defined above is an isomorphism of $T(\varphi_3)$ onto $T(\varphi_3\circ\psi)$. Hence there is a $\xi\in F_3$ with $r(\xi) = u$, $d(\xi) = v$ and

$$x\theta_1(u)\theta_2\circ\psi_1(u)\varphi_3\circ\psi(\xi) = y\theta_1(v)\theta_2\circ\varphi_1(v)\ .$$

Since $\psi = \psi_2\circ\psi_1|F_3$, the similarity equations give

$$\begin{aligned}\theta_1(u)\theta_2\circ\psi_1(u)\varphi_3\circ\psi_2\circ\psi_1(\xi) &= \theta_1(u)\varphi_2\circ\psi_1(\xi)\theta_2\circ\psi_1(v)\\ &= \varphi_3(\xi)\theta_1(v)\theta_2\circ\varphi_1(v)\ .\end{aligned}$$

Thus $(x, u)\xi = (y, v)$ in $T(\varphi_3)$, i.e., $(x, u) \sim (y, v)$. Hence f^{θ_1} induces an imbedding of $T(\varphi_3)/F_3$ into $T(\varphi_2)/F_2$, as a G-space.

The next proof includes the fact that immersiveness is equivalent to a sort of one-one-ness.

LEMMA 5.5. *Suppose $((F_1, [\mu_1]), \varphi_1)$ and $((F_2, [\mu_2]), \varphi_2)$ are similar elements of $\mathscr{M}(G)$. Then φ_1 is immersive iff φ_2 is.*

Proof. Again the symmetry means we need to prove only one implication. By passing to similar homomorphisms, as permitted by Definition 5.1 and Lemma 3.7, we may arrange that φ_1 and φ_2 take values in an i.c. on which ν is (r, d)-quasi-invariant and then that (φ_1, ψ_2) is composable. Next choose an i.c. F_4 of F_2 on which φ_2 and ψ_2 are strict, $\varphi_1 \circ \psi_2$ is strictly similar to φ_2 and φ_2 is strictly immersive. Then there is a choice of ψ_1 and an i.c. F_3 of F_1 such that $\varphi_1 | F_3$ is strict, $\psi_1 | F_3$ is strict, $\psi_1(F_3) \subseteq F_4$, $\psi_2 \circ \psi_1$ is strictly similar to the identity on F_3, and $\varphi_2 \circ \psi_1$ is strictly similar to φ_1 on F_3.

Now we will show that φ_2 is one-one on sets of the form $r^{-1}(v) \cap d^{-1}(u)$ for $u, v \in F_4^{(0)}$. Suppose $r(\xi) = r(\eta) = v$, $d(\xi) = d(\eta) = u$ and $d(x) = \varphi_2(v)$, and let $\varphi_2(\xi) = \varphi_2(\eta)$. Then $(x, v)\xi = (x, v)\eta$, so $\xi = \eta$.

From this we see that the same holds for φ_1 on F_3. Suppose $\xi, \eta \in F_1$, $d(\xi) = d(\eta)$, $r(\xi) = r(\eta)$ and $\varphi_1(\xi) = \varphi_1(\eta)$. Then $\varphi_2 \circ \psi_1(\xi) = \varphi_2 \circ \psi_1(\eta)$ because of the similarity. Hence $\psi_1(\xi) = \psi_1(\eta)$, so $\psi_2 \circ \psi_1(\xi) = \psi_2 \circ \psi_1(\eta)$. By use of the strict similarity of $\psi_2 \circ \psi_1$ with the identity on F_3, we see that $\xi = \eta$. By reversing the argument for φ_2 above, we see that $\varphi_1 | F_3$ is strictly immersive.

COROLLARY 5.6. *If $((F_1, [\mu_1]), \varphi_1)$ and $((F_2, [\mu_2]), \varphi_2)$ are similar elements of $\mathscr{M}(G)$, then φ_1 is an imbedding iff φ_2 is an imbedding.*

6. Some results about immersions, imbeddings, etc. A variety of questions arise naturally about the definitions of § 5. We prove that a composition of imbeddings is an imbedding and a composition of homomorphisms with dense range has dense range. We prove that the homomorphism φ' of Theorem 3.6 has dense range, i.e., that "the range of φ is dense in the closure of the range of φ." There are other results here, and some obvious questions are not answered. Our purpose is to develop some useful facts and answer enough of these questions to justify the definitions.

The first lemma is a rather obvious fact, and we tend to use it without explicit reference, but it may help to state it once. It says that a homomorphism which is an isomorphism of i.c.'s is actually a monomorphism in the sense of category theory.

LEMMA 6.1. *Let $(F, [\lambda])$, $(G, [\mu])$ and $(H, [\nu])$ be measured groupoids and let $\psi: (G, [\mu]) \to (H, [\nu])$ be a homomorphism such that there are* i.c.'s *G_0 of G and H_0 of H with $\psi | G_0$ a strict isomorphism of G_0 onto H_0. If φ_1, φ_2 are homomorphisms of $(F, [\lambda])$ into $(G, [\mu])$ with $[\psi] \circ [\varphi_1] = [\psi] \circ [\varphi_2]$, then $[\varphi_1] = [\varphi_2]$.*

Proof. We may assume that $\varphi_1(F) \cup \varphi_2(F) \subseteq G_0$, so $\psi \circ \varphi_1(F) \cup \psi \circ \varphi_2(F) \subseteq H_0$. In that case, a similarity θ of $\psi \circ \varphi_1$ to $\psi \circ \varphi_2$ must take values in H_0 and $(\psi | G_0)^{-1} \circ \theta$ is a similarity of φ_1 to φ_2.

Next we give the characterization of imbeddings in terms of groupoids given by actions. The first step is a lemma related to the notion of the kernel of a homomorphism. For homomorphisms of virtual groups F into compact groups G, the ergodic groupoids into which $(G \times F^{(0)})*F$ decomposes correspond to the kernel [13]. This relates to condition (b) of Lemma 6.2.

LEMMA 6.2. *Let $(F, [\mu])$ be a measured groupoid and let (T, λ) be an analytic strict $(F, [\mu])$-space. These conditions are equivalent*:

(a) *There are an* i.c. *F_1 of F and a conull analytic F_1-invariant set $T_1 \subseteq T$ such that T_1*F_1 is principal and T_1/F_1 is analytic.*

(b) *Almost every groupoid in an ergodic decomposition of $(T*F, [\lambda*\mu])$ is similar to the trivial group.*

Proof. To prove (a) $\Rightarrow$ (b) we may suppose $T_1 = T$, $F_1 = F$. Then $S = T/F$ is analytic. Let $q\colon T \to S$ be the quotient map and let S_1 be standard and $q_*(\lambda)$-conull. By the von Neumann selection lemma there are a Borel function $c\colon S \to T$ and a conull Borel set $S_0 \subseteq S_1$ such that $q \circ c$ is the identity on S_0. Denote the saturation of A by $[A]$ as usual and define $G_s = (T*F)\,|\,[c(s)]$ for $s \in S$. Now $q([c(S_0)]) = S_0$ and the level sets of q on $[c(S_0)]$ are exactly the F-orbits. Thus the decomposition of $T*F$ given by q produces transitive groupoids which are therefore ergodic, so q is an ergodic decomposition. Since F acts freely, $T*F$ is principal. Then the decomposition of $[c(S_0)]*F$ must produce principal groupoids. Since a principal transitive groupoid is similar to the trivial group, condition (a) implies condition (b).

For the converse, suppose $q\colon T \to S$ is an ergodic decomposition of $T*F$. Let S_0 be a conull set in S such that $G_s = (T*F)\,|\,q^{-1}(s)$ is similar to $\{1\}$ for $s \in S_0$. Then G_s is essentially transitive and essentially principal, so there is an equivalence class in $G_s^{(0)} = q^{-1}(s)$ which is conull and to which the contraction of G_s is principal. Let $\lambda = \int \lambda_s dq_*(\lambda)(s)$ be the decomposition of λ relative to q which we are using. Let $E = \{(t, tx) \in T \times T\colon (t, x) \in T*F\}$. Then $s \in S_0$ implies that λ_s is concentrated on some orbit, and that orbit is $[t]$ iff $\varepsilon_t \times \lambda_s(E) > 0$, and then $q(t) = s$. Choose Borel sets E_1, E_2 with $E_1 \subseteq E \subseteq E_2$ so that $\lambda*\lambda(E_2 - E_1) = 0$. Define $K = \{t \in T\colon \lambda_{q(t)}([t]) > 0\} = \{t \in T\colon \varepsilon_t \times \lambda_{q(t)}(E) > 0\}$, and define $K_i = \{t \in T\colon \varepsilon_t \times \lambda_{q(t)}(E_i) > 0\}$ $(i = 1, 2)$. We have $K_1 \subseteq K \subseteq K_2$ and $\lambda*\lambda = \int \varepsilon_t \times \lambda_{q(t)} d\lambda(t)$, so that $\varepsilon_t \times \lambda_{q(t)}(E_1) = \varepsilon_t \times \lambda_{q(t)}(E_2)$ for almost all t, so $\lambda(K_2 - K_1) = 0$ and $\lambda_s(K_2 - K_1) = 0$ for almost all s. Thus K is measurable for λ and for almost all λ_s and $t \in K$ implies $[t] \subseteq K$, so $s \in S_0$ implies $\lambda_s(X/K) = 0$. Hence K_1 is conull. Thus $q(K_1)$ is conull and the von Neumann selection lemma gives rise to a Borel function $c: S \to T$ such that the Borel set $S_1 = \{s \in S_0: q \circ c(s) \in K_1\}$ is conull. Then $T_1 = [c(S_1)]$ is contained in K, and T_1 is analytic, conull and invariant. The set $F_1 = \{\xi \in F: \xi$ and ξ^{-1} act on $T_1\}$ is an i.c. of F, $T_1 * F_1$ is principal and T_1/F_1 is Borel isomorphic to $q(T_1)$, which is analytic.

LEMMA 6.3. *Let (S, λ, p) be a $(G, [\nu])$ space and form $F = S*G$ and $\mu = \lambda*\nu$ and let $j: S*G \to G$ be the coordinate projection. Then j is an imbedding, and the space S_j is isomorphic to S.*

Proof. First, $F^{(0)}$ is the "graph" of p, which is naturally identified with S. Hence $G*F^{(0)} = T(j)$ is isomorphic to $\{(x, s) \in G \times S$: sx^{-1} is defined$\}$, and the action of $(s, y) \in F$ on $(x, s) \in T(j)$ is $(x, s)(s, y) = (xj(s, y), sy) = (xy, sy)$. Hence $(x, s)(s, x^{-1}) = (r(x), sx^{-1})$, so $X = \{(p(s), s): s \in S\}$ meets each orbit. Now if $(x_1, s_1)(s, y) = (x_2, s_2)$ then $s = s_1$ and $y = x_1^{-1}x_2$, so the action of F on $T(j)$ is free. It follows that X meets each orbit only once. Hence the quotient space $T(j)/F$ is isomorphic to X, and hence to S, so it is analytic.

THEOREM 6.4. *Let $(G, [\nu])$ be a measured groupoid and suppose $\mathscr{F} = ((F, [\mu]), \varphi) \in \mathscr{M}(G)$. Set $(S_\varphi, [\lambda]) = M(\mathscr{F})$ as in § 4 and $\mathscr{F}_\varphi = ((S_\varphi * G_0, [\lambda*\nu]), j)$ where j projects $S_\varphi * G_0$ onto G_0. Then φ is an imbedding iff there is a homomorphism $\psi: \mathscr{F}_\varphi \to \mathscr{F}$ such that (φ', ψ) is a similarity.*

Proof. If such a homomorphism exists, then Corollary 5.5 and Lemma 6.3 combine to show that φ is an imbedding. The rest of the proof is somewhat tedious, so to help keep the parts straight we shall announce the major divisions in the proof. We only need to find $\psi: \mathscr{F}_\varphi \to \mathscr{F}$ so that (φ', ψ) is a similarity of $(F, [\mu])$ with $(S_\varphi * G, [\lambda*\nu])$. By restricting to i.c.'s, we may assume φ is a strict imbedding.

The existence of ψ: Let $T = G*F^{(0)}$, let p be the quotient map of T onto T/F, and form ν_1 and $\lambda_1 = \nu_1 * \tilde{\mu}$ as before. By the proof of Lemma 5.3, p is an ergodic decomposition of $(T, [\lambda_1])$. Since the actions commute, T/F is already a G-space, and we can use it for S_φ. Set $F_\varphi = S_\varphi * G$. Recall that $\varphi'(\xi) = (p(\varphi \circ r(\xi), r(\xi)), \varphi(\xi))$ for $\xi \in F$. The von Neumann selection lemma gives us a Borel function $c: S_\varphi \to T$ such that the Borel set $S_1 = \{s \in S_\varphi: p \circ c(s) = s\}$ is conull relative to $p_*((\varphi \times i)_*(\tilde{\mu}) + \lambda_1)$. This latter measure is used because it gives weight to the image under p of the "graph of $\tilde{\varphi}$" in $G*F^{(0)}$. Let $T_1 = p^{-1}(S_1)$. Then T_1 is F-invariant, Borel and conull and $[c(S_1)] = T_1$ because $c \circ p(t) \sim t$ if $p(t) \in S_1$.

Now let $E = \{(t_1, t_2) \in T^2 : t_1 \sim t_2\}$, which is the one-one image of $T*F$ in T^2 under the map $(t, \xi) \to (t, t\xi)$. By composing the projection of $T*F$ onto F with the inverse of this function, we get a Borel function $f: E \to F$ such that $(t_1, t_2) \in E$ implies $t_1 f(t_1, t_2) = t_2$. Define θ on T_1 by $\theta(t) = f(c\circ p(t), t)$. Then for $t \in T_1$ we have $c\circ p(t)\theta(t) = t$, and θ is Borel. If $(s, x) \in F_\varphi | S_1$, then s and $sx \in S_1$, and $s = p(t)$ for some $t \in T_1$ with tx defined. Then $c(s)$ is in the F-orbit of t since $p(t) = p\circ c(s) = s$, so $c(s)x$ is defined. Then $p(c(s)x) = p(c(s))x = sx$, so $c(s)x \sim c(sx)$. Since the action of F is free, there is exactly one element of F which carries $c(s)x$ to $c(sx)$ and we shall call it $\psi(s, x)$. This defines ψ on $F_\varphi | S_1$. Let ψ be constant on the rest of F_φ. For $t \in T_1$, $c\circ p(t)\theta(t) = t$ so if $(p(t), x) \in F_\varphi | S_1$ we have $(c\circ p(t)x)\theta(t) = tx$. Hence $\psi(p(t), x) = \theta(t)\theta(tx)^{-1}$, so ψ is Borel on $F_\varphi | S_1$ and hence Borel.

ψ is a homomorphism of measurable groupoids: Suppose (s, x) and $(sx, y) \in F_\varphi | S_1$, and that $\xi, \eta \in F$ are such that $c(s)x\xi = c(sx)$ and $c(sx)y\eta = c(sxy)$. Then $c(s)xy\xi\eta = c(sxy)$, and by the uniqueness defining ψ we see that $\psi(s, xy) = \psi(s, x)\psi(sx, y)$. Thus ψ is algebraically a homomorphism of $F_\varphi | S_1$. For the measure theoretic part let $A \subseteq F^{(0)}$ be analytic, saturated and null for $\tilde{\mu}$. The set $G*A = \{(x, u) \in G \times A : d(x) = \varphi(u) \text{ and } u \in A\}$ is null in T and is invariant under both F and G. Thus $p(G*A)$ is null for $\lambda = p_*(\nu_1 * \mu)$. Now $c(s)x\psi(s, x)$ is defined for $(s, x) \in F_\varphi | S_1$, so $c(s)\psi(s, x)$ is defined and hence $\psi(s, p(s)) = r\circ\psi(s, x)$ is the second component of $c(s)$, so $\psi(s, p(s)) \in A$ iff $c(s) \in G*A$ iff $s \in p(G*A)$. Hence $\tilde{\psi}^{-1}(A)$ is null.

$[\varphi]\circ[\psi] = [j, F_\varphi]$ and $[\varphi']\circ[\psi] = [i, F]$: First notice that (φ, ψ) and (φ', ψ) are composable since φ and φ' are strict homomorphisms. Write $c = (a, b)$, so $a: S_\varphi \to G$, $b: S_\varphi \to F^{(0)}$ and for each s, $\varphi\circ b(s) = d\circ a(s)$. Also $(\varphi\circ b(s), b(s))a(s)^{-1} = (a(s), b(s)) = c(s)$, and $p(\varphi\circ b(s), b(s))a(s)^{-1}$ is defined and equal to $p\circ c(s)$, which is s if $s \in S_1$. Let $\theta_1(s) = (p(\varphi\circ b(s), b(s)), a(s)^{-1})$. Then θ_1 is Borel from S_φ to F_φ and $d\circ\theta_1(s) = s$ if $s \in S_1$, so $\theta_1(s)(s, x)\theta_1(sx)^{-1}$ makes sense if $(s, x) \in F_\varphi | S_1$. We must show this product is in fact $\varphi'\circ\psi(s, x)$. If $(y, u) = t \in T$, $s = p(t)$ and $(s, x) \in F_\varphi | S_1$, then with θ as used in constructing ψ we have $(a(s), b(s))\theta(t) = c(s)\theta(t) = t = (y, u)$ so $a(s)\varphi\circ\theta(t) = y$. Similarly $a(sx)\varphi\circ\theta(tx) = x^{-1}y$, so $\varphi\circ\psi(s, x) = \varphi\circ\theta(t)\varphi\circ\theta(tx)^{-1} = a(s)^{-1}xa(sx)$. This shows $\varphi\circ\psi \sim j$. Now $r\circ\psi(s, x) = r\circ\theta(t) = b(s)$ since $c(s)\theta(t)$ is defined, so

$$\begin{aligned}\theta_1(s)(s, x)\theta_1(sx)^{-1} &= (p(\varphi\circ b(s), b(s)), a(s)^{-1}xa(sx)) \\ &= (p(\varphi\circ r\circ\psi(s, x), r\circ\psi(s, x)), \varphi\circ\psi(s, x)) \\ &= \varphi'\circ\psi(s, x)\ ,\end{aligned}$$

as desired.

$[\psi]\circ[\varphi'] = [i, F]$: Since S_1 is $p_*((\tilde{\varphi} \times i)_*(\tilde{\mu}) + \lambda_1)$-conull, the set $U = \{u \in F^{(0)}: p(\varphi(u), u) \in S_1\} = \{u \in F^{(0)}: \varphi'(u) \in F_\varphi | S_1\}$ is conull in $F^{(0)}$. Hence (ψ, φ') is composable. The function θ from above is Borel on T_1 so $u \to \theta_2(u) = \theta(\varphi(u), u)$ is Borel on U. Also $c(p(\varphi(u), u))\theta_2(u) = (\varphi(u), u)$ by the definition of θ. Now if $\xi \in F|U$, then $\varphi(\xi) \in G$ and $\varphi'(\xi) \in F_\varphi | S_1$, and since the actions commute the following makes sense:

$$\begin{aligned}
c(p(\varphi\circ r(\xi),\ & r(\xi)))\varphi(\xi)(\theta_2\circ r(\xi)\xi\theta_2\circ d(\xi)^{-1}) \\
&= (\varphi\circ r(\xi), r(\xi))\varphi(\xi)(\xi\theta_2\circ d(\xi)^{-1}) \\
&= (\varphi(\xi)^{-1}, r(\xi))\xi\theta_2\circ d(\xi)^{-1} \\
&= (\varphi\circ d(\xi), d(\xi))\theta_2\circ d(\xi)^{-1} \\
&= c(p(\varphi\circ d(\xi), d(\xi))) \\
&= c(p(\varphi(\xi)^{-1}, r(\xi))) \\
&= c(p(\varphi\circ r(\xi), r(\xi))\varphi(\xi)) \,.
\end{aligned}$$

By the defining property of ψ, $\psi\circ\varphi'(\xi) = \theta_2\circ r(\xi)\xi\theta_2\circ d(\xi)^{-1}$.

It is desirable for a subobject of a subobject to be a subobject, in a natural way. The characterization of imbeddings given by Theorem 6.4 makes one form of this property relatively easy to establish, as we see below. Notice that having S_1*G a subobject of S_2*G involves a map of S_1 onto S_2 as G-spaces, as expected [16].

