TWO QUESTIONS ON WALLMAN RINGS

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In this paper we give an example of a Wallman ring \( \mathcal{A} \) on a topological space \( X \) such that the associated compactification \( \omega(X, Z(\mathcal{A})) \) is disconnected and \( \mathcal{A} \) is not a direct sum of any two proper ideals, herewith solving a question raised by H. L. Bentley and B. J. Taylor. Also, an example of a uniformly closed Wallman ring which is not a sublattice is given.

I. Introduction. Biles [2] has called a subring \( \mathcal{A} \) of the ring \( C(X) \), of all real-valued continuous functions on a topological space \( X \), a Wallman ring on \( X \) whenever \( Z(\mathcal{A}) \), the zero-sets of functions belonging to \( \mathcal{A} \), forms a normal base on \( X \) in the sense of Frink.

H. L. Bentley and B. J. Taylor [1] studied relationships between algebraic properties of a Wallman ring \( \mathcal{A} \) and topological properties of the compactification \( \omega(X, Z(\mathcal{A})) \) of \( X \). They proved that if \( \mathcal{A} \) is a Wallman ring on \( X \) such that \( \mathcal{A} = \mathcal{B} \oplus \mathcal{C} \) where \( \mathcal{B} \) and \( \mathcal{C} \) are proper ideals of \( \mathcal{A} \), then \( \omega(X, Z(\mathcal{A})) \) is disconnected. We shall prove that the converse of this result is not valid. But, when \( \omega(X, Z(\mathcal{A})) \) is disconnected we find a Wallman ring \( \mathcal{A}^\circ \), equivalent to \( \mathcal{A} \), which is a direct sum of any two proper ideals.

It is well-known that every closed subring of \( C^*(X) \), the ring of all bounded functions in \( C(X) \), that contains all the rational constants is a lattice. But this is not true for arbitrary closed subrings of \( C(X) \). We give an example of a uniformly closed Wallman ring on a space \( Y \) which is not a sublattice of \( C(Y) \). This corrects an assertion stated in ([1], p. 27).

II. Definitions and basic results. All topological spaces under consideration will be completely regular and Hausdorff. A nonempty collection \( \mathcal{F} \) of subsets of a nonempty set \( X \) is said to be a ring of sets if it is closed under the formation of finite unions and finite intersections. The collection \( \mathcal{F} \) is said to be disjunctive if for each closed set \( G \) in \( X \) and point \( x \in X \sim G \) there is a set \( F \in \mathcal{F} \) satisfying \( x \in F \) and \( F \cap G = \emptyset \). It is said to be normal if for \( F_1 \) and \( F_2 \) in \( \mathcal{F} \) with empty intersection there exist \( G_1 \) and \( G_2 \) which are complements of members of \( \mathcal{F} \) satisfying \( F_1 \subseteq G_1, F_2 \subseteq G_2 \) and \( G_1 \cap G_2 = \emptyset \). The collection \( \mathcal{F} \) is a normal base for the topological space \( X \) in case it is a normal, disjunctive, ring of sets that is a base for the closed sets of \( X \).

Throughout this section \( \mathcal{D} \) will denote a disjunctive ring of closed
sets in a topological space $X$ that is a base for the closed sets of $X$. Let $\omega(X, \mathcal{D})$ denote the collection of all $\mathcal{D}$-ultrafilters, and topologize them with a topology having as a base for the closed sets, sets of the form $D^* = \{Z \in \omega(X, \mathcal{D}) : D \in D\}$ where $D \in \mathcal{D}$. Then $X$ can be embedded in $\omega(X, \mathcal{D})$ as a dense subspace when it carries the relative topology. The embedding map takes each $x \in X$ into the unique $\mathcal{D}$-ultrafilter of supersets of $x$ in $\mathcal{D}$. The space $\omega(X, \mathcal{D})$ is a $T_1$-compactification of $X$ ([3], p. 122).

We now state some facts concerning the space $\omega(X, \mathcal{D})$ which will be needed. For a proof see ([3], p. 119, p. 123).

**Proposition 2.1.** The space $\omega(X, \mathcal{D})$ is Hausdorff if and only if $\mathcal{D}$ is a normal base on $X$.

The following result is an interesting characterization of $\omega(X, \mathcal{D})$ due to Sanin.