THEOREM 6.5. *Let $(G, [\mu])$ be a measured groupoid and let $(S_0, [\lambda_0])$ be a strict $(G, [\mu])$-space. Let $F = S_0*G$ so that $(F, [\lambda_0*\mu])$ is a measurable groupoid, and let $(S_1, [\lambda_1])$ be a strict $(F, [\lambda_0*\mu])$-space. Then $(S_1, [\lambda_1])$ is also a strict $(G, [\mu])$-space in such a way that $(S_1*F, [\lambda_1*(\lambda_0*\mu)])$ is isomorphic to $(S_1*G, [\lambda_1*\mu])$, by mean of an isomorphism φ such that $j_1\circ\varphi = j_0\circ j$, where $j_1: S_1*G \to G$, $j_0: S_0*G \to G$, and $j: S_1*F \to F$ are the natural projections.*

Proof. Let $p_0: S_0 \to G^{(0)}$ be such that sx is defined iff $p_0(s) = r(x)$, i.e., $S_0*G = \{(s, x) \in S_0 \times G: p_0(s) = r(x)\}$. Let $p_1: S_1 \to S_0$ be such that $s_1(s, x)$ is defined iff $p_1(s_1) = s$, for $s_1 \in S_1$ and $(s, x) \in F$. Set $p = p_0\circ p_1$. Now if $p(s_1) = r(x)$, then $(p_1(s_1), x) \in F$ and $s_1(p_1(s_1), x)$ is defined, so we can define $s_1x = s_1(p_1(s_1), x)$. Now in that case,

$$(s_1(p_1(s_1), x))(p_1(s_1)x, x^{-1})$$

is defined, so $p_1(s_1x) = p_1(s_1)x$. If $p(s_1) = r(x)$ and $d(x) = r(y)$ then $s_1(xy) = s_1(p_1(s_1), xy) = s_1((p_1(s_1), x)(p_1(s_1)x, y)) = (s_1(p_1(s_1), x))(p_1(s_1)x, y) = (s_1x)(p_1(s_1x), y) = (s_1x)y$, and if $p_1(s_1) = r(x)$, $p_1(s_1x) = r(y)$ then $d(x) = r(y)$

and the calculation is reversible. Thus we do have a strict action of G on S_1 ($p(S_1) = G^{(0)}$ because $p_0(S_0) = G^{(0)}$ and $p_1(S_1) = S_0$). The action is clearly Borel.

From $p_1*(\lambda_1) \sim \lambda_0$ and $p_0*(\lambda_0) \sim \tilde{\mu}$ it follows that $p_*(\lambda_1) \sim \tilde{\mu}$. By changing λ_0 and then λ_1 we may arrange that $p_1*(\lambda_1) = \lambda_0$ and $p_0*(\lambda_0) = \tilde{\mu}$. Then $p_*(\lambda_1) = \tilde{\mu}$. Let $\varphi(s_1, (p_1(s_1), x)) = (s_1, x)$; then φ is a Borel groupoid isomorphism of S_1*F onto S_1*G. If the measures agree then $(S_1*G, [\lambda_1*\mu])$ is a measured groupoid and the isomorphism statement is proved. Let $\mu = \int \mu^u d\tilde{\mu}(u)$ be a decomposition of μ relative to r. Then $\lambda_0*\mu = \int \varepsilon_s \times \mu^{p_0(s)} d\lambda_0(s)$ and this is the decomposition of $\lambda_0*\mu$ relative to r. Thus $\lambda_1*(\lambda_0*\mu) = \int \varepsilon_s \times (\varepsilon_{p_1(s)} \times \mu^{p(s)}) d\lambda_1(s)$ which maps to $\lambda_1*\mu = \int \varepsilon_s \times \mu^{p(s)} d\lambda_1(s)$ under φ.

To complete the proof, we observe that $j_1 \circ \varphi = j_0 \circ j$ is obvious.

For measured groupoids, similarities are like isomorphisms for many other categories. The next result shows that this idea is compatible with the idea that a surjective imbedding should be like an isomorphism, namely a similarity.

THEOREM 6.6. *Let $(F, [\lambda])$ and $(G, [\mu])$ be measured groupoids and let $\varphi: F \to G$ be a homomorphism. There is a homomorphism $\psi: G \to F$ such that (φ, ψ) is a similarity, iff φ is both surjective and an imbedding.*

Proof. If φ is an imbedding, Theorem 6.3 says there is a homomorphism $\psi_1: S(\varphi)*G \to F$ such that (φ', ψ_1) is a similarity. If φ is also surjective, then φ has dense range (by definition), so the inclusion $j: S(\varphi)*G \to G$ (i.e., coordinate projection) is a strict isomorphism of some S_0*G_0 onto G_0 where G_0 is an i.c. of G and S_0 is a conull strict G_0-space in $S(\varphi)$. Let $j_0 = j \mid (S_0*G_0)$ and take φ_0 to be similar to φ with $\varphi_0(F) \subseteq G_0$. Then $(\varphi_0, \psi_1 \circ j_0^{-1})$ is a similarity, so the desired ψ exists.

Now suppose (φ, ψ) is a similarity. We prove first that for $u, v \in F^{(0)}$, φ takes $r^{-1}(v) \cap d^{-1}(u)$ one-one onto $r^{-1}(\varphi(v)) \cap d^{-1}(\varphi(u))$. If $r(x) = v$ and $d(x) = u$, then $\psi \circ \varphi(x) = \theta(v) x \theta(u)^{-1}$. If $v_1 = \psi \circ \varphi(v)$ and $u_1 = \psi \circ \varphi(u)$, this shows that $\psi \circ \varphi$ takes $r^{-1}(v) \cap d^{-1}(u)$ one-one onto $r^{-1}(v_1) \cap d^{-1}(u_1)$. In particular φ is one-one on $r^{-1}(v) \cap d^{-1}(u)$. By symmetry, ψ is one-one on $r^{-1}(\varphi(v)) \cap d^{-1}(\varphi(u))$, but it also must take this set onto $r^{-1}(v_1) \cap d^{-1}(u_1)$. Thus $\varphi(r^{-1}(v) \cap d^{-1}(u)) = r^{-1}(\varphi(v)) \cap d^{-1}(\varphi(u))$.

Now let (x, u) and $(x_1, u_1) \in T(\varphi) = \{(y, v) \in G \times F^{(0)}: d(y) = \varphi(v)\}$, and suppose $r(x) = r(x_1)$. Then $x^{-1}x_1 \in r^{-1}(\varphi(u)) \cap d^{-1}(\varphi(u_1))$, so there is a $\xi \in r^{-1}(u) \cap d^{-1}(u_1)$ with $\varphi(\xi) = x^{-1}x_1$. Then $(x, u)\xi = (x_1, u_1)$. Thus the level sets of $r^+ (r^+(x, u) = r(x))$ are exactly the F-orbits. Hence

φ has a dense range and a closed range.

Thus $S(\varphi)*G$ is essentially isomorphic to G via the projection j. Under this isomorphism, φ corresponds to φ' and the fact that (φ, ψ) is a similarity means that $(\varphi', \psi\circ j)$ is a similarity. Thus φ is an imbedding, by Theorem 6.3.

The following theorem is another result which is not surprising in its basic content. We have defined separately the terms range closure and dense range. For $\varphi\colon F \to G$, $S_\varphi*G$ is the range closure and $\varphi = j\circ\varphi'$ (Theorem 3.5) is the formula that says φ factors through this subobject. We will discuss this further in § 7, but now we prove the fact that the range is dense in the range closure.

THEOREM 6.7. *Let* $(F, [\lambda])$ *and* $(G, [\mu])$ *be measured groupoids and let* φ *be a homomorphism from* $(F, [\lambda])$ *to* $(G, [\mu])$. *Take* $S(\varphi)$ *and* φ' *as in Theorem* 3.5. *Then* φ' *has dense range.*

Proof. Let $T(\varphi) = \{(x, u) \in G \times F^{(0)}\colon d(x) = \varphi(u)\}$. The method used in Theorem 3.6 was that of Theorem 3.2, which produces a map $f\colon T(\varphi) \to G^{(0)}*\mathscr{F}$. In fact we may take $S = S(\varphi) = f(T(\varphi))$. If $p\colon S \to G^{(0)}$ is the function such that sx is defined iff $p(s) = r(x)$, then $p\circ f(x, u) = r(x)$ for $(x, u) \in T(\varphi)$. Define $q(u) = f(\varphi(u), u)$ for $u \in F^{(0)}$. Then $\xi \in F$ implies $\varphi'(\xi) = (q(r(\xi)), \varphi(\xi))$, and

$$\begin{aligned} T(\varphi') &= \{(s, x, u) = ((s, x), u) \in (S*G) \times F^{(0)}\colon (sx, d(x)) = \varphi'(u)\} \\ &= \{(q(u)x^{-1}, x, u)\colon d(x) = \varphi(u)\}\ . \end{aligned}$$

If we define $g(s, x, u) = (x, u)$, g is a Borel space isomorphism of $T(\varphi')$ onto $T(\varphi)$. A simple calculation shows that g is a strict F-space isomorphism.

Now let us verify that g preserves the relevant measure classes. Let ν be the image on S of $\mu*\tilde{\lambda}$ under f. Then $\nu*\mu = \int_S \varepsilon_s \times \mu^{p(s)} d\nu(s)$ is the measure on $S*G$, so $(\nu*\mu)^{(s,p(s))} = \varepsilon_s \times \mu^{p(s)}$ is the integrand in the decomposition of $\nu*\mu$ relative to r over $\nu = r_*(\nu*\mu)$. Thus $(\varepsilon_s \times \mu^{p(s)})^{-1}$ is the integrand if we decompose $(\nu*\mu)^{-1}$ relative to d over $\nu = d_*((\nu*\mu)^{-1})$. Since $(\nu*\mu)^{-1} \sim \nu*\mu$, the measure class on $T(\varphi')$ is that of $\int_{F(0)} (\varepsilon_{q(u)} \times \mu^{\varphi(u)})^{-1} \times \varepsilon_u d\tilde{\lambda}(u)$. For almost all u we have $\mu^{\varphi(u)} \sim (\mu_{\varphi(u)})^{-1}$. Define h on $T(\varphi)$ by $h(x, u) = ((q(u), x^{-1})^{-1}, u)$: note that $(q(u), x^{-1}) \in S*G$. Then $h(\cdot, u)_*(\mu_{\varphi(u)}) \sim (\varepsilon_{q(u)} \times \mu^{\varphi(u)})^{-1} \times \varepsilon_u$ for almost all $u \in F^{(0)}$. Now $(q(u), x^{-1})^{-1} = (q(u)x^{-1}, x)$, so $g_*(h(\cdot, u)_*(\mu_{\varphi(u)})) = \mu_{\varphi(u)} \times \varepsilon_u$. Thus g is a measure class isomorphism, giving an isomorphism of $T(\varphi')*F$ onto $T(\varphi)*F$.

Define $f'(q(u)x^{-1}, x, u) = f(x, u)$. Then $f'\colon T(\varphi') \to S$ is an ergodic decomposition of $T(\varphi')*F$. Now S is a strict $S*G$ space by the formula $s(s, y) = sy$, and $T(\varphi')$ is a strict $S*G$-space by the formula $(s, x, u)(s, y) = (sy^{-1}, y^{-1}x, u)$. Now if $s = f(x, u)$, $sy = f(y^{-1}x, u)$. Thus f' is a strict $S*G$-space map. Hence we may use S as $S(\varphi')$. Let p' be the map of S to $(S*G)^{(0)} = S$ involved in the $S*G$-space structure. As above for $S*G$, we have $p'\circ f'(s, x, u) = s$. Also $s = q(u)x^{-1} = f(\varphi(u), u)x^{-1} = f((\varphi(u), u)x^{-1}) = f(x, u)$. Hence p' is the identity on S. Thus φ' has dense range.

The next theorem is of minor interest, and will not be used, so the proof is omitted. After that we have a useful technical lemma.

THEOREM 6.8. *Let $(F_1, [\mu_1])$, $(F_2, [\mu_2])$ be measured groupoids with associated equivalence relations $(E_1, [\nu_1])$ and $(E_2, [\nu_2])$, respectively. If $\psi: F_1 \to F_2$ has dense range, then $\psi_0 = (\tilde{\psi} \times \tilde{\psi})|E_1$ has range dense in E_2.*

LEMMA 6.9. *Let $(F_1, [\mu_1])$, $(F_2, [\mu_2])$ be measured groupoids and let $\varphi: F_1 \to F_2$ be a homomorphism with dense range. If $p: F_2^{(0)} \to S$ is an ergodic decomposition of F_2, then $p\circ\tilde{\varphi}$ is an ergodic decomposition of F_1.*

Proof. Define $q: T(\varphi) \to F_1^{(0)}$ by $q(x, u) = u$, $r^+: T(\varphi) \to F_2^{(0)}$ by $r^+(x, u) = r(x)$. We will use Lemma 2.8. Suppose $g: F_1^{(0)} \to Z$ is Borel and constant on equivalence classes, where Z is countably separated. If $x \in F_2$, $\xi \in F_1$ and $(x, r(\xi)) \in T(\varphi)$, then $q((x, r(\xi))\xi) = d(\xi) \sim r(\xi) = q(x, r(\xi))$. Hence $g\circ q$ is constant on equivalence classes, and there is a Borel function $g_1: F_2^{(0)} \to Z$ such that $g_1\circ r^+ = g\circ q$ a.e., because r^+ is an ergodic decomposition. If $(x, u) \in T(\varphi)$ and $r(x) = r(y)$, where $y \in F_2$, then $q((x, u)y) = q(y^{-1}x, u) = u = q(x, u)$, so $g\circ q((x, u)y) = g\circ q(x, u)$. Now $g_1\circ r^+ = g\circ q$ on some conull set $K \subseteq T(\varphi)$. Then K is $\mu_{2,\varphi(u)} \times \varepsilon_u$-conull for $\tilde{\mu}_1$-almost all u, and for any such u, g_1 is constant on $\{r(x): d(x) = \varphi(u) \text{ and } (x, u) \in K\}$. Thus g_1 is essentially constant on almost every orbit in $F_2^{(0)}$. There is a g_2 which agrees a.e. with g_1 and is constant on almost every orbit: regard $Z \subseteq [0, 1]$ and define $g_2(u) = \int g_1 dr_*(\mu_u)$ for almost all u. Now there is an $h: S \to Z$ with $h\circ p = g_2$ a.e. Then $h\circ p\circ r^+ = g_2\circ r^+ = g\circ q$ a.e. Let $T_1 = \{t \in T(\varphi): h\circ p\circ r^+(t) = g\circ q(t)\}$. Then T_1 is conull and invariant under both F_1 and F_2. If $\gamma(u) = (\varphi(u), u)$ for $u \in F_1^{(0)}$, this implies that $\gamma(u) \in T_1$ for almost all u. Thus $h\circ p\circ r^+\circ\gamma = g\circ q\circ\gamma = g$ a.e. Now $r^+\circ\gamma = \tilde{\varphi}$, so we have $g = h\circ p\circ\tilde{\varphi}$ a.e. Thus $p\circ\tilde{\varphi}$ has the factorization property and is, therefore, an ergodic decomposition, by Lemma 2.8.

LEMMA 6.10. *Let $(F_1, [\mu_1])$ and $(F_2, [\mu_2])$ be measured groupoids* *and let $\psi: F_1 \to F_2$ have dense range. Let $(S, [\nu])$ be an $(F_2, [\mu_2])$-space with $p: S \to F_2^{(0)}$ determining when sx is defined. Then ψ can be used to make an F_1-space of $S*F_1^{(0)} = \{(s, u): p(s) = \psi(u)\}$, and $S*F_1 = \{(s, x): p(s) = \psi \circ r(x)\}$ is a groupoid. The function $\psi^+: S*F_1 \to S*F_2$ defined by $\psi^+(s, x) = (s, \psi(x))$ is a homomorphism with dense range.*

Proof. First, we suppose ψ is strict. We define $(s_1, x_1)(s_2, x_2) = (s_1, x_1x_2)$ for (s_1, x_1) and $(s_2, x_2) \in S*F_1$, when $s_1\psi(x_1) = s_2$ and $d(x_1) = r(x_2)$. It is easy to verify that this makes a groupoid, and $(S*F_1)^{(0)} = S*F^{-(0)}$, while ψ^+ is algebraically a homomorphism.

Now let $\nu = \int \nu_u d\tilde{\mu}_2(u)$ be a decomposition of ν relative to p, and observe that $\{x: \nu_{r(x)}x \sim \nu_{d(x)}\}$ is closed under multiplication. It is conull because we assumed $[\nu]$ is invariant [19, Theorem 2.9]. Thus the set contains an i.c. Since we may replace ψ by a similar homomorphism, we may assume ψ takes values in this i.c. Then Theorem 2.9 of [19] shows that $[\nu]$ is F_1-invariant, and $(S*F_1, [\nu*\mu_1])$ is a measured groupoid.

Now $T(\psi^+) = \{((s, x), (sx, u)) \in (S*F_2) \times (S*F_1)^{(0)}: d(x) = \psi(u)\}$, and if $(s, x) \in S*F_2$, $\xi \in F_1$ and $\psi \circ r(\xi) = d(x)$, then

$$\begin{aligned}((s, x), (sx, r(\xi)))(sx, \xi) &= ((s, x)\psi^+(sx, \xi), ((sx)\psi(\xi), d(\xi))) \\ &= ((s, x\psi(\xi)), ((sx)\psi(\xi), d(\xi))) .\end{aligned}$$

Thus $T(\psi^+)$ is naturally isomorphic to $S*(F_2*F_1^{(0)})$, carrying the action to the one in which F_1 operates only on the factor $F_2*F_1^{(0)}$, as in the construction of $S(\psi)$. Thus every invariant set is of the form $A*B$ where $A \subseteq S$ and B is invariant in $F_2*F_1^{(0)}$. Hence $(s, x, u) \to (s, r(x))$ is an ergodic decomposition relative to F_1. Transferred to $T(\psi^+)$, this says ψ^+ has dense range.

THEOREM 6.11. *Let $(F_1, [\mu_1])$, $(F_2, [\mu_2])$ and $(G, [\lambda])$ be measured groupoids, and suppose that $\psi: F_1 \to F_2$, $\varphi_1: F_1 \to G$ and $\varphi_2: F_2 \to G$ are homomorphisms such that $[\varphi_2] \circ [\psi] = [\varphi_1]$. If ψ has dense range, then $M[\psi]$ is an isomorphism.*

Proof. By taking i.c.'s and replacing homomorphisms by similar ones, we may arrange that φ_2 and ψ are strict and that $\varphi_2 \circ \psi = \varphi_1$. Then φ_1 is also strict. Now $T(\varphi_2)$ is an F_2-space so $T(\varphi_2)*F_1^{(0)} = \{((x, v), u) \in T(\varphi_2) \times F_1^{(0)}: \psi(u) = v\}$ is an F_1-space, and carries an invariant measure class. This space is naturally isomorphic to $T(\varphi_1) = \{(x, u) \in G \times F_1^{(0)}: d(x) = \varphi(u)\}$ as an F_1-space, via g, where $g(x, u) = ((x, \psi(u)), u)$. The measure on $T(\varphi_1)$ is $\lambda*\tilde{\mu}_1 = \int \lambda_{\varphi(u)} \times \varepsilon_u d\tilde{\mu}_1(u)$, which g carries to $\int (\lambda_{\varphi(u)} \times \varepsilon_{\psi(u)}) \times \varepsilon_u d\tilde{\mu}_1(u)$. Now the measure we use on $T(\varphi_2)$ is $\int \lambda_v \times \varepsilon_v d\tilde{\mu}_2(v)$, so the measure we want on $T(\varphi_2)*F_1^{(0)}$, as in Lemma 6.10, is just $g_*(\lambda * \tilde{\mu}_1)$. Also, under this isomorphism with $T(\varphi_2)*F_1^{(0)}$, the function $(x, u) \mapsto (x, \psi(u))$, which has $M(\psi)$: $S(\varphi_1) \to S(\varphi_2)$ as a quotient, just corresponds to $(\psi^+)^\sim$. If $f\colon T(\varphi_2) \to S(\varphi_2)$ is a G-equivariant ergodic decomposition of $T(\varphi_2)*F_2$, then $f \circ (\psi^+)^\sim$ is a G-equivariant ergodic decomposition of $T(\varphi_1)*F_1$. Hence $(x, u) \mapsto f(x, \psi(u))$ is a G-equivariant ergodic decomposition of $T(\varphi_1)*F_1$ and may be used to establish $S(\varphi_2)$ as $S(\varphi_1)$, and $M(\psi)$ becomes the identity. Thus $M(\psi)$ is an isomorphism.

Consider now homomorphisms $\varphi\colon (F, [\lambda]) \to (G, [\mu])$ and $\psi\colon (G, [\mu]) \to (H, [\nu])$. The following theorem makes precise the intuitive content of the statement that the closure of the range of ψ-restricted-to-the-closure-of-the-range-of-φ is the closure of the range of $\psi \circ \varphi$, i.e., $\psi(\varphi(F)^-)^- = (\psi \circ \varphi(F))^-$.

THEOREM 6.12. *If $(F, [\lambda])$, $(G, [\mu])$ and $(H, [\nu])$ are measured groupoids and $\varphi\colon F \to G$, $\psi\colon G \to H$ are homomorphisms, then $S(\psi \circ \varphi)$ and $S(\psi \circ j)$ are isomorphic as H-spaces, where $j\colon S(\varphi)*G \to G$ is the inclusion homomorphism of the closure of the range of φ into G.*

Proof. We have $\psi \circ \varphi = \psi \circ j \circ \varphi'$, and φ' has dense range, so $M(\varphi')$ is an isomorphism of $S(\psi \circ \varphi)$ with $S(\psi \circ j)$ (mod null sets).

Now we can show that similar groupoids have the same actions, as isomorphic groups have the same actions.

THEOREM 6.13. *Let $(F_1, [\mu_1])$ and $(F_2, [\mu_2])$ be similar measured groupoids. Then there is a natural one-one correspondence between $(F_1, [\mu_1])$-spaces and $(F_2, [\mu_2])$-spaces.*

Proof. Let $\varphi_1\colon F_1 \to F_2$ and $\varphi_2\colon F_2 \to F_1$ be the similarity. If $(S_1, [\lambda_1])$ is an $(F_1, [\mu_1])$-space, let $j_1\colon S_1*F_1 \to F_1$ be the natural homomorphism. We define $\tau(S_1) = S_2$ to be $S(\varphi_1 \circ j_1)$. We define τ_2 the same way for $(F_2, [\mu_2])$-spaces. If $S_2 = \tau_1(S_1)$, then Theorem 6.12 shows that $\tau_2(S_2) = S(\varphi_2 \circ j_2) \cong S(\varphi_2 \circ \varphi_1 \circ j_1) \cong S(j_1) \cong S_1$, i.e., $\tau_2 \circ \tau_1(S_1) \cong S_1$. Similarly $\tau_1 \circ \tau_2(S_2)$ is always isomorphic to S_2.

Now if S_1 and S_1' are F_1-spaces, and $f\colon S_1 \to S_1'$ is equivariant, then $\varphi = f \times i$ is a homomorphism of S_1*F_1 to $S_1'*F_1$ with $\varphi_1 \circ j_1' \circ \varphi = \varphi_1 \circ j_1$. Then φ induces an F_2-space map of $\tau_1(S_1)$ to $\tau_1(S_1')$.

The next lemma and Theorem 6.16 are additional ways of saying that containment is transitive for measured groupoids.

LEMMA 6.14. *Let $\varphi\colon (F, [\lambda]) \to (G, [\mu])$ be an extensive homo-*

morphism of measured groupoids and let (S, ν) *be an* $(F, [\lambda])$*-space. If* φ *is an imbedding, so is* $\varphi\circ j$*, where* j *is the inclusion of* $S*F$ *into* F.

Proof. We identify the units of $S*F$ with S. Then $T(\varphi\circ j) = G*S = \{(\xi, s): d(\xi) = \varphi\circ j(s, p(s)) = \varphi(p(s))\}$ and $S*F$ acts on $G*S$ as follows: $(\xi, s)(s, x) = (\xi\varphi(x), sx)$. We also have F acting on $G*S$ by $(\xi, s)x = (\xi\varphi(x), sx)$ when $p(s) = r(x)$, so the orbits in $G*S$ are the same for the action of $S*F$ as they are for the action of F. Also, the function f taking (ξ, s) to $(\xi, p(s))$ is algebraically strictly F-equivariant from $T(\varphi\circ j)$ onto $T(\varphi)$. The measure on $T(\varphi\circ j)$ is $\int\mu_{\varphi(p(s))} \times \varepsilon_s d\nu(s)$ and on $T(\varphi)$ we have $\int\mu_{\varphi(u)} \times \varepsilon_u d\tilde{\lambda}(u)$. Since $p_*(\nu)\sim \tilde{\lambda}$, f is strictly F-equivariant and normalized.

Suppose now that F_1 is an i.c. of F and T_2 is a conull F_1-invariant set in $T(\varphi)$ such that F_1 acts freely on T_2 and T_2/F_1 is analytic. Let $S_1 = p^{-1}(F_1^{(0)})$ and $T_1 = f^{-1}(T_2)$. Then T_1 is conull and S_1*F_1-invariant.

If $(\xi, s) \in T_1$, $(s, x) \in S_1*F_1$ and $(\xi, s)(s, x) = (\xi, s)$, then $sx = s$ so $r(x) = p(s) = d(x)$, and $\xi\varphi(x) = \xi$ so $\varphi(x)$ is a unit. Because φ is an imbedding, x is a unit, namely $p(s)$. Hence S_1*F_1 acts freely on T_1.

Since T_2/F_1 is analytic, there is a cross-section $\gamma: T_2/F_1 \to T_2$ which will give rise to a Borel set $B \subseteq T_2$ whose saturation is conull and which meets each orbit at most once. Suppose (ξ, s) and $(\xi, s)(s, x) = (\xi, s)x$ are both in $f^{-1}(B)$. Then $f(\xi, s)$ and $f((\xi, s)x) = f(\xi, s)x$ are both in B. Hence x is a unit, so $f^{-1}(B)$ meets each orbit only once. Now the saturation of $f^{-1}(B)$ is $f^{-1}([B])$, which is conull. Another contraction of F_1, to the image in $F_1^{(0)}$ of $[B]$ will complete the argument.

LEMMA 6.15. *Let* $\varphi: (F, [\lambda]) \to (G, [\mu])$ *and* $\psi: (G, [\mu]) \to (H, [\nu])$ *be composable homomorphisms. Then* $(\psi\circ\varphi)' = (\psi\circ j_\varphi)'\circ\varphi'$.

Proof. We may assume that ψ and φ are strict homomorphisms. Let q_1: $T(\psi\circ j_\varphi) \to S(\psi\circ j_\varphi)$ be a suitable ergodic decomposition. As in Lemma 4.1 and Theorem 6.11, $(x, u) \mapsto (x, \varphi'(u))$ takes $T(\psi\circ\varphi)$ to $T(\psi\circ j_\varphi)$ and the function q_1: $T(\psi\circ\varphi) \to S(\psi\circ j_\varphi)$ defined by $q_1(x, u) = q_2(x, \varphi'(u))$ is a G-equivariant ergodic decomposition. Using q_1, we may take $S(\psi\circ j_\varphi)$ as $S(\psi\circ\varphi)$. Then according to the way we define $(\;\;)'$, Theorem 3.6, for $\xi \in F$ we have

$$\begin{aligned}(\psi\circ\varphi)'(\xi) &= (q_2(\psi\circ\varphi(r(\xi)), r(\xi)), \psi\circ\varphi(\xi)) \\ &= (q_1(\psi\circ j_\varphi(\varphi'(r(\xi))), \varphi'(r(\xi))), \psi\circ j_\varphi\circ\varphi'(\xi)) \\ &= (\psi\circ j_\varphi)'(\varphi'(\xi))\;.\end{aligned}$$

THEOREM 6.16. *Let* $\varphi: (F, [\lambda]) \to (G, [\mu])$ *and* $\psi: (G, [\mu]) \to (H, [\nu])$ *be composable homomorphisms. If both of them are imbeddings, so is* $\psi \circ \varphi$. *If both of them have dense range, so does* $\psi \circ \varphi$.

Proof. $(\psi \circ \varphi)' = (\psi \circ j_\varphi)' \circ \varphi'$, and in the first case each of φ' and $(\psi \circ j_\varphi)'$ is half a similarity, so $(\psi \circ \varphi)'$ is half a similarity.

In the second case, j_ϕ is an isomorphism, so $S(\psi \circ j_\varphi) \cong S(\psi)$ as H-spaces, and $S(\psi) \cong H^{(0)}$ as an H-space because ψ has dense range. Also we have $S(\psi \circ \varphi) \cong S(\psi \circ j_\varphi)$ by Theorem 6.12, so $S(\psi \circ \varphi) \cong H^{(0)}$ as an H-space.

7. Order among subobjects and some category theory.

For virtual subgroups $S \times G$ and $T \times G$ of a group G, Mackey defined $S \times G$ to be smaller than $T \times G$ if there is a G-equivariant map of S onto T. This is a definition by extension: if $S = G/H$ and $T = G/K$, H is conjugate to a subgroup of K iff such a map exists. This does not behave as well as ordinary containment for subgroups, but there are a number of facts which can be formulated in terms of this ordering in a congenial way.

In this section we want to develop some of these facts and to relate some of the properties of Section 5 to notions from category theory. Some of the results we state are due to Caroline Series [21]. She studied homomorphisms in terms of the size of kernel or range closure, and gave several of the definitions we use here [21, Chapter II, Section 3 and Section 4].

We begin with three definitions. The first and third are as formulated by Series and the second is equivalent to one of hers. Following the definitions we will discuss them and their relationships.

DEFINITION 7.1. Let $\varphi: (F, [\lambda]) \to (G, [\mu])$ be a homomorphism of measured groupoids.

a) φ is *trivial* if it is similar to a homomorphism φ' such that $\varphi'(F) \subseteq G^{(0)}$.

b) If (S, ν) is a $(G, [\mu])$-space, we say φ *takes values in* $S*G$ provided there is a homomorphism $\psi: (F, [\lambda]) \to (S*G, [\nu*\mu])$ such that $\varphi = j_S \circ \psi$.

c) The *kernel of* φ is $(T(\varphi)*F, [(\mu*\tilde{\lambda})^*\lambda])$, denoted $\operatorname{Ker}(\varphi)$.

This property of triviality for φ depends only on the similarity class, and generalizes the one for group homomorphisms. However, at first it seems to have a difficulty, as follows. If $\varphi(F) \subseteq G^{(0)}$, then $\tilde{\varphi}$ is constant on each equivalence class. If F were ergodic, $\tilde{\varphi}$ would be essentially constant, so G would be essentially transitive. We might want the kernel of φ to be ergodic, and then it appears that in general φ could not be trivial on its kernel. Mackey pointed out in section 7 of [16] that for a homomorphism φ into a compact group G, $T(\varphi)*F$ decomposes into isomorphic ergodic groupoids, any one of which is a good candidate to be called the kernel of φ. If this worked in general we would face a choice between ergodic kernels and having φ trivial on its kernel. However an unpublished example of Series shows that what we have called Ker φ can have distinct integrands. Therefore it is easier to decide to allow Ker φ to remain as given here.

This definition of taking values in the subobject $S*G$ is motivated by the fact that for groups, φ takes ordinary function values in the subgroup H iff it factors through the inclusion homomorphism of H. Next we want to show that this definition agrees with that of Series, and that the property is invariant under similarity.

LEMMA 7.2. *φ takes values in $S*G$ iff there is a Borel function $\beta: F^{(0)} \to S$ such that*

(i) *for some* i.c. *F_0 of F, $x \in F_0$ implies $\beta(d(x)) = \beta(r(x))\varphi(x)$ makes sense and is true.*

(ii) *$\beta^{-1}(E)$ is null if $E \subseteq S$ is negligible.*

If φ takes values in $S*G$ and φ' is similar to φ, then φ' takes values in $S*G$.

Proof. If such a β exists, define $\psi(x) = (\beta(r(x)), \varphi(x))$ for $x \in F$. Let p be the mapping of S to $G^{(0)}$ such that sx is defined iff $p(s) = r(x)$. Then condition (i) on β implies that ψ carries F_0 into $S*G$. $\psi | F_0$ is a homomorphism because φ is and G acts on S, and because $E \subseteq S$ is negligible iff $\{(s, p(s)): s \in E\}$ is negligible in $(S*G)^{(0)}$, while $\beta^{-1}(E) = \tilde{\psi}^{-1}(\{(s, p(s)): s \in E\})$.

For the converse, suppose $\varphi = j_S \circ \psi$. Then there is a function $\beta_1: F \to S$ such that $\psi(x) = (\beta_1(x), \varphi(x))$ for $x \in F$. If $d(x) = r(y)$, then $\psi(x)\psi(y)$ is defined, so $\beta_1(x)\varphi(x) = \beta_1(y)$. Thus $\beta_1(y)$ depends only on $r(y)$, and there is a $\beta: F^{(0)} \to S$ such that $\beta_1 = \beta \circ r$. If ψ is strict on F_0, condition (i) follows easily, and so does (ii).

For the last statement, suppose β is given and that θ is a similarity of φ to φ'. Define $\beta'(u) = \beta(u)\theta(u)^{-1}$ for $u \in F^{(0)}$. It is not hard to show that this makes sense, and that β' satisfies (i) and (ii) for φ'.

Next we want to show that these notions are properly related. In Theorem 7.8, we give another result of the same kind. Part of the idea involved here is that $G*G$ "is" the trivial subobject of G (see Theorem 7.9). Also, one can show $G*G \approx G^{(0)}$.

LEMMA 7.3. (a) $\varphi: (F, [\lambda]) \to (G, [\mu])$ *takes values in $G*G$ iff φ*

is trivial.

(b) $\varphi|\operatorname{Ker}\varphi$, i.e., $\varphi\circ j_{T(\varphi)}$, *is trivial.*

Proof. (a) Notice that if (ψ_1, ψ_2) is a composable pair of homomorphisms and either is trivial, so is $\psi_1\circ\psi_2$. If $\varphi = j_G\circ\psi$, we can therefore prove φ is trivial by proving j_G is trivial. Define $\theta(s) = s$ for $s \in G$. Then $\theta(s)j_G(s, x)\theta(sx)^{-1} = r(s)$ for $(s, x) \in G*G$, which implies j_G is trivial.

If φ is trivial define $\beta = \tilde{\varphi}$ to show that φ takes values in $G*G$.

(b) Given $\varphi: F \to G$, define $\theta: T(\varphi) \to G$ by $\theta(x, u) = x$. Then for $\xi \in F$ and $x \in G$ with

$$(x, r(\xi)) \in T(\varphi),\ \theta(x, r(\xi))\varphi\circ j_{T(\varphi)}((x, r(\xi)), \xi)\theta(x\varphi(\xi), d(\xi))^{-1} = r(x)\ .$$

Thus $\varphi\circ j_{T(\varphi)}$ is trivial.

We remark that φ is an imbedding iff $\operatorname{Ker}\varphi$ is trivial in a certain sense, according to Lemma 6.2. Thus $\operatorname{Ker}\varphi$ is sensitive to more than just whether φ is immersive. Our next results need definitions of order or "containment" among groupoids. One of these is due to Mackey [15] and the other to Series [21].

DEFINITION 7.4. Let (S, λ) and (T, μ) be $(G, [\nu])$-spaces.

a) $S*G < T*G$ iff there is a normalized $(G, [\nu])$-equivariant map $f: S \to T$.

b) $S*G \prec T*G$ iff there is a $(G, [\nu])$-equivariant map $f: S \to T$.

c) S is quasi-equivalent to T if $I \times S$ and $I \times T$ are isomorphic, where $I = [0, 1]$ and G acts trivially on I.

d) $S*G \lesssim T*G$ iff $(I \times S)*G < (I \times T)*G$.

One difficulty with these order relations is that we can have $S*G < T*G < S*G$ without having S equivalent to T. In fact, let $A = \prod_{n\in Z} Z/4Z$ and let Z act on A by coordinate shifts, which are automorphisms, and form $G = A \circledS Z$. Let $H = \{x \in A: x_n = 0$ for $n < 0\}$ and let $K = \{x \in A: x_n = 0$ for $n < 0$ and $x_0 = 0$ or $2\}$. Then H is conjugate to a subgroup of K and vice versa, but they are not conjugate. Series has given another example [21, page 33].

Leaving that aside, we want to exhibit some more affirmative results. We follow the notation used by Series for types of standard measure spaces. For $n = \infty, 1, 2, \cdots$, J_n is a space with n atoms. For $n = 0$, $J_n = I$, the unit interval, with Lebesgue measure. For $n = -\infty, -1, -2, \cdots$, $J_n = I \cup J_{-n}$. We will say a space is of type J if we do not want to specify a particular J_n.

LEMMA 7.5. [21, *Proposition* 13.6]. *Let X be an analytic Borel space with atom-free probability measure μ. Let f be a Borel* *function into an analytic Borel space Y. Let $\mu = \int \mu_y df_*(\mu)(y)$ be a decomposition of μ relative to f and suppose that almost every μ_y is of the same type J. Let p be the coordinate projection of $J \times Y$ onto Y. Then X is isomorphic (mod null sets) to $J \times Y$ via a Borel function $g: X \to J \times Y$ such that $p \circ g = f$.*

The proof is omitted (see [21]), but a comment or two may help the reader. The discrete parts of the μ_y can be dealt with using the von Neumann selection lemma and an exhaustion argument. For the continuous case one can take $X \subseteq I$ and regard all the measures as being on I. Then $h(x, y) = \mu_y([0, x])$ defines a Borel function and the necessary function g can be defined by $g(x) = (h(x, f(x)), f(x))$.

A lemma we will use in conjunction with Lemma 7.5 is a structure theorem for quotient mappings, as follows. It also is proved using cross-sections and an exhaustion argument.

LEMMA 7.6. *Let X be an analytic Borel space with probability measure μ. Let f be Borel from X to an analytic space Y and let $\mu = \int \mu_y df_*(\mu)(y)$ be a decomposition of μ relative to f. Then there are disjoint Borel sets $Y_n \subseteq Y$ for $n \in \mathbf{Z} \cup \{+\infty, -\infty\}$ whose union is conull and such that if $y \in Y_n$ then μ_y is of type J_n and is concentrated on $f^{-1}(y)$.*

LEMMA 7.7. *Let X be an analytic Borel space with probability measure μ and let f be a Borel function from X into an analytic Borel space Y. Let m be Lebesgue measure on I, form $I \times Y$ and let $p: I \times Y \to Y$ be the projection. Then there is a Borel function $g: I \times Y \to X$ such that $f \circ g = p$* a.e. *and $g_*(m \times f_*(\mu)) \sim \mu$.*

Proof. This is easy if $X = J \times Y$, so we may apply Lemma 7.6 and Lemma 7.5.

THEOREM 7.8. *A homomorphism $\varphi: (F, [\lambda]) \to (G, [\mu])$ is trivial iff $I \times F = (I \times F^{(0)}) * F < T(\varphi) * F = \mathrm{Ker}\,(\varphi)$.*

Proof. If φ is trivial, then for $(x, r(\xi)) \in T(\varphi)$ and $\xi \in F$ we have $(x, r(\xi))\xi = (x\varphi(\xi), d(\xi)) = (x, d(\xi))$, so the action of F on $T(\varphi) = G * F^{(0)}$ is essentially that of the action of F on $F^{(0)}$ with "multiplicity" added by the fibers. By Lemma 7.7 there is a Borel function $g: I \times F^{(0)} \to T(\varphi)$ such that $p(g(\alpha, u)) = u$ for almost all pairs $(\alpha, u) \in I \times F^{(0)}$ and $g_*(m \times \tilde{\lambda}) \sim \mu_1 * \tilde{\lambda}$, where m is Lebesgue measure (see the proof of Theorem A3.5 regarding $\mu_1 * \tilde{\lambda}$). Then g is almost equivariant, so we can choose it to be equivariant, and we do have $I \times F < \mathrm{Ker}\,\varphi$.

Let $j\colon \mathrm{Ker}\,\varphi \to F$ be the inclusion, $j_{T(\varphi)}$, and let $h\colon I \times F^{(0)} \to T(\varphi)$ be equivariant and take $m \times \tilde{\lambda}$ to $\mu_1 * \tilde{\lambda}$. Then $\varphi \circ j \circ (h \times i)$ is trivial because $\varphi \circ j$ is. Suppose $\theta\colon I \times F^{(0)} \to G$ is a Borel function for which $\varphi'(\alpha, \xi) = \theta(\alpha, r(\xi))\varphi(\xi)\theta(\alpha, d(\xi))^{-1}$ is almost always in $G^{(0)}$. Then there is an α such that $\varphi'(\alpha, \xi) \in G^{(0)}$ for almost all ξ. If we define $\theta_0(u) = \theta(\alpha, u)$, $\theta_0(r(\xi))\varphi(\xi)\theta_0(d(\xi))^{-1}$ defines a homomorphism with values in $G^{(0)}$ a.e., so φ is trivial.

For a group G, the trivial subgroup corresponds to the action of G on itself by translation. Thus, if S is a transitive G-space we have $G \times G < S \times G$. For groupoids, one G-space is $I \times G$ and we might not have $G * G < (I \times G) * G$. Thus the following theorem is a reasonable analogue of the idea that the subgroupoid corresponding to $G * G$ is the smallest one.