**Theorem 2.2.** The space $S = \omega(X, \mathcal{D})$ is uniquely determined (in the usual sense) among $T_1$-compactifications of $X$ by its properties

(a) $\{\text{cl}_S D : D \in \mathcal{D}\}$ is a base for the closed sets of $\omega(X, \mathcal{D})$.

(b) For $F_1, F_2$ in $\mathcal{D}$, $\text{cl}_S F_1 \cap \text{cl}_S F_2 = \text{cl}_S (F_1 \cap F_2)$.

According to the Proposition 2.1 if any Hausdorff compactification of $X$ satisfies (a) and (b), then $\mathcal{D}$ is a normal base on $X$.

**III. Disconnectedness of $\omega(X, Z(\mathcal{A}))$.** The next result is a necessary and sufficient condition for the disconnectedness of $\omega(X, Z(\mathcal{A}))$ being $\mathcal{A}$ a Wallman ring on $X$.

**Theorem 3.1.** Let $\mathcal{A}$ be a Wallman ring on a space $X$. Then $\omega(X, Z(\mathcal{A}))$ is disconnected if and only if there is a Wallman ring $\mathcal{A}^\circ$, equivalent to $\mathcal{A}$ (i.e., $\omega(X, Z(\mathcal{A})) = \omega(X, Z(\mathcal{A}^\circ)))$, which is the direct sum of any two proper ideals.

**Proof.** The sufficiency has been proved in ([1], Theorem 3.14) with $\mathcal{A} = \mathcal{A}^\circ$. Necessity. Suppose that $S = \omega(X, Z(\mathcal{A}))$ is disconnected. Then there exist nonempty disjoint closed subsets $A$ and $B$ of $S$ whose union is $S$. Since $A$ is a closed set of $S$,

$$A = \bigcap \{\text{cl}_S Z : A \subset \text{cl}_S Z, Z \in Z(\mathcal{A})\}.$$

It follows from $A \cap B = \emptyset$ that $\{B, \text{cl}_S Z : A \subset \text{cl}_S Z, Z \in Z(\mathcal{A})\}$ does not have the finite intersection property. Therefore $B \cap \text{cl}_S Z_1 \cap \cdots \cap \text{cl}_S Z_n = \emptyset$, for some $Z_i \in Z(\mathcal{A})$, $A \subset \text{cl}_S Z_i$, $1 \leq i \leq n$. This implies $A = \bigcap \{\text{cl}_S Z_i : 1 \leq i \leq n\} = \text{cl}_S \cap \{Z_i : 1 \leq i \leq n\}$. So $A = \text{cl}_S Z(f)$ where
In the same way we find that $B = \text{cl}_K Z(g), g \in \mathcal{A}$.

The set $\mathcal{A}^\circ = \{h/s: h, s \in \mathcal{A}, Z(s) = \emptyset\}$ is a subring of $C(X)$ such that $Z(\mathcal{A}) = Z(\mathcal{A}^\circ)$. So $\mathcal{A}^\circ$ is a Wallman ring on $X$ equivalent to $\mathcal{A}$. The functions $h_i = f^i/(f^2 + g^i), h_2 = g^2/(f^2 + g^2)$ belong to $\mathcal{A}^\circ$ and they are the characteristic functions of the zero-sets $Z(g)$ and $Z(f)$, respectively. Since $Z(f) \cap Z(g) = \emptyset$, the ideal $(h_i)$ of $\mathcal{A}^\circ$ generated by $h_i$ is proper, $1 \leq i \leq 2$. On the other hand, $1 = h_1 + h_2$ implies that $\mathcal{A}^\circ = (h_1) \oplus (h_2)$.

The following is an example of Wallman ring which cannot be expressed as the direct sum of nontrivial ideals.

**Example 3.2.** Let $X = [0, 1) \cup [2, 3), \mathcal{B} = \{f \in C(X): \text{for some compact set } K \subset X, f \text{ is an integer constant on } X \sim K\}$. Since $X$ is locally compact, $Z(\mathcal{B})$ is a disjunctive base for the closed sets of $X$.