THEOREM 7.9. *Let $(G, [\mu])$ be a measured groupoid and let (S, λ, p) be an analytic Borel $(G, [\mu])$-space. Then $(I \times G) * G < S * G$.*

Proof. Let $\lambda = \int \lambda_u d\tilde{\mu}(u)$ be a decomposition of λ relative to p over $\tilde{\mu}$, and apply Lemma 7.8 to S, λ, p and $G^{(0)}$, to get a Borel function $g\colon I \times G^{(0)} \to S$ such that $p \circ g(t, u) = u$ for (t, u) in a set $K \subseteq I \times G^{(0)}$ conull relative to $m \times \tilde{\mu}$, and $g_*(m \times \tilde{\mu}) \sim \lambda$. Then $g_*(m \times \varepsilon_u) \sim \lambda_u$ for almost all u, because $g_*(m \times \tilde{\mu}) = \int g_*(m \times \varepsilon_u) d\tilde{\mu}(u)$ and $g_*(m \times \varepsilon_u)$ is almost always concentrated on $p^{-1}(u)$. Now extend g to $I \times G$ as follows: if $(t, r(x)) \in K$, let $g(t, x) = g(t, r(x))x$; if $(t, r(x)) \notin K$, let $g(t, x) = g(t, r(x))$. If $(t, r(x)) \in K$, and $d(x) = r(y)$, then $g(t, xy) = g(t, r(x))(xy) = g(t, x)y$. Thus g is algebraically almost equivariant, so there is an equivariant $f\colon I \times G \to S$ which agrees with g a.e.

Now let us show that g is normalized, so f is. Let G_1 be an i.c. such that $x \in G_1$ implies $\lambda_{r(x)}x \sim \lambda_{d(x)}$, such that $u \in G_1^{(0)}$ implies $g_*(m \times \varepsilon_u) \sim \lambda_u$, and such that for $u \in G_1^{(0)}$ the u-section K_u is m-conull in I. Then $g_*(m \times \varepsilon_x) \sim \lambda_{r(x)}x \sim \lambda_{d(x)}$ for $x \in G_1$. Hence

$$
\begin{aligned}
g_*(m \times \mu) &= \iint g_*(m \times \varepsilon_x) d\mu_u(x) d\tilde{\mu}(u) \\
&\sim \iint \lambda_u d\mu_u(x) d\tilde{\mu}(u) \\
&= \int \lambda_u d\tilde{\mu}(u) \\
&\sim \lambda\,.
\end{aligned}
$$

DEFINITION 7.10. If (T, λ) is a $(G, [\mu])$-space, we call it trivial iff for every $(G, [\mu])$-space (S, ν) we have $T*G \lesssim S*G$.

The next theorem gives another way to construct trivial subgroupoids of G, because $(I \times T)*G \lesssim S*G$ implies $(T*G) \lesssim S*G$.

THEOREM 7.11. *Let $(G, [\mu])$ be a measured groupoid, let G_1 be an* i.c. *on which μ has a right quasi-invariant decomposition, and let A be a Borel set in $G^{(0)}$ with $[A]=G^{(0)}$. Then there is a measure λ on $G^{(0)}$ concentrated on A such that $\lambda(B) = 0$ iff $\tilde{\mu}(B) = 0$ for saturated Borel sets B, and if we set $\nu=\int \mu^u d\lambda(u)$, ν is quasi-invariant on $r^{-1}(A)$ and $(I\times r^{-1}(A))*G < G*G$.*

Proof. The existence of λ was proved in the proof of Theorem 6.17 of [18]. Then ν is quasi-invariant by Lemma 3.4, and $d_*(\nu) = \int d_*(\mu^u)d\lambda(u) \sim \tilde{\mu}$.

By Lemma 7.8 there is a Borel function $f\colon I \times A \rightarrow d^{-1}(A)$ such that $f_*(m \times \varepsilon_u) \sim \mu_u$ a.e. and $f_*(m \times \lambda) \sim \int \mu_u d\lambda(u)$, and there is a conull set $X \subseteqq I \times A$ such that $d\circ f(t, u) = u \in G_1^{(0)}$ for $(t, u) \in X$ and $r\circ f(t, u) \in G_1^{(0)}$ for $(t, u) \in X$.

Let $S = r^{-1}(A)$ and let $Y = \{(t, s) \in I \times S\colon (t, r(s)) \in X\}$. Then Y is conull in $I \times S$ relative to $m \times \nu$. Define $g\colon I \times S \rightarrow G$ by taking $g(t, s) = f(t, r(s))s$ when $(t, s) \in Y$ and $g(t, s) = f(t, r(s))$ if $(t, s) \notin Y$. Then g is Borel and almost algebraically equivariant. By Lemma 1.4, we only need to prove $g_*(m \times \nu) \sim \mu$.

Now $g_*(m \times \varepsilon_x) \sim \mu_{r(x)}x \sim \mu_{d(x)}$ for $x \in G_1$, so $u \in G_1^0$ implies

$$\begin{aligned} g_*(m \times \mu^u) &\sim \int \mu_{d(x)} d\mu^u(x) \\ &= \int \mu_v d(d_*(\mu^u))(v) . \end{aligned}$$

Hence

$$\begin{aligned} g_*(m \times \nu) &\sim \iint \mu_v d(d_*(\mu^u))(v) d\lambda(u) \\ &\sim \int \mu_v d\tilde{\mu}(v) . \end{aligned}$$

The next lemma characterizes trivial subobjects of measured groupoids. It is closely related to Lemma 6.1. Notice that the proof of the "only if" part of the lemma does not require the mapping f to be normalized.

THEOREM 7.12. *Let $(F, [\lambda])$ be a measured groupoid and let (S, μ, p, a) be an $(F, [\lambda])$-space. Then $(I \times S)*F < F*F$ iff there are* *an i.c.* F_1 *of* F *and a conull analytic* F_1*-invariant set* $S_1 \subseteq S$ *such that* $S_1 * F_1$ *is principal and the orbit space* S_1/F_1 *is analytic.*

Proof. First, suppose $f: I \times S \to F$ is equivariant, and let $U \subseteq I \times S$ and $V \subseteq F^{(0)}$ be conull sets such that $p(U) = V$, U is $F|V$-invariant, and $f|U$ is strictly $F|V$-equivariant. There will be a $t \in T$ such that the t-section U_t is conull in S. Then $p(U_t) \subseteq V$ and is conull, so there is a conull Borel set $V_1 \subseteq p(U_t)$. Now U_t is $F|V$-invariant, so if we take $F_1 = F|V_1$ and $S_1 = p^{-1}(V_1) \cap U_t$, then S_1 is conull and F_1-invariant. Define $g(s) = f(t, s)$ for $s \in S_1$. Then g is strictly F_1-equivariant.

Now suppose $(s, x) \in S_1 * F_1$ and $sx = s$. Then $x = g(s)^{-1}(g(s)x) = g(s)^{-1}g(sx) = g(s)^{-1}g(s)$, so x is a unit. Thus F_1 acts freely on S_1, i.e., $S_1 * F_1$ is principal.

To show that S_1/F_1 is analytic, we will show that $g^{-1}(F_1^{(0)})$ is a Borel set meeting erch orbit exactly once. In fact, if $s \in S_1$, then $g(sg(s)^{-1}) = g(s)g(s)^{-1} \in F_1^{(0)}$. Also, if $g(s_1) = g(s_2) \in F_1^{(0)}$ and there is an x with $s_1 x = s_2$, then $g(s_2) = g(s_1 x) = g(s_1)x = g(s_2)x$, so x is a unit and $s_1 = s_2$.

For the converse, begin with S_1, F_1, and let $Y = S_1/F_1$ with $q: S_1 \to Y$ the quotient map. Let $\gamma: Y \to S_1$ be a measurable cross-section and let Y_0 be a conull Borel set on which γ is a Borel function. Let $S_0 = q^{-1}(Y_0)$, $F_0 = F|p(S_0)$. Now S_0 is a union of orbits so $p(S_0)$ is saturated in $F_1^{(0)}$, and S_0 is conull, so $p(S_0)$ is conull in $F_1^{(0)}$. Since F_1 acts freely on S_1, F_0 acts freely on S_0. Thus $g: (y, x) \mapsto \gamma(y)x$ is one-one and Borel from $Y * F_0 = \{(y, x) \in Y \times F_0 : p \circ \gamma(y) = r(x)\}$ onto S_0. For $(y, x) \in Y * F_0$, let $p_1(y, x) = d(x)$, and let $(y, x)x_1 = (y, xx_1)$ if $r(x_1) = d(x)$. Then $Y * F_0$ is an F_0-space and g is algebraically strictly equivariant. Going from S_0 to $Y * F_0$ by g^{-1} and then projecting to F_0 gives an algebraically strictly F_0-equivariant Borel function f: $f(s) = x$ iff $\gamma(q(s))x = s$. Hence the proof will be complete if g preserves the measure class.

The measure on $S_0 * F_0$ is $\int \varepsilon_s \times \lambda^{p(s)} d\mu(s)$, which "is" its decomposition relative to r (taking $(S_0 * F_0)^{(0)} = S_0$). We may assume the decomposition of λ is left quasi-invariant, so the decomposition for $S_0 * F_0$ is also. Then the measures $a_*(\varepsilon_s \times \lambda^{p(s)})$ are in the same class as long as s varies only within one F_0-orbit. Thus, if $q(s) = y$, $a_*(\varepsilon_s \times \lambda^{p(s)}) = a(s, \cdot)_*(\lambda^{p(s)}) \sim a(\gamma(y), \cdot)_*(\lambda^{p(\gamma(y))})$. Hence

$$\mu \sim \int a(s, \cdot)_*(\lambda^{p(s)}) d\mu(s)$$

$$\sim \int a(\gamma \circ q(s), \cdot)_*(\lambda^{p(\gamma(q(s)))})) d\mu(s)$$

$$\sim \int a(\gamma(y), \cdot)_*(\lambda^{p(\gamma(y))})d(q_*(\mu))(y)$$
$$= \int g(y, \cdot)_*(\lambda^{q(\gamma(y))})d(q_*(\mu))(y)$$
$$= g_*\left(\int \varepsilon_y \times \lambda^{p(\gamma(y))}d(q_*(\mu))(y)\right).$$

Thus g carries $q_*(\mu)*\lambda$ to a measure equivalent to μ, which is what we wanted to know.

THEOREM 7.13. *Let* $(F, [\lambda])$ *and* $(G, [\mu])$ *be measured groupoids and let* $\varphi: (F, [\lambda]) \to (G, [\mu])$ *be a homomorphism. Then* φ *is trivial iff the range closure of* φ *is trivial, and this occurs iff* φ *takes values in a trivial subobject of* G.

Proof. Let $j_\varphi: S(\varphi)*G \to G$ and $\varphi': F \to S(\varphi)*G$ be as in Theorem 3.5. Then $\varphi = j_\varphi \circ \varphi'$, so φ takes values in a trivial subobject of G if the range closure of φ is trivial. If $j: S*G \to G$ is an inclusion and $\psi: F \to S*G$ is a homomorphism with $\varphi = j \circ \psi$, then $M(\psi): S(\varphi) \to S(j) = S$ is equivariant, by Lemma 4.1. Hence $S(\varphi)*G < S*G$ (see also [21, Proposition 4.5 of Chapter II]). This establishes the last equivalence.

Now suppose $\varphi(F) \subseteq G^{(0)}$. Then the action of F on $T(\varphi)$ is trivial, so the function $g: T(\varphi) \to G$ taking (x, u) to x^{-1} is constant on F-orbits. Also, g is strictly G-equivariant, and g carries the measure used on $T(\varphi)$, namely $\int \mu_{\varphi(u)} \times \varepsilon_u d\tilde{\lambda}(u)$, to a measure equivalent to $\int \mu^v d\varphi_*(\tilde{\lambda})(v)$, which is quasi-invariant on $S = r^{-1}(\varphi(F^{(0)}))$ and relative to which S is a G-space with $S*G \lesssim G*G$. If $q: T(\varphi) \to S(\varphi)$ is a strictly G-equivariant ergodic decomposition of $T(\varphi)*F$, there is a G-equivariant $f: S(\varphi) \to S$ with $f \circ q = g$. Then f is normalized.

Now suppose $S*G$ is trivial, j is its inclusion and $\psi: F \to S*G$ is such that $\varphi = j \circ \psi$. To show that φ is trivial it will suffice to show that j is trivial. That $S*G$ is trivial means there is an equivariant normalized equivariant function $f: I \times S \to G$. Since f is strictly equivariant on a conull set, there is a t with $g = f(t, \cdot)$ equivariant from S to G. Define $\theta: S \to G$ by $\theta(s) = g(s)^{-1}$. Then $\theta(sx)^{-1} = g(sx) = g(s)x$ for almost all (s, x), so $j(s, x) = \theta(s)\theta(sx)^{-1}$ a.e., as we wanted to show.

Rephrasing a result of Series [21, Proposition 4.6 of Chapter II], we can characterize trivial homomorphisms in terms of kernels.

THEOREM 7.14. *Let* $(F, [\lambda])$ *and* $(G, [\mu])$ *be measured groupoids* *and let* $\varphi: F \to G$ *be a homomorphism. Then* φ *is trivial iff* $I \times F = (I \times F^{(0)}) * F < \operatorname{Ker} \varphi$.

Proof. To say φ is trivial means that φ is trivial on $F^{(0)} * F$ (which is F). Then by Proposition 4.6 of Chapter II of [21], we have $(I \times F^{(0)}) * F < \operatorname{Ker} \varphi$.

Now if $(I \times F^{(0)}) * F < \operatorname{Ker} \varphi$, Proposition 4.6 of Chapter II of [21] says that there is a conull set $U \subseteq I \times F^{(0)}$ and a Borel $\theta: U \to G$ such that if $(t, r(x))$ and $(t, d(x)) \in U$, then $\varphi(x) = \theta(t, r(x))\theta(t, d(x))^{-1}$. Choose t such that the section U_t is conull in $F^{(0)}$, and let $F_1 = F | U_t$, $\theta_1 = \theta(t, \cdot)$. Then $x \in F_1$ implies $\varphi(x) = \theta_1(r(x))\theta_1(d(x))^{-1}$, so φ is trivial.

Now for the other extreme, the kernel can be used to characterize imbeddings.

THEOREM 7.15. *Let* $\varphi: (F, [\lambda]) \to (G, [\mu])$ *be a homomorphism of measured groupoids. Then* φ *is an imbedding iff* $\operatorname{Ker} \varphi$ *is a trivial subgroupoid of* F.

Proof. If we take $S = T(\varphi)$ in Theorem 7.12, this result is immediate.

THEOREM 7.16. *Homomorphisms with dense range are epimorphisms in the sense of category theory.*

Proof. Let $(F, [\lambda])$, $(G, [\mu])$ and $(H, [\nu])$ be measured groupoids and let $\psi: F \to G$ be a homomorphism with dense range. Let $\varphi_1, \varphi_2: G \to H$ be homomorphisms such that $[\varphi_1] \circ [\psi] = [\varphi_2] \circ [\psi]$. By taking an i.c. of G, replacing ψ by a similar homomorphism, and then replacing F by an i.c., we may arrange that $\varphi_1, \varphi_2, \psi$ are strict (then $\varphi_1 \circ \psi$ and $\varphi_2 \circ \psi$ exist) and that $\varphi_1 \circ \psi$ and $\varphi_2 \circ \psi$ are strictly similar. Let $\theta: F^{(0)} \to H$ be a Borel function such that for every $\xi \in F$, $\theta \circ r(\xi) \varphi_1 \circ \psi(\xi) \theta \circ d(\xi)^{-1} = \varphi_2 \circ \psi(\xi)$.

Now if $(x, u) \in T(\psi)$, i.e., $x \in G$, $u \in F^{(0)}$ and $d(x) = \psi(u)$, then $d \circ \varphi_1(x) = \varphi_1 \circ d(x) = \varphi_1 \circ \psi(u) = d \circ \theta(u)$, so $\theta(u)\varphi_1(x)^{-1}$ is defined. Also $r \circ \theta(u) = \varphi_2 \circ \psi(u) = \varphi_2 \circ d(x) = d \circ \varphi_2(x)$, so $\varphi_2(x)\theta(u)$ is defined. Thus we can define $g(x, u) = \varphi_2(x)\theta(u)\varphi_1(x)^{-1}$ for $(x, u) \in T(\psi)$. Then g is constant on F-orbits, because if $(x, r(\xi)) \in T(\psi)$, then

$$\begin{aligned} g(x\psi(\xi), d(\xi)) &= \varphi_2(x)\varphi_2 \circ \psi(\xi)\theta \circ d(\xi)\varphi_1 \circ \psi(\xi)^{-1}\varphi_1(x)^{-1} \\ &= \varphi_2(x)\theta \circ r(\xi)\varphi_1(x)^{-1} \\ &= g(x, r(\xi)) \,. \end{aligned}$$

The assumption that ψ has dense range just means that the function $f: (x, u) \mapsto r(x)$ is an ergodic decomposition of $T(\psi)$ relative to the action of F. Hence there is a Borel function $h: G^{(0)} \to H$ such that $g(x, u) = h(r(x))$ for almost all (x, u). We have G acting on $T(\psi)$ by $x(y, u) = (xy, u)$ if $d(x) = r(y)$, and on $G^{(0)}$ by $x*d(x) = r(x)$. Also G has a weak (left) action on a subset of H, given by

$$x*\xi = \varphi_2(x)\xi\varphi_1(x)^{-1}$$

when the product is defined. Then g and f are both equivariant, so we can take h to be equivariant on a conull set $U \subseteq G^{(0)}$. Thus for $x \in G|U$, $h(r(x)) = \varphi_2(x)h(d(x))\varphi_1(x)^{-1}$. Thus φ_1 and φ_2 are similar via h.

The method used to prove this theorem also works to prove a theorem about representation of groupoids. If L_1 and L_2 are representations of $(G, [\mu])$, i.e., Borel homomorphisms to the groups of unitary operators on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, then the space of intertwining operators $R(L_1, L_2)$ is defined to be the set of bounded Borel functions $A: G^{(0)} \to \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ such that $A(r(x))L_1(x) = L_2(x)A(d(x))$ for x in some i.c. of G. We identify functions which agree a.e., and supply the essential sup norm. We say L_1 and L_2 are equivalent if there is a unitary valued A in $R(L_1, L_2)$; they are disjoint if $R(L_1, L_2) = \{0\}$; L_1 is irreducible if $R(L_1, L_1)$ has dimension 1. Of course $A^*(u) = A(u)^*$ defines an element of $R(L_2, L_1)$ and pointwise composition maps $R(L_2, L_3) \times R(L_1, L_2)$ to $R(L_1, L_3)$. A special case of the following theorem was proved and used on page 47 of [20]. This strengthens a theorem of Peter Hahn [6, Theorem 5.19].

THEOREM 7.17. *Let $\psi: (F, [\lambda]) \to (G, [\mu])$ be a homomorphism with dense range, and let L_1, L_2 be strict representations of $(G, [\mu])$. For $A \in R(L_1, L_2)$ define $\psi'(A) = A \circ \tilde{\psi}$. Then ψ' is an isomorphism of $R(L_1, L_2)$ onto $R(L_1 \circ \psi, L_2 \circ \psi)$. This operation preserves sums, products and adjoints. In particular, if L is irreducible so is $L \circ \psi$.*

Proof. Most of this is easy, so we only discuss the fact that ψ' is onto. If $B \in R(L_1 \circ \psi, L_2 \circ \psi)$, we can define g on $T(\psi)$ by $g(x, u) = L_2(x)B(u)L_1(x)^{-1}$. As in Theorem 7.16, there is a Borel A on $G^{(0)}$ such that $A(r(x)) = L_2(x)B(u)L_1(x)^{-1}$ a.e. in $T(\psi)$, and A can be chosen to be equivariant. Thus $A \in R(L_1, L_2)$. Hence, $L_2(x)A \circ \tilde{\psi}(u)L_1(x)^{-1} = A(r(x)) = L_2(x)B(u)L_1(x)^{-1}$, so $A \circ \tilde{\psi} = B$ a.e.

It was suggested by Peter Hahn that the method we have used in the last two proofs could be used to generalize a result of Robert Zimmer on amenability [Theorem 3.3, 23]. Zimmer defined amenability of group actions in a way which can apply to groupoids [8], using an analog of the fixed-point property, and he proved that a group action which is the range closure of a homomorphism defined on an amenable group action is also amenable.

The definition goes as follows. Let E be a separable Banach space and let A be the group of isometric isomorphisms of E, with the strong operator topology and Borel structure. Let E^* be the dual of E, with the weak* topology, and let E_1^* be the unit ball in E^*. If G is a measured groupoid, let $\gamma: G \to A$ be a Borel homomorphism and define $\gamma^*(x) = \gamma(x^{-1})^*$ for $x \in G$. A function assigning a compact convex set $K_u \subseteq E_1^*$ to each $u \in G^{(0)}$ is called an *invariant Borel field of compact convex sets* if $\{(u, f) \in G^{(0)} \times E_1^*: f \in K_u\}$ is Borel and $\gamma^*(x)K_{d(x)} = K_{r(x)}$ for almost every x. Then there is an i.c. of x's for which $\gamma^*(x)K_{d(x)} = K_{r(x)}$. This is the appropriate analog of an action of a group on a compact convex set, and the analog of a fixed point is an *invariant section*, i.e., a Borel function $\sigma: G^{(0)} \to E_1^*$ such that $\sigma(u) \in K_u$ for almost all u, and $\gamma^*(x)\sigma(d(x)) = \sigma(r(x))$ for almost all x (again an i.c. of x's will satisfy the condition).

We say G is *amenable* if every invariant Borel field of compact convex sets has an invariant section.

Given any homomorphism φ, the φ' of Theorem 3.5 has dense range by Theorem 6.7, so to deal with range closures it is sufficient to prove a result about homomorphisms with dense range. Hence our next theorem does generalize Theorem 3.3 of Zimmer [23].

THEOREM 7.18. *Suppose* $(F, [\lambda])$ *is amenable and there is a homomorphism* $\psi: (F, [\lambda]) \to (G, [\mu])$ *which has dense range. Then* $(G, [\mu])$ *is amenable.*

Proof. Take E, γ, K as in the definition just above. For $u \in F^{(0)}$, let $C_u = K_{\psi(u)}$ and $\beta = \gamma \circ \psi$. Let G_0 be an i.c. of G such that $x \in G_0$ implies $\gamma^*(x)K_{d(x)} = K_{r(x)}$ and $\gamma | G_0$ is strict. By passing to an equivalent ψ and an i.c. of F, we may assume ψ is a strict homomorphism and carries F into G_0. We may also suppose that $q: (x, u) \to r(x)$ is an ergodic decomposition projecting $T(\psi) = \{(x, u) \in G \times F^{(0)}: d(x) = \psi(u)\}$ onto a conull subset of $G_0^{(0)}$, since that is essentially what "dense range" means. Then it is clear that C is an invariant Borel field of compact convex sets for β, and since F is amenable there must be an invariant section ρ. By replacing F by an i.c., we may arrange that $\beta(\xi)\rho(d(\xi)) = \rho(r(\xi))$ for all ξ in F.

Now define $g: T(\psi) \to E_1^*$ by $g(x, u) = \gamma^*(x)\rho(u)$. Then $g(xy, u) = \gamma^*(x)\gamma^*(y)\rho(u) = \gamma^*(x)g(y, u)$ if $r(y) = d(x)$ and $\psi(u) = d(y)$. It follows, by using x^{-1}, that g is strictly equivariant from $T(\psi)$ to E_1^*. We also have

$$\begin{aligned} g(x\psi(\xi), d(\xi)) &= \gamma^*(x)\beta(\xi)\rho(d(\xi)) \\ &= \gamma^*(x)\rho(r(\xi)) \\ &= g(x, r(\xi)) \end{aligned}$$

whenever $d(x) = \psi(r(x))$, so g is constant on F orbits in $T(\psi)$. Hence g factors through $\sigma: G^{(0)} \to E_1^*$, i.e., there is a Borel σ such that $\sigma(r(x)) = g(x, u)$ for almost all $(x, u) \in T(\psi)$. By Lemma A1.2, there is an equivariant choice of σ, i.e., $\sigma(r(x)) = \gamma^*(x)\sigma(d(x))$ for almost all x. Thus G is amenable. For $\tilde{\lambda}$-almost every $u \in F^{(0)}$, we have $\sigma(r(x)) = \gamma^*(x)\rho(u)$ for $\mu_{\psi(u)}$-almost every x. In particular, for almost every u there is such an x for which $\sigma(r(x)) = \gamma^*(x)\sigma(d(x)) = \gamma^*(x)\sigma(\psi(u))$. Hence $\rho = \sigma \circ \tilde{\psi}$ a.e.

Here is another result on epimorphisms, whose proof is omitted.

THEOREM 7.19. *Let $\psi: (F, [\lambda]) \to (G, [\mu])$ be a homomorphism such that for $u \in F^{(0)}$, ψ takes $r^{-1}(u) \cap d^{-1}(u)$ onto $r^{-1}(\psi(u)) \cap d^{-1}(\psi(u))$. Then ψ is an epimorphism.*

Finally, we have one result on imbeddings which is in the direction of saying that imbeddings are monomorphisms. This may be the closest to that statement that is true. Even it fails for immersions, as we see from examples with groups.

THEOREM 7.20. *Let $\varphi: (F, [\lambda]) \to (G, [\mu])$ be a homomorphism and let $\psi: (G, [\mu]) \to (H, [\nu])$ be an imbedding. If $[\psi]\circ[\varphi]$ is trivial, so is $[\varphi]$.*

Proof. $S(\psi\circ\varphi) \cong S(\psi\circ j_\varphi)$ as H-spaces and $S(\varphi)*G \sim S(\psi\circ j_\varphi)*H$ as groupoids. Then $S(\varphi)*G$ is principal and $S(\varphi)/G$ is analytic (up to a null set), so φ is trivial by Theorems 7.12 and 7.13.

Appendix. The four sections of the appendix give proofs of results in the first four sections of the body of the paper.

LEMMA A1.1 (*Lemma* 1.4). *Let $(G, [\mu])$ be a measured groupoid, let (S, λ, p) be an analytic Borel $(G, [\mu])$-space and let T be a strict analytic Borel $(G, [\mu])$-space. If $f_1: S \to T$ is almost $(G, [\mu])$-equivariant, then there is a $(G, [\mu])$-equivariant function $f: S \to T$ which agrees with f_1* a.e. *Furthermore, $f_{1*}(\lambda) = f_*(\lambda)$ and is quasi-invariant. The function f exists even if T is a weak G-space.*

Proof. There is no loss in generality if we assume to begin with that (S, λ, p) is a strict $(G, [\mu])$-space and that $\mu = \int \mu(r, u) d\tilde{\mu}(u)$ is a left quasi-invariant decomposition of μ into probability measures. To see this, use Lemma 6.2 of [19] and Definition 1.1 together with the remarks preceding Definition 1.2.

Let S_1 be a conull Borel set in S such that if $s \in S_1$ then $f_1(sx) = f_1(s)x$ for $\mu^{p(s)}$-almost almost all x. Decompose $\lambda = \int \lambda_u d\tilde{\mu}(u)$ relative to p and let U be a conull Borel set in $G^{(0)}$ such that $u \in U$ implies that λ_u is a nontrivial measure concentrated on $p^{-1}(u) \cap S_1$. In the groupoid $S*G$, the set of units is the graph of p and can be identified with S, and the decomposition of the measure relative to r has integrands $\varepsilon_s \times \mu(r, p(s))$. Then $d_*(\varepsilon_s \times \mu(r, p(s)))$ is identified with a measure concentrated on $\{sx: r(x) = p(s)\}$, which is the equivalence class of s in S. Since the decomposition of μ relative to r is quasi-invariant, $\int d\lambda(s) d_*(\varepsilon_s \times \mu(r, p(s)))$ is equivalent to λ (i.e., the image of λ in the graph of p), so we can also assume U is chosen so that S_1 is $d_*(\varepsilon_s \times \mu(r, p(s)))$-conull for λ_u-almost every s when $u \in U$.

Now let $G_0 = G | U$ and let $S_0 = \{s \in S: p(s) \in U$ and S_1 is $d_*(\varepsilon_s \times \mu(r, p(s)))$-conull$\}$. Then S_0 is an invariant conull Borel set by the proof of Lemma 6.3 of [19]. Now take $T \subseteq [0, 1]$ and define $f(s) = \int f_1(sx)x^{-1} d\mu(r, p(s))(x)$ for $s \in S_0$. Notice that if $s \in S_1$ then $f_1(sx) = f_1(s)x$ for $\mu(r, p(s))$-almost all x so that $f_1(sx)x^{-1}$ is defined and equal to $f_1(s)$ for almost all x in $r^{-1}(p(s))$. Thus $f = f_1$ on $S_1 \cap S_0$. If $s \in S_0$, there is a y with $sy \in S_1$. Then $sx = (sy)y^{-1}x$ and $f_1(sx)(y^{-1}x)^{-1}$ is defined for almost all x, so $f_1(sx)x^{-1}$ is defined for almost all x. If we define $F_1(s, x) = f_1(sx)x^{-1}$ when this is valid and $F_1(s, x) = 0$ otherwise, F_1 is Borel and $f(s) = \int F_1(s, x) d\mu(r, p(s))(x)$, so f is well defined and Borel from S_0 to $[0, 1]$. To see that $f(S_0) \subseteq T$ and f is equivariant, let $s \in S_0$ and choose y with $sy \in S_1$. Since S_0 is invariant, we have $sy \in S_1 \cap S_0$. Now if $r(x) = p(s)$, then $r(x) = r(y)$, and by quasi-invariance we have $f_1((sy)(y^{-1}xz)) = f_1(sy)(y^{-1}xz)$ for $\mu(r, d(x))$-almost all z. Thus $f(sx) = \int f_1((sy)(y^{-1}xz))z^{-1} d\mu(r, d(x))(z) = f_1(sy)y^{-1}x$. If we take $x = p(s)$ this gives $f(s) = f_1(sy)y^{-1} \in T$, and applying it again we get $f(sx) = f(s)x$ for $(s, x) \in S_0*G_0$. Observe that this proof is valid if T is even a weak G-space.

To see that $f_*(\lambda)$ is quasi-invariant, notice that $\lambda*\mu$ is mapped to $f_*(\lambda)*\mu$ by $(s, x) \to (f(s), x)$ and (sx, x^{-1}) goes to $(f(s)x, x^{-1})$ under this function. Thus quasi-invariance of λ implies the same for $f_*(\lambda)$.

The next lemma is useful in constructing equivariant functions.

LEMMA A1.2. *Let $(G, [\mu])$ be a measured groupoid and let (S_1, p_1), (S_2, p_2) and (S_3, p_3) be analytic strict $(G, [\mu])$-spaces. Suppose*

λ is a finite quasi-invariant measure on S_1. Let $f: S_1 \to S_2$ and $g: S_1 \to S_3$ be equivariant, and suppose $h_1: S_2 \to S_3$ is a Borel function with $h_1 \circ f = g$ a.e. *Then $(S_2, f_*(\lambda), p_2)$ is a strict $(G, [\mu])$-space and there is an equivariant $h: S_2 \to S_3$ which agrees with h_1* a.e. *relative to $f_*(\lambda)$.*

Proof. By Lemma A1.1 we know that $f_*(\lambda)$ is quasi-invariant, and that it suffices to prove that h_1 is almost equivariant.

Let $\lambda = \int \lambda_s df_*(\lambda)(s)$ be a decomposition of λ relative to f and let $E_1 = \{s \in S_1: h_1 \circ f(s) = g(s)\}$. Then there is a conull Borel set $E_2 \subseteq S_2$ such that for $s \in E_2$ the measure λ_s is a probability measure concentrated on $E_1 \cap f^{-1}(s)$. By Theorem 2.13 of [19], we also know that $\{(s, x) \in S_2 * G: \lambda_s x \sim \lambda_{sx}\}$ is conull. Hence $\{(s, x) \in S_2 * G: s \in E_2, sx \in E_2$ and $\lambda_s x \sim \lambda_{sx}\}$ is conull. If (s, x) is in this set, then E_1 is conull for λ_s and $\lambda_s x$, so there is an $s_1 \in E_1 \cap f^{-1}(s)$ with $s_1 x \in E_1$. Then

$$\begin{aligned} h_1(sx) &= h_1(f(s_1)x) \\ &= h_1(f(s_1 x)) \\ &= g(s_1 x) \\ &= g(s_1)x \\ &= h_1(s)x , \end{aligned}$$

which completes the proof.

Now we want to inquire whether the requirement that an equivariant function be normalized is very stringent. For homomorphisms of groupoids it eliminates many [18, § 4], but between G-spaces it is equivalent to an apppaently weaker condition. We begin with an easy lemma about invariant sets.

The next two lemmas show that all equivariant functions (Definition 1.3(a)) are normalized in the sense used by C. Series [21].

LEMMA A1.3. *Let $(G, [\mu])$ be a measured groupoid, let (S, λ, p) be a strict $(G, [\mu])$-space, and let G_1 be an* i.c. *of G. If $N \subseteq p^{-1}(G_1^{(0)})$ is analytic, null and G_1-invariant, then its G-saturation, $[N]$, is also null.*

Proof. Let $s \in N$, $x \in G$ with $r(x) = p(s)$ and $sx \notin N$. Now $p(sx) = d(x)$ and if $d(x) \in G_1^{(0)}$ we would have $x \in G_1$ so $sx \in N$. Thus $sx \notin p^{-1}(G_1^{(0)})$. Thus $[N] - N \subseteq S - p^{-1}(G_1^{(0)})$, which is of measure 0.

LEMMA A1.4. *Let $(G, [\mu])$ be a measured groupoid, let $(S_1, \lambda_1, p_1, a_1)$ and $(S_2, \lambda_2, p_2, a_2)$ be strict analytic $(G, [\mu])$-spaces, and let*

$f: S_1 \to S_2$ be strictly $(G, [\mu])$-equivariant. Then $f(S_1)$ is G-invariant and $f_*(\lambda_1) \sim \lambda_2$.

Proof. It is easy to show that $f(S_1)$ is invariant. The rest of the proof is based on the uniqueness of the measure classes in the equivalence classes of units.

Let $\mu = \int \mu(r, u) d\tilde{\mu}(u)$ be a decomposition of μ relative to r. By Lemma 6.2 of [19], there is an i.c., G_1, of G such that $u \in G_1^{(0)}$ implies $\mu(r, u)$ is a probability measure concentrated on $G_1 \cap r^{-1}(u)$, and this decomposition is quasi-invariant under G_1. Let $S_3 = p_1^{-1}(G_1^{(0)})$, $S_4 = p_2^{-1}(G_1^{(0)})$. Then $\lambda_1 * \mu = \int \varepsilon_s \times \mu(r, p_1(s)) d\lambda_1(s)$ and $\lambda_2 * \mu = \int \varepsilon_s \times \mu(r, p_2(s)) d\lambda_2(s)$ are quasi-invariant on $S_3 * G_1$ and $S_4 * G_1$. For $s \in S_3$, the measure $\nu_s^1 = a_1 * (\varepsilon_s \times \mu(r, p_1(s)))$ is concentrated on its orbit, and the class $[\nu_s^1]$ is the same for all s in a given orbit. Also, $f_*(\nu_s^1) = a_2 * (\varepsilon_{f(s)} \times \mu(r, p_2(f(s))))$, which we denote by ν_s^2.

For a Borel set $B \subseteq S_2$, B is null iff $B \cap S_4$ is null and this happens iff the saturated Borel set $Q = \{s \in S_4 : \nu_s^2(B) > 0\}$ is null. If $A = f^{-1}(B)$, let $P = \{s \in S_3 : \nu_s^1(A) > 0\}$. Then $P = f^{-1}(Q)$, and P is also saturated and Borel. Our hypothesis about the measure implies that $\lambda_1(P) = 0$ iff $\lambda_2(Q) = 0$, so $\lambda_1(A) = 0$ iff $\lambda_2(B) = 0$, as desired.

LEMMA A1.5. *Suppose f, g are weakly equivariant and $\beta_1: S_1 \to G$ is Borel, with $f(s)\beta_1(s) = g(s)$ for almost all s and $\beta_1(sx) = x^{-1}\beta_1(s)x$ for $\lambda_1 * \mu$-almost every (s, x). Then there are a Borel function $\beta: S_1 \to G$, an* i.c. *G_1 and an analytic conull strict $(G_1, [\mu])$-space $S_3 \subseteq S_1$ such that $\beta = \beta_1$* a.e., *$\beta(sx) = x^{-1}\beta(s)x$ for $(s, x) \in S_3 * G_1$ and $f(s)\beta(s) = g(s)$ for $s \in S_3$.*

REMARK. If the action of G on S_2 is free, one can prove that β_1 satisfies what is required of β, but the proof fails otherwise.

Proof. Choose β via Lemma A1.1, and choose G_1 and S_3^* so that β, f, g are strictly equivariant on S_3^*. Now $f(s)\beta(s)$ is still defined for almost all s, and it is not hard to see that the set of $s \in S_3^*$ for which the product is defined is invariant. On that set $f(s)\beta(s) = g(s)$ a.e., and since both functions are Borel and equivariant, the set S_3 where they agree is invariant.

The next three results are useful in establishing the existence of strictly quasi-invariant decompositions of measures.

LEMMA A1.6. *If $(G, [\mu])$ is a measured groupoid and μ has a strictly left quasi-invariant decomposition $\mu = \int \mu^u d\tilde{\mu}(u)$, then there is also a strictly right quasi-invariant decomposition.*

Proof. Set $\lambda = (\mu)^{-1}(\lambda(A) = \mu(\{x^{-1}: x \in A\}))$. Then $d_*(\lambda) = r_*(\mu) = \tilde{\mu}$, and $\lambda = \int (\mu^u)^{-1} d\tilde{\mu}(u)$ is a strictly right quasi-invariant decomposition. Set $\mu^+ = d_*(\mu)$. Then $\mu^+ \sim \tilde{\mu}$ and $\lambda \sim \mu$, so we can choose strictly positive and finite Radon-Nikodym derivatives $f = d\tilde{\mu}/d\mu^+$ and $g = d\mu/d\lambda$. Then

$$\mu = g\lambda = \int g(\mu^u)^{-1} d\tilde{\mu}(u) = \int (g)(f \circ d)(\mu^u)^{-1} d\mu^+(u) .$$

Take $\mu_u = (g)(f \circ d)(\mu^u)^{-1}$.

COROLLARY A1.7. *Let H be a locally compact group with m a probability measure equivalent to Haar measure. If ν is a quasi-invariant measure on an H-space S, then $(S \times H, [\nu \times m])$ has strictly quasi-invariant decompositions on both sides.*

LEMMA A1.8. *Let $(G, [\mu])$ be a measured groupoid with strictly quasi-invariant decompositions. If (S, λ, p) is a strict $(G, [\mu])$-space, then λ has a strictly quasi-invariant decomposition.*

Proof. Let $\mu = \int \mu_u d\tilde{\mu}(u)$ be a strictly quasi-invariant decomposition, and let $\lambda = \int \lambda_u d\tilde{\mu}(u)$ be any decomposition of λ relative to p. We have assumed that the Borel set $G_1 = \{x \in G: (\lambda_{r(x)})x \sim \lambda_{d(x)}\}$ is conull, so the Borel set $U_1 = \{u \in G^{(0)}: \mu_u(G_1) = 1\}$ is conull in $G^{(0)}$. Set $U_2 = [U_1]$, $G_2 = G | U_2$. We shall construct a $\lambda' \sim \lambda$ with a strictly quasi-invariant decomposition. For $u \notin U_2$, let $\lambda'_u = 0$. For $u \in U_2$, let $\lambda'_u = \int (\lambda_{r(x)}) x d\mu_u(x)$. For $u \in U_1$, $\lambda'_u \sim \lambda_u$. Also, $u \mapsto \lambda'_u$ is Borel, so we can form $\lambda' = \int \lambda'_u d\tilde{\mu}(u)$. Then $\lambda' \sim \lambda$, so we can choose a Radon-Nikodym derivative $g = d\lambda/d\lambda'$ which is positive and finite everywhere.

If $v \in U$, there is an x such that $d(x) = v$ and $u = r(x)$ is in U_1. Then $\{y: (\lambda_{r(y)})y \sim \lambda'_u\}$ is μ_u-conull. Since $(\mu_u)x \sim \mu_v$,

$$(\lambda'_u)x = \int (\lambda'_{r(y)}) yx d\mu_u(y) \sim \int (\lambda_{r(z)}) z d\mu_v(z) = \lambda'_v .$$

If we also have $w \in U_2$ and $z \in r^{-1}(v) \cap d^{-1}(w)$, then $(\lambda'_u)xz \sim \lambda'_w$ by the same argument. Hence $(\lambda'_v)z \sim \lambda'_w$. If $v = r(z)$ and $w = d(z)$ are not in $U_?$, $\lambda'_v = 0$, $\lambda'_w = 0$, so $(\lambda'_v)z = \lambda'_w$.