Consider the following functions in $C(X)$

$$
\varphi_1(x) = e, \ x \in [0, 1) \quad \varphi_1(x) = 0, \ x \in [2, 3),
$$

$$
\varphi_2(x) = 0, \ x \in [0, 1) \quad \varphi_2(x) = e, \ x \in [2, 3).
$$

Let $\mathcal{A}$ be the subring of $C(X)$ generated by $\mathcal{B} \cup \{\varphi_1, \varphi_2\}$. Since $\varphi_1 \varphi_2 = 0$, a function of $\mathcal{A}$ will be of the form

$$
f = g_{00} + g_{10} \varphi_1 + g_{20} \varphi_1^2 + \cdots + g_{m0} \varphi_1^m + g_{01} \varphi_2 + \cdots + g_{0j} \varphi_2^j
$$

where $g_{ij}$ belong to $\mathcal{B}$ and $m, j$ are nonnegative integers.

From the definition of $\mathcal{B}$, there exist compact sets $K_1 \subset [0, 1)$ and $K_2 \subset [2, 3)$ such that if $x \in X \sim (K_1 \cup K_2)$ then $g_{ik}(x) = \alpha_{ik} \in Z$ (the set of integer numbers). Therefore

$$
f(x) = \alpha_{00} + \alpha_{10} e + \cdots + \alpha_{m0} e^m, \quad x \in [0, 1) \sim K_1
$$

$$
f(x) = \alpha_{00} + \alpha_{01} e + \cdots + \alpha_{0j} e^j, \quad x \in [2, 3) \sim K_2.
$$

Since $Z(\mathcal{B}) \subset Z(\mathcal{A})$ it follows that $Z(\mathcal{A})$ is a disjunctive base for the closed sets of $X$ and a ring of sets.

Now, we will show that $K = [0, 1] \cup [2, 3]$ is a compactification of $X$ equivalent to $\omega(X, Z(\mathcal{A}))$. According to Theorem 2.2 it suffices to show that: (a) The family $\{\text{cl}_K Z: Z \in Z(\mathcal{A})\}$ is a base for the closed sets of $K$ (b) For $Z_1, Z_2$ in $Z(\mathcal{A}), \text{cl}_K (Z_1 \cap Z_2) = \text{cl}_K Z_1 \cap \text{cl}_K Z_2$.

(a) If $C$ is a closed set in $K$ and $1 \in C$, then the set $C \cap [0, 1]$ is compact and $1 \in C \cap [0, 1]$. Let $\beta$ be a point in $[0, 1)$ such that $C \cap [\beta, 1] = \emptyset$. Then, there exists a function $f \in C(K)$ such that $f([\beta, 1] \cup [2, 3]) = \{1\}$ and $f(C \cap [0, 1]) = \{0\}$. If $g$ is the restriction of $f$ to $X$, then $g \in \mathcal{B}, h = \varphi_1 g \in \mathcal{A}, C \subset \text{cl}_K Z(h)$ and $1 \in \text{cl}_K Z(h)$. With the point $3$ a similar argument can be used (also in (b)).
(b) Let \( f, g \in \mathcal{A} \) and suppose that \( 1 \in \text{cl}_K Z(f) \cap \text{cl}_K Z(g) \). From (*) there exists \( \beta \in [0, 1) \) such that \( f(x) = m_1 \) and \( g(x) = m_2 \) for every \( x \in [\beta, 1) \). By our assumption \( m_1 = m_2 = 0 \), therefore \( 1 \in \text{cl}_K (Z(f) \cap Z(g)) \).

Then \( K = \omega(X, Z(\mathcal{A})) \), hence \( Z(\mathcal{A}) \) is a normal base on \( X \) and \( \mathcal{A} \) is a Wallman ring.

Now, we will show that the characteristic function of the interval \([0, 1)\) is not in \( \mathcal{A} \). Let \( h \in \mathcal{A} \). From (*) there exist \( \beta \in [0, 1), \gamma \in [2, 3) \) and \( \alpha_{ik} \in \mathbb{Z}, 0 \leq i \leq m, 0 \leq k \leq j \) such that

\[
\begin{align*}
    h(x) &= \alpha_{00} + \alpha_{01}e + \cdots + \alpha_{0\ell}e^\ell, & x \in [\gamma, 3) \\
    h(x) &= \alpha_{00} + \alpha_{10}e + \cdots + \alpha_{m0}e^m, & x \in [\beta, 1).
\end{align*}
\]

If \([2, 3) \subset Z(h)\), then \( \alpha_{00} = \alpha_{01} = \cdots = \alpha_{\ell} = 0 \) because \( e \) is a transcendental number. Therefore \( h(x) = \alpha_{10}e + \cdots + \alpha_{m0}e^m \neq 1 \) if \( x \in [\beta, 1) \).

Finally, we will show that \( \mathcal{A} \) cannot be expressed as the direct sum of nontrivial ideals. Suppose that \( \mathcal{A} = \mathcal{C} \oplus \mathcal{H} \) where \( \mathcal{C} \) and \( \mathcal{H} \) are proper ideals of \( \mathcal{A} \). Then \( 1 \in \mathcal{A} \) implies that there exist \( f \in \mathcal{C} \) and \( g \in \mathcal{H} \) such that \( 1 = f + g \) and \( fg = 0 \). Hence \( \{Z(g), Z(f)\} \) is a partition on \( X \). On the other hand, since \( \mathcal{C} \) and \( \mathcal{H} \) are proper ideals, the zero-sets \( Z(f) \) and \( Z(g) \) are nonempty, so \( [0, 1) = Z(f) \) and \([2, 3) = Z(g)\). Therefore \( g \in \mathcal{A} \) is the characteristic function of the interval \([0, 1)\), which is a contradiction.

IV. An example of a closed Wallman ring which is not a lattice. Let \( N \) denote the set of natural numbers. By a sublattice of \( C(X) \) we mean a subset of \( C(X) \) which contains the supremum and infimum of each pair of its elements. By a closed subring of \( C(X) \) we mean a subring of \( C(X) \) which is closed in the uniform topology on \( C(X) \).

**Example 4.1.** Let \( \mathcal{B} \) be the set \( \{f \in C(N): \text{for some finite subset } M \subset N, f \text{ is an integer constant on } N \sim M \} \). Then \( \mathcal{B} \) is a subring of \( C(N) \) and \( Z(\mathcal{B}) = \{B \subset N: B \text{ or } N \sim B \text{ is finite} \} \). It is well-known that \( \mathcal{B} \) is a Wallman ring on \( N \) such that \( \omega(N, Z(\mathcal{B})) \) is the one-point compactification of \( N \).

Let \( \varphi \) be the function defined \( \varphi(2n) = n, \varphi(2n - 1) = -n, n = 1, 2, \ldots \). Let \( \mathcal{A} \) be the subring of \( C(N) \) generated by \( \mathcal{B} \cup \{\varphi\} \). Obviously \( Z(\mathcal{B}) \subset Z(\mathcal{A}) \). To show that \( Z(\mathcal{A}) \subset Z(\mathcal{B}) \), let \( f \in \mathcal{A} \). Then \( f = g_0 + g_1 \varphi + \cdots + g_m \varphi^m \), where \( g_i \in \mathcal{B}, 0 \leq i \leq m \). From the definition of \( \mathcal{B} \), there exist \( n_0 \in N, \alpha_i \in Z, 0 \leq i \leq m \) such that \( g_i(2n - 1) = g_i(2n) = \alpha_i, 0 \leq i \leq m \) for every \( n \geq n_0 \). If \( \alpha_1 = \cdots = \alpha_m = 0 \), then \( f(2n - 1) = f(2n) = \alpha_0 \) for every \( n \geq n_0 \) and therefore
Suppose $\alpha_{i_0} \neq 0$ for some $i_0 \geq 1$. Then, if $n \geq n_0$, $f(2n) = \alpha_0 + n\alpha_1 + \cdots + n^m\alpha_m$ and $f(2n - 1) = \alpha_0 - n\alpha_1 + \cdots + (-1)^m n^m\alpha_m$. So $Z(f)$ is finite and $Z(f) \in Z(\mathcal{B})$. Hence $\mathcal{A}$ is a Wallman ring on $X$.

If $\varphi^+ = \varphi \vee 0$, then $Z(\varphi^+) = \{1, 3, 5, \cdots\} \in Z(\mathcal{A})$. Therefore $\varphi^+ \notin \mathcal{A}$ and $\mathcal{A}$ is not a lattice. Finally, since the functions of $\mathcal{A}$ are integer-valued, it follows that $\mathcal{A}$ is uniformly closed in $C(N)$.

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