Now we can replace λ_u by $g\lambda'_u$ for each $u \in G^{(0)}$ and get a strictly quasi-invariant decomposition of λ.

REMARK. This generalizes Proposition 2.6 of C.C. Moore [1, Chapter 2]. The next two lemmas show that similar G-spaces are isomorphic.

LEMMA A1.9. *Let (S, λ, p) be an analytic strict $(G, [\mu])$-space and suppose $\beta: S \to G$ is Borel, $s\beta(s)$ is defined for $s \in S$, and $\beta(sx) = x^{-1}\beta(s)x$ holds for all $(s, x) \in S*G$. Define $\beta^+(s) = s\beta(s)$ for all $s \in S$. Then β^+ is a G-automorphism of (S, p) preserving $[\lambda]$.*

Proof. To show that β^+ preserves $[\lambda]$, we show first that λ is quasi-invariant under another groupoid. Let $G' = \{x \in G: r(x) = d(x)\}$. This is a G-space under the action $x*y = y^{-1}xy$ which is defined when $d(x) = r(y)$. Thus $d: G' \to G^{(0)}$ is the projection we need. We have assumed that $\beta: S \to G'$ is strictly equivariant. By Lemma 1.4, $\beta_*(\lambda)$ is quasi-invariant, so we can make $G'*G$ a measured groupoid with the measure $\beta_*(\lambda)*\mu$. We can define $s(\beta(s), x) = sx$ if $s \in S$ and $(\beta(s), x) \in G'*G$, because in that case $p(s) = d\circ\beta(s) = r(x)$. This makes (S, β) a strict $\beta(S)*G$-space, and $S*(\beta(S)*G)$ is a groupoid, naturally isomorphic with $S*G$. (This occurs whenever we have a strictly equivariant map of G-spaces.)

The measures also agree: $p_*(\lambda) = \tilde{\mu}$ and $\lambda*\mu = \int \varepsilon_s \times \mu^{p(s)} d\lambda(s)$, while $\beta_*(\lambda)*\mu = \int \varepsilon_x \times \mu^{d(x)} d\beta_*(\lambda)(x)$. The latter gives the decomposition of $\beta_*(\lambda)*\mu$ relative to r in $\beta(S)*G$. Hence

$$\lambda*(\beta_*(\lambda)*\mu) = \int \varepsilon_s \times (\varepsilon_{\beta(s)} x \mu^{d\cdot\beta(s)}) d\lambda(s) .$$

Since $d\circ\beta = p$, $(s, x) \mapsto (s, (\beta(s), x))$ takes $\lambda*\mu$ to $\lambda*(\beta_*(\lambda)^*\mu)$.

Since λ is quasi-invariant, $[\lambda*\mu]$ is symmetric. Hence $[\lambda*(\beta_*(\lambda)*\mu)]$ is symmetric, so λ is quasi-invariant for $\beta(S)*G$. Hence there is a strictly quasi-invariant decomposition $\lambda = \int \lambda(\beta, x) d\beta_*(\lambda)(x)$ relative to β.

We must see what this implies for the strictly quasi-invariant decomposition $\lambda = \int \lambda_u d\tilde{\mu}(u)$ relative to p. For each u,

$$\int \lambda(\beta, x) d\beta_*(\lambda_u)(x)$$

is concentrated on $p^{-1}(u)$, because $p = d\circ\beta$. Also,

$$\iint \lambda(\beta, x) d\beta_*(\lambda_u)(x) d\tilde{\mu}(u) = \int \lambda(\beta, x) d\beta_*(\lambda)(x) = \lambda .$$

($\beta_*(\lambda) = \int \beta_*(\lambda_u) d\tilde{\mu}(u)$, by Lemma 1.2 of [19], since $p = d\circ\beta$.) Thus for almost all u, $\lambda_u = \int \lambda(\beta, x) d\beta_*(\lambda_u)(x)$. Also, if $d(x) = r(y)$, $(\lambda(\beta, x)y \sim \lambda(\beta, y^{-1}xy)$, since this is a strictly quasi-invariant decomposition, so for each x we have $\beta_*^+(\lambda(\beta, x)) = (\lambda(\beta, x))x \sim \lambda(\beta, x)$. Now $p \circ \beta^+ = p$, so Lemma 1.2 of [19] gives

$$\beta_*^+\Big(\int \lambda(\beta, x) d\beta_*(\lambda_u)(x)\Big) = \int \beta_*^+(\lambda(\beta, x)) d\beta_*(\lambda_u)(x)$$
$$\sim \int \lambda(\beta, x) d\beta_*(\lambda_u)(x) ,$$

for each u. Hence $\beta_*^+(\lambda_u) \sim \lambda_u$ for almost all u, so $\beta_*^+(\lambda) \sim \lambda$, again by the same Lemma 1.2.

The next lemma is the same as Lemma 1.6.

LEMMA A1.10. *Let $(G, [\mu])$ be a measurable groupoid and let $(S_1, [\lambda_1])$ and $(S_2, [\lambda_2])$ be analytic $(G, [\mu])$-spaces. Suppose $f: S_1 \to S_2$ and $g: S_2 \to S_1$ are equivariant maps with $f \circ g$ similar to the identity on S_2 and $g \circ f$ similar to the identity on S_1. Then $(S_1, [\lambda_1])$ and $(S_2, [\lambda_2])$ are isomorphic.*

Proof. Let G_0 be an i.c. and let S_3 be analytic, G_0-invariant and conull in S_1 and suppose f is equivariant on S_3 and $g \circ f$ is strictly similar to the identity on S_3. By looking at $g^{-1}(S_3)$, $f^{-1}(g^{-1}(S_3)) \cap S_3, \cdots$ we see that S_3 may be chosen so that $f(S_3) \subseteq g^{-1}(S_3)$. In the same way, there is an analytic, invariant conull $S_4 \subseteq S_2$ such that $f \circ g$ is strictly similar to the identity on S_4 and $g(S_4) \subseteq f^{-1}(S_4)$. Set $S_5 = S_3 \cap f^{-1}(S_4)$ and $S_6 = g^{-1}(S_3) \cap S_4$. Then $f(S_5) \subseteq S_6$ and $g(S_6) \subseteq S_5$, and $(f | S_5) \circ (g | S_6)$ is strictly similar to i on S_6 while $(g | S_6) \circ (f | S_5)$ is strictly similar to i on S_5. Thus we may assume the original similarities were strict. Then by Lemma A1.7 there are G-automorphisms γ_1 of $(S_1, [\lambda_1])$ and γ_2 of $(S_2, [\lambda_2])$ such that $f \circ g = \gamma_2$ and $g \circ f = \gamma_1$. Then $f \circ g \circ \gamma_2^{-1}$ is the identity on S_2 and $\gamma_1^{-1} \circ g \circ f$ is the identity on S_1. Thus f is an isomorphism and $f^{-1} = g \circ \gamma_2^{-1} = \gamma_1^{-1} \circ g$.

We take $\mathscr{F}$ and $G^{(0)} * \mathscr{F}$ as in § 1. The next lemma is the same as Lemma 1.7 and gives the existence of a "universal G-space".

LEMMA A1.11. *$G^{(0)} * \mathscr{F}$ is an analytic G-space, provided the given decomposition of μ relative to r is quasi-invariant.*

Proof. Everything is simple except possibly the fact that the action is Borel. To prove that, we make use of another way of seeing what the Borel structure is. If f is a bounded Borel function, defined at least on $r^{-1}(u)$, then let M_f denote the bounded operator on $\mathscr{H}(u) = L^2(\mu^2(r, u))$ given by multiplication by f. Then $[f]_u \to M_f$ is one-one from $\mathscr{F}(u)$ onto the operators of multiplication by a $[0, 1]$-valued function on $\mathscr{H}(u)$. Let $G^{(0)} * \mathscr{L}(\mathscr{H}) = \cup \{\{u\} \times$

$\mathscr{L}(\mathscr{H}(u)): u \in G^{(0)}\}$ have the smallest Borel structure for which the projection onto $G^{(0)}$ is Borel, along with all the functions $\psi_{g,h}$ (g, h bounded Borel) where $\psi_{g,h}(u, A) = (A[g]_u : [h]_u)$, the inner product being computed in $\mathscr{H}(u)$. By reducing to the case of constant $\mathscr{H}$ [20, § 1], we can see that $G^{(0)}*\mathscr{L}(\mathscr{H})$ has an analytic or standard Borel structure if $G^{(0)}*\mathscr{H}$ does, i.e., if $G^{(0)}$ does. Now $G^{(0)}*\mathscr{F}$ is isomorphic to a Borel subset of $G^{(0)}*\mathscr{L}(\mathscr{H})$ because $\psi_{g,h}(u, M_f) = \psi_g(u, [f]_{u\bar{h}})$. Thus if the action of G on $G^{(0)}*\mathscr{L}(\mathscr{H})$ is Borel and the map $(u, [f]_u) \rightarrow (u, M_f)$ is equivariant we will be through with the proof. The action of G on $G^{(0)}*\mathscr{H}$ is as follows: For each x there is a positive function $\rho(x, \cdot)$ on $r^{-1}(d(x))$ such that $(U_x g)(y) = \rho(x, y)g(xy)$ defines a unitary operator from $\mathscr{H}(d(x))$ onto $\mathscr{H}(r(x))$. This gives a right action on $G^{(0)}*\mathscr{H}$. Now $(r(x), A)x = (d(x), U_x A U_x^{-1})$ defines an action of G on $G^0*\mathscr{L}(\mathscr{H})$, and if we reduce to the case where $\dim \mathscr{H}(u)$ is constant and pass to a bundle of the form $G^{(0)} \times \mathscr{K}$ [20, § 1], then it is clear that the action of G on $G^{(0)} \times \mathscr{L}(\mathscr{K})$ is Borel since $x \rightarrow U_x$ is Borel [20, Proposition 3.4].

A2. Ergodic decompositions of measurable groupoids. The numbers in this section agree with these of § 2. Another approach to this material is found in Theorem 6.1 of [7].

LEMMA 2.1. *The measurable groupoid (G, C) is ergodic iff $\mathscr{H}_r \cap \mathscr{H}_d$ is one-dimensional.*

Proof. If (G, C) is not ergodic, let A be a saturated Borel set in $G^{(0)}$ for which A and $B = G^{(0)} \setminus A$ both have positive measure [19, Corollary 6.4]. Then $\varphi_A \circ r = \varphi_A \circ d$ and $\varphi_B \circ r = \varphi_B \circ d$, and these are orthogonal elements of $\mathscr{H}_r \cap \mathscr{H}_d$ but neither is zero.

Now if $\mathscr{H}_r \cap \mathscr{H}_d$ has dimension greater than 1, there is a nonzero element $g \in \mathscr{H}_r \cap \mathscr{H}_d$ which is orthogonal to the constant functions. Then there are Borel functions f_1, f_2 in $L^2(\tilde{\lambda})$ such that $f_1 \circ r = f_2 \circ d = g$ a.e. Thus for almost every $u \in G^{(0)}$, $f_1 \circ r(x) = f_2 \circ d(x)$ for λ_u-almost all x, i.e., $f_1 \circ r(x) = f_2(u)$ for λ_u-almost all x. Hence, for $\tilde{\lambda}$-almost every v this holds for $r_*(\lambda_v)$-almost every u. Thus f_1 is almost always constant a.e. relative to $[r_*(\lambda_v)]$ (which is the same as $[r_*(\lambda_u)]$ if $u \in [\nu]$) and for $r_*(\lambda_v)$-almost every u that constant is $f_2(u)$. This proves that $f_1 = f_2$ a.e., so $f_1 \circ d = f_2 \circ d$ a.e. and hence $f_1 \circ r = f_1 \circ d$ a.e. But f_1 is not constant a.e. since $f_1 \circ r$ is not, so (G, C) is not ergodic.

DEFINITION A2.2. Let $(G, [\lambda])$ be a measurable groupoid. A strict ergodic decomposition of $(G, [\lambda])$ is a mapping q of $G^{(0)}$ into an analytic Borel space T such that if $\nu = q_*(\tilde{\lambda})$ and $\tilde{\lambda} = \int \tilde{\lambda}(q, t) d\nu(t)$ is a decomposition of $\tilde{\lambda}$ relative to q, then for v-almost all t, $q^{-1}(t)$ is saturated and $(G|q^{-1}(t), [\lambda^t])$ is an ergodic groupoid, where $\lambda^t = \int \lambda_u d(\tilde{\lambda}(q, t))(u)$. An ergodic decomposition of $(G, [\lambda])$ is a Borel mapping q of $G^{(0)}$ into an analytic space T such that for some conull Borel set $U \subseteq G^{(0)}$, $q|U$ is a strict ergodic decomposition of $(G|U, [\lambda])$.

LEMMA A2.3. *Let $(G, [\mu])$ be a measured groupoid, and let a Borel function q from $G^{(0)}$ to an analytic space T be an ergodic decomposition. If a Borel function g from $G^{(0)}$ to an analytic space Z is constant on equivalence classes, then there is a Borel $h: T \to Z$ such that $h \circ q = g$* a.e. *Such an h is determined* a.e. *relative to $\mu = q_*(\tilde{\lambda})$.*

Proof. Suppose q is an ergodic decomposition and let $\tilde{\lambda} = \int \tilde{\lambda}_t d\mu(t)$ be a decomposition of $\tilde{\lambda}$ relative to q. Take $Z \subseteq [0, 1]$. Because q is an ergodic decomposition, for almost all t we have g constant a.e. relative to $\tilde{\lambda}_t$. Therefore we can define $h_1: T \to [0, 1]$ by $h_1(t) = \int g d\tilde{\lambda}_t$ and get a Borel function with $h_1 \circ q = g$ a.e. Then h_1 takes values in Z a.e., so the desired h exists. The uniqueness is easy.

THEOREM A2.4. (*Uniqueness of ergodic decompositions*). *Let $q_1: G^{(0)} \to T_1$ and $q_2: G^{(0)} \to T_2$ be ergodic decompositions of the measured groupoid $(G, [\lambda])$. Then there are a conull Borel set $U \subseteq G^{(0)}$ and a Borel isomorphism $f: q_1(U) \to q_2(U)$ such that $q_2 = f \circ q_1$ on U. Also, q_1 and q_2 have the same level sets in U. If q_1 and q_2 are strict decompositions, U may be taken to be saturated.*

Proof. Let $\mu_1 = q_{1*}(\tilde{\lambda})$ and $\mu_2 = q_{2*}(\tilde{\lambda})$. By Lemma A2.2 there exist Borel functions $f_1: T_1 \to T_2$ and $f_2: T_2 \to T_1$ such that $q_2 = f_1 \circ q_1$ a.e. and $q_1 = f_2 \circ q_2$ a.e. By the uniqueness part of Lemma A2.3, the Borel set $T_3 = \{t \in T_1: f_2 \circ f_1(t) = t\}$ is μ_1-conull. Now f_1 is one-one from T_3 onto an analytic set $T_4 \subseteq T_2$ and $f_2|T_4$ is the inverse of $f_1|T_3$. We have $f_{1*}(\mu_1) = \mu_2$, so T_4 is μ_2-conull.

Let V be a conull Borel set such that $q_1|V$ and $q_2|V$ are strict ergodic decompositions and $q_2 = f_1 \circ q_1$ on V. Let $T_5 \subseteq T_3$ be a conull set for which the conditions in Definition 2.2 are satisfied for q_1. Define $U = V \cap q_1^{-1}(T_5)$. It is easy to verify that $f = f_1|q_1(U)$ does what is needed.

If q_1 and q_2 are strict decompositions, we can take V to be saturated because $\{u: q_2(u) = f \circ q_1(u)\}$ contains a conull saturated set. Also, the set $q_1^{-1}(T_5)$ is saturated, so U is then saturated.

THEOREM A2.5. *If $(G, [\lambda])$ is a measured groupoid, then $(G, [\lambda])$ has an ergodic decomposition. If λ has a (right or left) quasi-invariant decomposition, then $(G, [\lambda])$ has a strict ergodic decomposition.*

Proof. Since $(G, [\lambda])$ has an i.c. on which λ has a quasi-invariant decomposition, it suffices to prove the second part of the theorem. To begin, let $M = M(\tilde{\lambda})$ be the measure algebra of Borel sets in $G^{(0)}$ modulo $\tilde{\lambda}$-null sets and let M_0 be the sub σ-algebra in M of equivalence classes of saturated Borel sets. Let $q: \mathrm{Bor}(G^{(0)}) \to M$ be the quotient homomorphism and let $\mathscr{A}_0$ be a countable algebra of saturated Borel sets for which $\{q(A): A \in \mathscr{A}_0\}$ is dense in M_0. Then $\mathscr{A}_0$ determines an analytic quotient space T of $G^{(0)}$: if p is the quotient map, $p(u) = p(v)$ iff $\{A \in \mathscr{A}_0: u \in A\} = \{A \in \mathscr{A}_0: v \in A\}$. Let $\nu = p_*(\tilde{\lambda})$ and decompose $\tilde{\lambda} = \int \tilde{\lambda}_t d\nu(t)$, then define $\lambda^t = \int \lambda_u d\lambda_t(u)$ for $t \in T$ and set $G_t = d^{-1}(p^{-1}(t)) = r^{-1}(p^{-1}(t)) = r^{-1}(p^{-1}(t)) = G | p^{-1}(t)$. It seems plausible that this should give an ergodic decomposition of (G, C) [1, pages 112–117]. By construction, each $p^{-1}(t)$ is saturated, so it suffices to show that for ν-almost every t in T, $(G_t, [\lambda^t])$ is a virtual group.

Each G_t is a Borel subset of G and hence is an analytic Borel groupoid. For almost every t the measure $\tilde{\lambda}_t$ is concentrated on $p^{-1}(t)$. For almost every u in $G^{(0)}$ the measure λ_u is a probability measure concentrated on $d^{-1}(u)$. Combining these two facts, we see that for almost every $t \in T$ the measure λ^t is concentrated on G_t and $d_*(\lambda^t) = \tilde{\lambda}_t$. Thus we may regard λ^t as a measure on G_t with a right quasi-invariant decomposition, so that $[\lambda^t]$ is right invariant. Since λ is symmetric, it follows that λ^t is symmetric for almost every t. Thus almost every $(G_t, [\lambda^t])$ is a measurable groupoid.

Now we must show that almost every $(G_t, [\lambda^t])$ is ergodic. Since T is analytic, there is a conull Borel set T_0 which is standard in the relative Borel structure, and T_0 can be chosen such that $t \in T_0$ implies that $(G_t, [\lambda^t])$ is a measurable groupoid, and all λ^t, $\tilde{\lambda}_t$ are probability measures, with $\tilde{\lambda}_t$ concentrated on $p^{-1}(t)$. If $(G_t, [\lambda^t])$ is ergodic for almost every $t \in T_0$ then it is for almost every $t \in T$, so there is no loss of generality in replacing G by $G | p^{-1}(T_0)$ and T by T_0, i.e., we may suppose T is standard and $(G_t, [\lambda^t])$ is a measurable groupoid for every $t \in T$. We seek to apply Lemma A2.1.

Now define Hilbert bundles over T as follows: $\mathscr{H}(t) = L^2(\lambda^t)$, $\mathscr{H}_r(t) = \{f \circ r: f \in L^2(\lambda_t)\}$, $\mathscr{H}_d(t) = \{f \circ d: f \in L^2(\tilde{\lambda}_t)\}$, $\mathscr{H}'(t) = L^2(\tilde{\lambda}_t)$. For a bounded Borel function g on G, let $g_d(u) = \int g d\lambda_u$ and $g_r(u) = \int g\, d\lambda^u$

where $\lambda^u(E)=\lambda_u(\{x^{-1}: x\in E\})$ for $u\in G^{(0)}$. Then g_r and g_d are bounded and Borel in $G^{(0)}$. The Borel structure on $T*\mathscr{H}$ is the smallest for which the projection onto T is Borel along with all functions ψ_g for bounded Borel g, where $\psi_g(t,f)=\int f(x)g(x)d\lambda^t(x)$. The same procedure is used for $T*\mathscr{H}'$. Now if f is in $L^2(\tilde{\lambda}_t)$ and g is bounded and Borel on G, $\int f\circ r(x)g(x)d\lambda^t(x)=\int f(u)g_r(u)d\tilde{\lambda}_t(u)$. Hence $(t,f)\to(t,f\circ r)$ is Borel from $T*\mathscr{H}'$ to $T*\mathscr{H}$. It is one-one since it is an isometry on each $\mathscr{H}'(t)$, so the image is a Borel set. This image is $T*\mathscr{H}_r$. Similarly, $T*\mathscr{H}_d$ is a Borel subset of $T*\mathscr{H}$. Hence $T*\mathscr{H}_r\cap T*\mathscr{H}_d=T*(\mathscr{H}_r\cap\mathscr{H}_d)$ is a Borel set in $T*\mathscr{H}$.

Now let C be the set of points (t,f) in $T*(\mathscr{H}_r\cap\mathscr{H}_d)$ such that $f\neq 0$ and the vector f in $\mathscr{H}_r(t)\cap\mathscr{H}_d(t)$ is orthogonal to the vector represented by the constant function which is everywhere 1. This is $\{(t,f): f\neq 0 \text{ and } \psi_1(t,f)=0\}$, so it is a Borel set. Let D be the projection of C into T. By the von Neumann selection lemma there is a Borel cross-section f of $T*(\mathscr{H}_r\cap\mathscr{H}_d)$ such that $f(t)=0$ for almost all $t\notin D$ and $f(t)\in C$ for almost all $t\in D$. Taking real and imaginary parts is a Borel operation, and the real and imaginary parts of each $f(t)$ are orthogonal to 1, so we may suppose each $f(t)$ is real and orthogonal to 1, and $f(t)\neq 0$ for almost all $t\in D$.

Now $L^2(\lambda)$ is isometric to the direct integral of the $\mathscr{H}(t)$'s, so there is a Borel function g on G in $L^2(\lambda)$ such that $f(t)$ is the class of g in $L^2(\lambda^t)$ for almost all t. There are Borel functions f_1 and f_2 on $G^{(0)}$ such that $g=f_1\circ r=f_2\circ d$ a.e., because $f(t)\in\mathscr{H}_r(t)\cap\mathscr{H}_d(t)$ always, i.e., f is a cross-section of both images of $T*\mathscr{H}'$ and hence is an image of two cross-sections.

As in Lemma A2.1, $f_1\circ r=f_1\circ d$ a.e.; by passing to an equivalent function, we may suppose the sets $A_1=\{u\in G^{(0)}: f_1(u)>0\}$ and $A_2=\{u\in G^{(0)}: f_1(u)<0\}$ are saturated. Now f_1 is orthogonal to 1 relative to $\tilde{\lambda}_t$ for ν-almost all t so $\{t: \tilde{\lambda}_t(A_1)>0\}$ and $\{t: \tilde{\lambda}_t(A_2)>0\}$ differ by a null set. Also, these sets differ from D by a null set, since f_1 is nontrivial relative to $\tilde{\lambda}_t$ essentially for t in D. Since A_1 is saturated, $\tilde{\lambda}_t(A_1)$ is 0 or 1 a.e.; $\tilde{\lambda}_t(A_2)=0$ or 1 a.e. also, and $\{t: \tilde{\lambda}_t(A_1)=1\}$ differs from $\{t: \tilde{\lambda}_t(A_2)=1\}$ by a null set. Thus both sets are null, so f_1 is null, and therefore D is null. This proves the theorem.

Definition A2.6. Let $(G,[\mu])$ be a measurable groupoid and let (S,λ) be an analytic Borel G-space with q.i. measure. The measure λ is ergodic iff $(S*G,[\lambda*\mu])$ is an ergodic groupoid. An ergodic decomposition of (S,λ) relative to G is a Borel mapping q of S into an analytic Borel space T such that if $\lambda=\int\lambda_t dq_*(\lambda)(t)$ is a decomposition of λ relative to q then for $q_*(\lambda)$-almost all t in T the set

$q^{-1}(t)$ is invariant and the measure λ_t is concentrated on $q^{-1}(t)$ and is q.i. and ergodic.

COROLLARY A2.7. *If (S, λ) is an analytic G-space with a quasi-invariant measure for a measurable groupoid (G, C) and C has an element with a left quasi-invariant decomposition then S has a decomposition into ergodic parts, which is essentially unique.*

LEMMA A2.8. *The converse of Lemma* A2.3 *is true.*

Proof. Take g to be some ergodic decomposition. Then modulo null sets, $\{g^{-1}(B)\colon B \text{ is Borel in } Z\} = \{q^{-1}(h^{-1}(B))\colon B \text{ is Borel in } Z\} \subseteq \{q^{-1}(B)\colon B \text{ is Borel in } T\}$. Thus the latter set is dense in the saturated Borel sets, and by the proof of Theorem A2.5 we see that q is an ergodic decomposition.

A3. Commuting groupoid actions and closing of ranges of homomorphisms. The numbers in this section agree with those of § 3.

DEFINITION A3.1. If S is an F-space and a G-space, we say the actions commute iff for $s \in S$, $\xi \in F$ and $x \in G$, if sx and $s\xi$ are defined then so are $(sx)\xi$ and $(s\xi)x$ and they are equal.

THEOREM A3.2. *Let $(F, [\mu])$ and $(G, [\nu])$ be measured groupoids and let (S, λ, p) and (S, λ, q) be strict $(F, [\mu])$- and $(G, [\nu])$-spaces respectively. Suppose these actions commute. Then there is a strictly G-equivariant function $f\colon S \to G^{(0)} * \mathscr{F}$ which is an ergodic decomposition of $S*F$. If S' is an analytic $(G, [\nu])$-space and $f'\colon S \to S'$ is a $(G, [\nu])$-equivariant ergodic decomposition of $S*F$, then $(G^{(0)}*\mathscr{F}, f_*(\lambda))$ and $(S', f'_*(\lambda))$ are isomorphic $(G, [\nu])$-spaces.*

Proof. First we describe a general method for constructing strictly G-equivariant functions from S to $G^{(0)}*\mathscr{F}$ and then show how to choose the ingredients to achieve the desired goal. Let $I = [0, 1]$ and let $g\colon S \to I$ be any Borel function. For $s \in S, x \in G$ define $h(s)(x) = g(sx)$ if $p(s) = r(x)$ and 0 otherwise, and let $f(s)$ be the element of $\mathscr{F}(p(s))$ which is the equivalence class of $h(s)$. From the fact that $(sx)y = s(xy)$ when either side exists, it follows that $g((sx)y) = g(s(xy))$ if $(sx)y$ exists, and hence that $f(sx) = f(s)x$ if $(s, x) \in S*G$. If k is a bounded Borel function on G, the function taking $(s, x) \in S \times G$ to $h(s)(x)k(x)$ is Borel, so the function taking $s \in S$ to $\int h(s)(x)k(x)d\nu(r, p(s))(x)$ is Borel. Thus f is Borel.

Now we want to find a g such that the resulting f will be an ergodic decomposition. Let F_0 be an i.c. of F on which μ has a q.i. decomposition and let $S_0 = p^{-1}(F_0^{(0)})$. Then S_0 is conull and is a strict F_0-space. If $s \in S$, $x \in G$ and $q(s) = r(x)$, while $\xi \in F$ and $p(s) = r(\xi)$, then $p(sx) = r(\xi) = p(s)$, because the actions commute. Hence S_0 is G invariant and is a strict $G|q(S_0)$-space. Let $G_0 = G|q(S_0)$.

Sets of the form $q^{-1}(A)$ for $A \in G^{(0)}$ are also F-invariant so in constructing a countable algebra $\mathscr{A}$ of F_0-invariant Borel sets in S_0 to produce a strict ergodic decomposition of $S_0 * F_0$ as in the proof of Theorem A2.4, we may assume $\mathscr{A} \supseteq \{p^{-1}(A): A \in \mathscr{A}_0\}$, where $\mathscr{A}_0$ is an countable generating algebra of Borel sets in $G_0^{(0)}$. Suppose $\pi: S_0 \to T$ is a strict ergodic decomposition of $S_0 * F_0$ so obtained. Then $\pi(s_1) = \pi(s_2)$ implies $p(s_1) = p(s_2)$, so there is a Borel function $q: T \to G_0^{(0)}$ such that $q \circ \pi = p$. Then q is automatically onto, and if we let $\lambda' = \pi_*(\lambda)$, $q_*(\lambda') \sim r_*(\nu) = \tilde{\nu}$.

Now let T' be λ'-conull and Borel in T with the property that $((S_0 * F_0)|\pi^{-1}(t), [(\lambda * \mu)^t])$ is an ergodic groupoid for $t \in T'$ (see the proof of Theorem A2.4). Notice that $(S_0 * F_0)|\pi^{-1}(t)$ and $(S_0 * G_0) * F_0|(\pi^{-1}(t) \times \{x\})$ are isomorphic if $q(t) = r(x)$, and $\pi \times i$ takes $S_0 * G$ onto $T * G = \{(t, x) \in T \times G: q(t) = r(x)\}$. Let $\lambda = \int \lambda_t d\lambda'(t)$ be a decomposition of λ relative to π. If λ_t is concentrated on $\pi^{-1}(t)$ then $\nu^{p(s)} = \nu^{q(t)}$ for λ_t-almost all s so $\int \varepsilon_s \times \nu^{p(s)} d\lambda_t(s) = \lambda_t \times \nu^{q(t)}$. For each such t, $\pi_*(\lambda_t) = \varepsilon_t$ and thus $(\pi \times i)_*(\lambda_t \times \nu^{q(t)}) = \varepsilon_t \times \nu^{q(t)}$. By Lemma 1.2 of [13], $(\pi \times i)_*(\lambda * \nu) = \lambda' * \nu$. Then we see that

$$\int (\lambda_t \times \varepsilon_x) d(\lambda' * \nu)(t, x) = \iint (\lambda_t \times \varepsilon_x) d(\varepsilon_t \times \nu^{q(t)})(t', x) d\lambda'(t)$$
$$= \int \lambda_t \times \nu^{q(t)} d\lambda'(t) = \lambda * \nu .$$

Since $\lambda_t \times \varepsilon_x$ is concentrated on $\pi^{-1}(t) \times \{x\}$ if λ_t is concentrated on $\pi^{-1}(t)$, we see that $\pi \times i$ is an ergodic decomposition of $(S * G) * F$.

Now $\tau(s, x) = (sx, x^{-1})$ defines a measure class preserving Borel automorphism of $S * G$ which commutes with the action of F, so if A is F-invariant so is $\tau(A)$. If $\mathscr{B}$ is a countable generating algebra of Borel sets in $T * G$, then $\mathscr{A}^+ = \{(\pi \times i)^{-1}(B): B \in \mathscr{B}\}$ is a countable algebra of Borel sets in $S * G$. Since $\pi \times i$ gives an ergodic decomposition, $\mathscr{A}^+$ must be dense in the F-invariant sets in $S * G$. Let $\mathscr{A}^\tau$ be the smallest algebra containing $\mathscr{A}^+$ and invariant under τ. Then $\mathscr{A}^\tau$ is countable and dense in the F-invariant sets so it gives rise to an ergodic decomposition π' of $(S * G) * F$. Now π' and $\pi \times i$ have the same level sets on some conull Borel set $Z \subseteq S * G$, and π' and $\pi' \circ \tau$ have the same level sets by the τ-invariance of $\mathscr{A}^\tau$. Thus $\pi \times i$ and $(\pi \times i) \circ \tau$ have the same level sets on Z. Let S_1 be a conull Borel set in S_0 such that Z is $\varepsilon_s \times \nu^{p(s)}$-conull for $s \in S_1$. Define $g: S \to I$ by letting $j: T \to I$ be an imbedding and taking g to be some extension of $j \circ \pi$.

We have $f(s_1) = f(s_2)$ iff $p(s_1) = p(s_2)$ and $g(s_1 x) = g(s_2 x)$ for $\nu^{p(s_1)}$-almost all x. For $s_1, s_2 \in S_1$, if $p(s_1) = p(s_2) = u$, then the set $X = \{x \in r^{-1}(u)$: (s_1, x) and $(s_2, x) \in Z\}$ is ν^u-conull. Thus for $s_1, s_2 \in S_1$, $f(s_1) = f(s_2)$ implies $\{x \in X: g(s_1 x) = g(s_2 x)\}$ is ν^u-conull. Since $x \in X$ implies (s_1, x) and $(s_2, x) \in Z$, for $x \in X$ we have $(g(s_1 x), x^{-1}) = (g(s_2 x), x^{-1})$ iff $(g(s_1), x) = (g(s_2), x)$, so $f(s_1) = f(s_2)$ and $s_1, s_2 \in S_1$ together imply $g(s_1) = g(s_2)$. Conversely, let $s_1, s_2 \in S_1$ with $g(s_1) = g(s_2)$. Then $p(s_1) = p(s_2)$, which we call u, and take X as before. Then $g(s_1 x) = g(s_2 x)$ for $x \in X$. Thus $f(s_1) = f(s_2)$. Hence f and g have the same level sets on S_1.

There is a conull set S_2 such that if $s \in S_2$, $\xi \in F$ and $s\xi$ is defined and in S_2, then $g(s) = g(s\xi)$. We may assume that S_1 is chosen so that $s \in S_1$ implies $a_*(\varepsilon_s \times \nu^{p(s)})$ is concentrated on S_2. Then suppose $s \in S_1$, $\xi \in F$ and $s\xi \in S_1$. In that case, $\{x \in r^{-1}(p(s))$: sx and $(s\xi)x = (sx)\xi$ are in $S_1\}$ is $\nu^{p(s)}$-conull, so $h(s) = h(s\xi)$ a.e., i.e., $f(s) = f(s\xi)$. Hence f is an ergodic decomposition of $S*F$.

If f' is taken as in the statement of the theorem, then by Lemma A2.2 there are Borel functions $h: G^{(0)}*\mathscr{F} \to S'$ and $h': S' \to G^{(0)}*\mathscr{F}$ with $h \circ f = f'$ a.e. and $h' \circ f' = f$ a.e. By Lemma 1.5, h and h' may be taken to be equivariant. By the uniqueness in Lemma 2.3, $h \circ h'$ and $h' \circ h$ are the identities on conull sets.

In the process of constructing the closure of the range of a homomorphism, it will be necessary to construct some quasi-invariant measures. The next lemma gives one of the basic ingredients. First some preparation is needed.

Let $(G, [\nu])$ be a measured groupoid and let E be the equivalence relation on $G^{(0)}$ induced by G, i.e., $E = (r, d)(G) \subseteq G^{(0)} \times G^{(0)}$. Let $\nu' = (r, d)_*(\nu)$.

DEFINITION A3.3. We shall say that ν is (r, d)-quasi-invariant if it has decompositions $\nu = \int \nu_u d\tilde{\nu}(u)$ and $\nu = \int \nu_{v,u} d\nu'(v, u)$ such that

(a) for $(v, u) \in E$, $\nu_{v,u}$ is concentrated on $r^{-1}(v) \cap d^{-1}(u)$,

(b) for $(v, u) \in E$, $(\nu_{v,u})^{-1} \sim \nu$,

(c) if $r(x) \sim u$, then $\nu_{u,r(x)} \cdot x \sim \nu_{u,d(x)}$ and $x \cdot \nu_{d(x),u} \sim \nu_{r(x),u}$, and

(d) for $u \in G^{(0)}$, $\nu_u = \int \nu_{v,u} d(r_*(\nu_u))(v)$.

As we explained just after Definition 3.3, G always has an i.c. on which the restricted measure is (r, d)-quasi-invariant.

LEMMA A3.4. *Let $(G, [\nu])$ be a measured groupoid and suppose ν is (r, d)-quasi-invariant. Let λ be a finite measure on $G^{(0)}$ such* *that* $\lambda(A) = 0$ *iff* $\tilde{\nu}(A) = 0$ *for Borel analytic sets* $A \subseteq G^{(0)}$. *Let* $\nu_1 = \int \nu_u d\lambda(u)$, *and let* $y \in G$ *act on* $x \in G$ *by* $x * y = y^{-1}x$ *provided* $r(x) = r(y)$. *Then* ν_1 *is quasi-invariant.*

Proof. If $A \subseteq G^{(0)}$ is Borel, then $U = \{u \in G^{(0)}: r_*(\nu_u)(A) = 0\}$ is a saturated Borel set. Now $r_*(\nu_1)(A) = \int r_*(\nu_u)(A) d\lambda(u)$ which is 0 iff U is λ-conull iff U is $\tilde{\nu}$-conull iff $\tilde{\nu}(A) = \int r_*(\nu_u)(A) d\tilde{\nu}(u)$ is 0. Hence $r_*(\nu_1) \sim \tilde{\nu}$ and we can decompose $\nu_1 = \int \nu_1^u d\tilde{\nu}(u)$ over $\tilde{\nu}$ relative to r. The proof will be complete if $\nu_1^{r(y)} * y = y^{-1}\nu_1^{r(y)} \sim \nu_1^{d(y)}$ whenever $y \in G$.

To this end, we seek another more convenient way to write the decomposition. First of all, define $\lambda_v = \int d_*(\nu_1^u) d(r_*(\nu_v))(u)$ for $v \in G^{(0)}$. Then $v \sim w$ implies $\lambda_v \sim \lambda_w$ because $r_*(\nu_v) \sim r_*(\nu_w)$. Also

$$\begin{aligned}
\int \lambda_v d\tilde{\nu}(v) &= \iint d_*(\nu_1^u) d(r_*(\nu_v))(u) d\tilde{\nu}(v) \\
&= \int d_*(\nu_1^u) d\tilde{\nu}(u) \\
&= d_*\Big(\int \nu_1^u d\tilde{\nu}(u)\Big) \\
&= \lambda \, .
\end{aligned}$$

Now define $\nu_2^v = \int \nu_{v,u} d\lambda_v(u)$. Then for any $w \in G^{(0)}$,

$$\begin{aligned}
\int \nu_2^v d(r_*(\nu_w))(v) &\sim \iint \nu_{v,u} d\lambda_w(u) d(r_*(\nu_w))(v) \\
&= \iint \nu_{v,u} d(r_*(\nu_w))(v) d\lambda_w(u) \\
&\sim \iint \nu_{v,u} d(r_*(\nu_u))(v) d\lambda_w(u) \\
&= \int \nu_u d\lambda_w(u) \, .
\end{aligned}$$

Hence

$$\begin{aligned}
\int \nu_2^v d\tilde{\nu}(v) &\sim \iint \nu_u d\lambda_w(u) d\tilde{\nu}(w) \\
&= \int \nu_u d\lambda(u) \\
&= \nu_1 \, .
\end{aligned}$$

Now ν_2^v is concentrated on $r^{-1}(v)$, so this is a decomposition of a measure equivalent to ν_1 and by essential uniqueness we have $\nu_2^v \sim \nu_1^v$ a.e. relative to $\tilde{\nu}$, so we may as well assume $\nu_1^v \sim \nu_2^v$ always. Then for $y \in G$ we have

$$\begin{aligned} y\cdot\nu_1^{d(y)} &\sim y\cdot\int\nu_{d(y),u}d\lambda_{d(y)}(u)\\ &= \int y\cdot\nu_{d(y),u}d\lambda_{d(y)}(u)\\ &\sim \int\nu_{r(y),u}d\lambda_{r(y)}(u)\\ &\sim \nu_1^{r(y)}\ . \end{aligned}$$

THEOREM A3.5. *Let* $(F, [\mu])$ *be a measured groupoid, let* $(G, [\nu])$ *be a measured groupoid for which* ν *is* (r, d)*-quasi-invariant and let* $\varphi: F \to G$ *be a homomorphism. Then there are* i.c.'s F_0 *and* G_0 *of* F *and* G, *a strict* $(G_0, [\nu])$*-space* (S_φ, λ) *and a strict homomorphism* $\varphi': F_0 \to S_\varphi * G_0$ *such that* $\varphi | F_0 = j\circ\varphi'$, *where* $j: S_\varphi * G_0 \to G_0$ *is the inclusion (coordinate projection).*

DEFINITION A3.6. We call $(S_\varphi * G, [\lambda * \nu])$ the closure of the range of φ, and will denote j by j_φ when necessary to identify its connection with φ.

Proof of theorem. First replace F by an i.c. so that φ is strict and let $T = G * F^{(0)} = \{(x, u): d(x) = \tilde{\varphi}(u)\}$. By passing to an i.c. if necessary, we may have $G^{(0)} = [\varphi(F^{(0)})]$. Then let G, F act on T by $(x, u)y = (y^{-1}x, u)$ if $r(x) = r(y)$ and $(x, u)\xi = (x\varphi(\xi), d(\xi))$ if $u = r(\xi)$. It is easy to see that these actions commute and that $p(x, u) = r(x)$ defines the projection of T into $G^{(0)}$ involved in making T a G-space, while $p'(x, u) = u$ defines the one for F.

Next we must construct a suitable measure on T. First form $\nu_1 = \int\nu_u d(\varphi_*(\tilde{\mu}))(u)$. Then ν_1 is carried on $X = d^{-1}(\varphi(F^{(0)}))$ and X is the projection of T into G. By Lemma A3.4, ν_1 is quasi-invariant under the action of G on X given by letting $y \in G$ act on $x \in X$ if $r(x) = r(y)$ and then $x * y = y^{-1}x$. Also, the coordinate projection of T onto X partitions the action of G on T over X. The measure we need is $\nu_1 * \tilde{\mu} = \int(\varepsilon_x \times \tilde{\mu}_{d(x)})d\nu_1(x) = \int(\nu_{\tilde{\varphi}(u)} \times \varepsilon_u)d\tilde{\mu}(u)$.

To see that $\nu_1 * \tilde{\mu}$ is G-quasi-invariant, use the first formula for it. Clearly the decomposition of $\nu_1 * \tilde{\mu}$ given above is the relevant one for the partition of T over X and $(\varepsilon_x \times \tilde{\mu}_{d(x)})y = \varepsilon_{y^{-1}x} \times \tilde{\mu}_{d(y^{-1}x)}$ so $\nu_1 * \tilde{\mu}$ is quasi-invariant by Theorem 2.9 of [19].

To see that $\nu_1 * \tilde{\mu}$ is F-quasi-invariant, use the second formula. The coordinate projection of $G * F^{(0)}$ onto $F^{(0)}$ partitions the action. If $r(\xi) = u$ and $d(\xi) = v$, for $\xi \in F$, then $(x, u) \mapsto (x\varphi(\xi), v) = (x, u)\xi$ maps $d^{-1}(\varphi(u)) \times \{u\}$ one-one onto $d^{-1}(\varphi(v)) \times \{v\}$ and carries $\nu_{\varphi(u)} \times \varepsilon_u$ to $(\nu_{\varphi(u)}\varphi(\xi)) \times \varepsilon_v$ which is equivalent to $\nu_{\varphi(v)} \times \varepsilon_v$ because $d(\varphi(\xi)) = \varphi(v)$ and $\nu_{r(x)}x \sim \nu_{d(x)}$ always. Again by Theorem 2.9 of [19], $\nu_1 * \tilde{\mu}$

is F-quasi-invariant.

By Theorem A3.2 there are an i.c. G_1 of G, a $\nu_1*\tilde{\mu}$-conull G_1-invariant set $T_1 \subseteqq T$ and a Borel function $f\colon T \to G^{(0)}*\mathscr{F}$ such that f is strictly G-equivariant and an ergodic decomposition of $(T*F)$ and $f(t\xi) = f(t)$ whenever $t \in T_1$, $\xi \in F$ and $t\xi$ is defined. Then for $x, y \in G_1$ and $u \in F^{(0)}$ with $r(x) = r(y)$ and $(x, u) \in T_1$ we have $(y^{-1}x, u) \in T_1$. If $d(z) = d(x)$ and $z \in G_1$, we can take $y = xz^{-1}$ to show that $(z, u) \in T_1$. Let V_1 be the projection of T_1 onto $F^{(0)}$. We have proved that $T_1 = d^{-1}(\varphi(V_1))*V_1$. Now T_1 is conull so V_1 is $\tilde{\mu}$-conull and hence $\varphi(V_1)$ is $\varphi_*(\tilde{\mu})$-conull. Let U_0 be a $\varphi_*(\tilde{\mu})$-conull Borel set contained in $\varphi(V_1)$. Then let $V_0 = \tilde{\varphi}^{-1}(U_0)$, $F_0 = F|V_0$, $G_0 = G|[U_0]$, and $T_0 = d^{-1}(U_0)*V_0$. Then T_0 is G_0-invariant and conull and $f\colon T_0 \to G_0^{(0)}*\mathscr{F}$ "$\subseteqq$" $G^{(0)}*\mathscr{F}$ is equivariant. Also $(T*F)|T_0 = T_0*F_0$. If we set $S_\varphi = G_0^{(0)}*\mathscr{F}$ and $\lambda = (f|T_0)_*(\nu_1*\tilde{\mu})$, then λ is quasi-invariant since $\lambda*\nu$ is the image of $(\nu_1*\tilde{\mu})*\nu$.

The next consideration is the strict homomorphism $\varphi'\colon (F_0, [\mu]) \to (S_\varphi*G_0, [\lambda*\nu])$. We want to define $\varphi'(\xi) = (f(\varphi(r(\xi)), r(\xi)), \varphi(\xi))$ as in Theorem 7.8 of [18]. This gives $j\circ\varphi' = \varphi$ on F_0, and we must verify several facts. First let $q\colon S_\varphi \to G_0^{(0)}$ be defined by $q\circ f = p$; of course q is also the natural projection of $G_0^{(0)}*\mathscr{F}$ onto $G_0^{(0)}$. Then for $\xi \in F_0$, $q(f(\varphi(r(\xi)), r(\xi))) = p(\varphi(r(\xi)), r(\xi)) = \varphi(r(\xi)) = r(\varphi(\xi))$ so $(f(\varphi(r(\xi)), r(\xi)), \varphi(\xi)) \in S_\varphi*G$, i.e., $\varphi'(F_0) \subseteqq S_\varphi*G$. Next, $f(\varphi(r(\xi)), r(\xi))\varphi(\xi) = f((\varphi(r(\xi)), r(\xi))\varphi(\xi)) = f(\varphi(\xi)^{-1}, r(\xi)) = f((\varphi(d(\xi)), d(\xi))\xi^{-1}) = f(\varphi(d(\xi)), d(\xi))$. From this it follows easily that φ' is algebraically a homomorphism. Clearly φ' is Borel. To prove φ' has the proper measure theoretic behavior, let E be saturated in S_φ. Then $f^{-1}(E)$ is a Borel set and is invariant under both F and G, so its projection, V, into $F_0^{(0)}$ is analytic and $f^{-1}(E) = d^{-1}(\varphi(V))*V$. Since almost every ν_u is a probability measure, $\nu_1*\tilde{\mu}(f^{-1}(E)) = \tilde{\mu}(V)$. Thus E is null iff V is. Since $V = (\tilde{\varphi}')^{-1}(E)$, we have the desired result.

We have constructed the closure of the range of a homomorphism of virtual groups if it takes values in a groupoid with an (r, d)-quasi-invariant measure. For the general case, we observe that $(G, [\nu])$ always has an i.c. G_0 on which ν is (r, d) quasi-invariant, and φ is similar to a homomorphism φ_0 taking values in G_0. We need to see that S_{φ_0} does not really depend on the choice of φ_0, as the following lemma shows.

LEMMA A3.7. *Let $(G, [\nu])$ be a measured groupoid in which ν is (r, d)-quasi-invariant and let φ_1, φ_2 be similar homomorphisms of a measurable groupoid $(F, [\mu])$ into $(G, [\nu])$. Let $T_1 = T(\varphi_1) = \{(x, u) \in G\times F^{(0)}\colon d(x) = \varphi_1(u)\}$ and take the measure $\nu_1 = \int \nu_u d(\varphi_{1*}(\tilde{\mu}))(u)$ on $d^{-1}(\varphi_1(F^0))$ and $\nu_1*\tilde{\mu}$ on T_1. Similarly form $T_2 = T(\varphi_2)$, ν_2 and*

$\nu_2 * \tilde{\mu}$. *Then there are* i.c.'s F_0 *and* G_0 *of* F *and* G *and* F_0 *and* G_0-*invariant conull analytic sets* $T_1^* \subseteq T_1$ *and* $T_2^* \subseteq T_2$ *which are strictly isomorphic as* F_0 *and* G_0-*spaces under a measureclass-preserving function* f. *Hence* $(S_{\varphi_1}, \lambda_1)$ *and* $(S_{\varphi_2}, \lambda_2)$ *have strictly isomorphic analytic conull* G_0-*invariant subspaces.*

Proof. Suppose $\theta: F^{(0)} \to G$ is Borel and $\theta \circ r(\xi)\varphi_2(\xi) = \varphi_1(\xi)\theta \circ d(\xi)$ for almost every $\xi \in F$. Then there is an i.c. F_1 of such that φ_1, φ_2 and the similarity are all strict on F_1. Set $G_0 = G \mid ([\varphi_1(F_1^{(0)})] \cap [\varphi_2(F_1^{(0)})])$ and set $F_0 = F \mid (\tilde{\varphi}_1^{-1}(G_0^{(0)}) \cap \tilde{\varphi}_2^{-1}(G_0^{(0)}))$. Then $[\varphi_1(F_0^{(0)})] = [\varphi_2(F_0^{(0)})] = G_0^{(0)}$, and $T_1^* = d^{-1}(\varphi_1(F_0^{(0)})) * F_0^{(0)}$ is conull in T_1 while $T_2^* = d^{-1}(\varphi_2(F_0^{(0)})) * F_0$ is conull in T_2.

Now define $f(x, u) = (x\theta(u), u)$ for $(x, u) \in T_1^*$ and $g(x, u) = (x\theta(u)^{-1}, u)$ for $(x, u) \in T_2^*$, as we can since $r \circ \theta = \tilde{\varphi}_1$ and $d \circ \theta = \tilde{\varphi}_2$. These are mutual inverses, so each is one-one and onto; each is clearly Borel. The similarity equation forces f to be F_0-equivariant and f is clearly G-equivariant. Now $f_*(\nu_{\varphi_1(u)} \times \varepsilon_u) = (\nu_{\varphi_1(u)}\theta(u)) \times \varepsilon_u \sim \nu_{\varphi_2(u)} \times \varepsilon_u$ for each $u \in F_0^{(0)}$, so $f_*(\nu_1 * \tilde{\mu}) \sim \nu_2 * \tilde{\mu}$, as desired. Since T_1^* and T_2^* are isomorphic, we can carry the quotient mapping of T_1^* onto S_{φ_1} over to T_2^* via f and get a quotient mapping of T_2^* onto S_{φ_1} which is an ergodic decomposition of $T_2^* * F_0$. Thus S_{φ_1} may be used for S_{φ_2}, ending the proof.

A4. Functorial properties of the range closure construction. The numbers in this section agree with those in § 4.

LEMMA A4.1. *Suppose* $\mathscr{F}_1 = ((F_1, [\lambda_1]), \varphi_1)$ *and* $\mathscr{F}_2 = ((F_2, [\lambda_2]), \varphi_2)$ *are in* $\mathscr{M}(G)$, φ_2 *is strict,* ψ *is a homomorphism of* $\mathscr{F}_1$ *to* $\mathscr{F}_2$ *and* $\theta: F_1^{(0)} \to G$ *is a Borel function for which* $\theta \circ r(\xi)\varphi_2 \circ \psi(\xi) = \varphi_1(\xi)\theta \circ d(\xi)$ *for almost all* ξ. *Then there is a* G-*equivariant normalized* $h = M(\psi, \theta): S_{\varphi_1} \to S_{\varphi_2}$ *obtained as the essential quotient of the function* f^θ *from* $T_1 = G * F_1^{(0)}$ *to* $T_2 = G * F_2^{(0)}$ *defined by* $f^\theta(x, u) = (x\theta(u), \psi(u))$.

Proof, There is no loss of generality in supposing ψ and the similarity θ of $\varphi_2 \circ \psi$ with φ_1 are strict. Then $r \circ \theta = \tilde{\varphi}_1$ implies that $x\theta(u)$ is defined when $(x, u) \in T_1$, and $d \circ \theta = (\varphi_2 \circ \psi)^\sim$ implies that $f^\theta(x, u) \in T_2$. Furthermore, if $r(\xi) = u$ then $f^\theta((x, u)\xi) = f^\theta(x\varphi_1(\xi), d(\xi)) = (x\varphi_1(\xi)\theta \circ d(\xi), \psi \circ d(\xi)) = (x\theta \circ r(\xi)\varphi_2 \circ \psi(\xi), \psi \circ d(\xi)) = f^\theta(x, u)\psi(\xi)$, while $r(y) = r(x)$ implies $f^\theta((x, u)y) = f^\theta(x, u)y$. Now suppose G_0 is an i.c. of G and $g_1: T_1 \to G^{(0)} * \mathscr{F}$ and $g_2: T_2 \to G^{(0)} * \mathscr{F}$ are ergodic decompositions of the actions of F_1 and F_2 which are strictly G_0-equivariant on conull analytic G_0-invariant sets $X_1 \subseteq T_1$ and $X_2 \subseteq T_2$ and have F_1 or F_2 invariant level sets on X_1 and X_2. Then $g_2 \circ f^\theta$ is constant on all F_1-orbits in X_1 so by Lemma A2.3 there is a Borel function h from $G^{(0)}*\mathscr{F}$ to $G^{(0)}\circ\mathscr{F}$ such that $h\circ g_1 = g_2\circ f^\theta$ a.e. Since g_1, g_2, f^θ are equivariant, by Lemma A1.2 we may suppose h is algebraically strictly equivariant on a conull analytic G_1-invariant set for some i.c. $G_1 \subseteq G_0$. We may as well suppose $G_1 = G_0$.

Now to show that h^{-1} has the proper behavior on saturated sets, let A be analytic and G_0-invariant in S_{φ_2}. Then $g_2^{-1}(A)$ is analytic in T_2 and $B = g_1^{-1}(A) \cap X_2$ is G_0-invariant and also F_2-invariant relative to X_2. Now X_2 is invariant under some i.c. of F_2, so by passing to another i.c. we may suppose X_2 is invariant. Then B is F_2-invariant. Now use the fact that μ has an (r, d)-quasi-invariant decomposition on G_0, and $\mu = \int \mu_u d\tilde{\mu}(u)$. In that case, for $v = r(\xi)$ with $\xi \in F_2$, $(\mu_{\varphi_2(v)} \times \varepsilon_v)\xi \sim \mu_{\varphi_2(d(\xi))} \times \varepsilon_{d(\xi)}$. Hence the set $V = \{v \in F_2^{(0)}: (\mu_{\varphi_2(v)} \times \varepsilon_v)(B) > 0\}$ is invariant. If A is a null set, $g_2^{-1}(A)$ is null, so B is null, and hence V is null. Because ψ is a homomorphism, $\tilde{\psi}^{-1}(V)$ is null. For $u \in F_1^{(0)}$, the u-section of $(f^\theta)^{-1}(B)$ is $B_{\psi(u)}\theta(u)^{-1}$ (a translate of the $\psi(u)$-section of B) which is null unless $u \in \tilde{\psi}^{-1}(V)$ because the decomposition is quasi-invariant. Thus $(f^\theta)^{-1}(B)$ is null. Now $g_1^{-1}(h^{-1}(A))$ differs from $(f^\theta)^{-1}(g_2^{-1}(A))$ by a null set and $(f^\theta)^{-1}(T_2 - X_2)$ is null, by the argument just used, so $h^{-1}(A)$ is null.,

On the other hand, if A has positive measure, so does B, so V has positive measure. It follows that the set $\tilde{\psi}^{-1}(V)$ has positive measure. The u-section of $(f^\theta)^{-1}(B)$ will have positive measure for $u \in \tilde{\psi}^{-1}(V)$, so $(f^\theta)^{-1}(B)$ has positive measure, and $h^{-1}(A)$ has positive measure.

LEMMA A4.2. *Under the hypotheses of Lemma A4.1, if δ is another similarity of $\varphi_2\circ\psi$ with φ_1 and φ_2 is strict, then $M(\psi, \delta)$ is similar to $M(\psi, \theta)$.*

Proof. Let F_3 be an i.c. of F_1 on which both similarities are strict. Then $T_3 = d^{-1}(\varphi_1(F_3^{(0)}))*F_3^{(0)}$ is F_3 invariant in $T_1 = T(\varphi_1)$ and is also $G_1 = G|[\varphi(F_3^{(0)})]$ invariant. Hence the quotient of T_3 in S_{φ_1} is G_1-invariant and conull, so we may suppose the similarities were strict on F_1. Then define $\alpha(x, u) = x\theta(u)\delta(u)^{-1}x^{-1}$. It is easily seen that the product does exist and that α is Borel from T_1 to G. Also $f^\theta(x, u)\alpha(x, u)$ is always defined and equal to $f^\delta(x, u)$, while $\alpha((x, u)\xi) = \alpha(x, u)$ for $\xi \in F_1$ if $(x, u)\xi$ is defined, and $\alpha((x, u)y) = y^{-1}\alpha(x, u)y$ if $r(y) = r(x)$. Using Lemma A2.3 we see that there is a Borel $\beta: S_{\varphi_1} \to G$ such that $\beta\circ g_1 = \alpha$ a.e. Lemma A1.2 says that there is a choice of β for which $\beta(sy) = y^{-1}\beta(sy)y$ as long as s is in a certain conull analytic saturated set, i.e., G_0-invariant for some i.c. G_0. It is not hard to see that $M(\psi, \theta)(s)\beta(s) = M(\psi, \delta)(s)$ for almost all $s \in S_{\varphi_1}$, so $[M(\psi, \theta)] = [M(\psi, \delta)]$.

DEFINITION A4.3. Call this class of maps $[M(\psi)]$.

LEMMA A4.4. *If μ is (r, d)-quasi-invariant on G and $\psi_1: \mathscr{F}_1 \to \mathscr{F}_2$ is a homomorphism, where $\mathscr{F}_2 = ((F_2, [\lambda_2], \varphi_2)$ with φ_2 strict, and $\psi_2: (F_1, [\lambda_1]) \to (F_2, [\lambda_2])$ is a homomorphism with $[\psi_2] = [\psi_1]$ then $\psi_2: \mathscr{F}_1 \to \mathscr{F}_2$ is a homomorphism and $[M(\psi_1)] = [M(\psi_2)]$.*

Proof. The first assertion follows immediately from the definition of homomorphism. We may suppose, as before, that θ_1 is a strict similarity of $\varphi_2 \circ \psi_1$ with ψ_1 and that θ is a strict similarity of ψ_2 with ψ_1. If $\theta_2(u) = \theta_1(u)\varphi_2 \circ \theta(u)$, then for $\xi \in F_1$ we have $\theta_2 \circ r(\xi)\varphi_2 \circ \psi_2(\xi) = \varphi_1(\xi)\theta_2 \circ d(\xi)$. Let $f_1^{\theta_1}(x, u) = (x\theta_1(u), \psi_1(u))$, $f_2^{\theta_2}(x, u) = (x\theta_2(u), \psi_2(u))$ for $(x, u) \in T_1$. Then $f_1^{\theta_1}(x, u)\theta(u) = (x\theta_1(u)\varphi_2 \circ \theta(u), d \circ \theta(u)) = f_2^{\theta_2}(x, u)$ because $r \circ \theta = \tilde{\psi}_1$ and $d \circ \theta = \tilde{\psi}_2$. Hence $g_2 \circ f_1^{\theta_1} = g_2 \circ f_2^{\theta_2}$ which implies that $M(\psi_1, \theta_1) \circ g_1 = M(\psi_2, \theta_2) \circ g_1$ a.e. and hence that $[M(\psi_1)] = [M(\psi_1, \theta_1)] = [M(\psi_2, \theta_2)] = [M(\psi_2)]$.

For a definition of $M[\psi]$, see § 4.

LEMMA A4.5. *If $\psi_1: \mathscr{F}_1 \to \mathscr{F}_2$ and $\psi_2: \mathscr{F}_2 \to \mathscr{F}_3$ are homomorphism, for $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$ in $\mathscr{M}(G)$, then $M([\psi_2] \circ [\psi_1]) = M[\psi_2] \circ M[\psi_3]$.*

Proof. We may assume that μ is (r, d)-quasi-invariant. By taking i.c.'s in the proper order we may suppose (φ_3, ψ_2), (φ_2, ψ_1) and (ψ_2, ψ_1) are composable and that we have strict similarities θ_1 of $\varphi_2 \circ \psi_1$ with φ_1 and θ_2 of $\varphi_3 \circ \psi_2$ with φ_2. Then $\theta(u) = \theta_1(u)\theta_2 \circ \tilde{\psi}_1(u)$ defines a strict similarity of $\varphi_3 \circ \psi$ with φ_1, where $\psi = \psi_2 \circ \psi_1$. Then $f^\theta = f_2^{\theta_2} \circ f_1^{\theta_1}$. Now $X = \{t \in G * F_2^{(0)}: M(\psi_2, \theta_2) \circ g_2(t) = g_3 \circ f_2^{\theta_2}(t)\}$ contains a conull invariant Borel set since both functions are equivariant and Borel and they agree a.e. Hence $(f_1^{\theta_1})^{-1}(X)$ has the same property, so we see that $M(\psi_2, \theta_2) \circ M(\psi_1, \theta_1) \circ g_1 = M(\psi_2, \theta_2) \circ g_2 \circ f_1^{\theta_1} = g_3 \circ f^\theta$ a.e. Hence $M(\psi_2, \theta_2) \circ M(\psi_1, \theta_1) = M(\psi, \theta)$ a.e. which gives the desired result.

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