Pacific Journal of Mathematics

TWO QUESTIONS ON WALLMAN RINGS

José L. Blasco Olcina

Vol. 88, No. 1 March 1980

PACIFIC JOURNAL OF MATHEMATICS
Vol. 88, No. 1, 1980

TWO QUESTIONS ON WALLMAN RINGS

JOSE L. BLASCO

In this paper we give an example of a Wallman ring $\mathscr{A}$ on a topological space X such that the associated compactification $\omega(X, Z(\mathscr{A}))$ is disconnected and $\mathscr{A}$ is not a direct sum of any two proper ideals, herewith solving a question raised by H. L. Bentley and B. J. Taylor. Also, an example of a uniformly closed Wallman ring which is not a sublattice is given.

I. **Introduction.** Biles [2] has called a subring $\mathscr{A}$ of the ring $C(X)$, of all real-valued continuous functions on a topological space X, a Wallman ring on X whenever $Z(\mathscr{A})$, the zero-sets of functions beloning to $\mathscr{A}$, forms a normal base on X in the sense of Frink.

H. L. Bentley and B. J. Taylor [1] studied relationships between algebraic properties of a Wallman ring $\mathscr{A}$ and topological properties of the compactification $\omega(X, Z(\mathscr{A}))$ of X. They proved that if $\mathscr{A}$ is a Wallman ring on X such that $\mathscr{A} = \mathscr{B} \oplus \mathscr{C}$ where $\mathscr{B}$ and $\mathscr{C}$ are proper ideals of $\mathscr{A}$, then $\omega(X, Z(\mathscr{A}))$ is disconnected. We shall prove that the converse of this result is not valid. But, when $\omega(X, Z(\mathscr{A}))$ is disconnected we find a Wallman ring $\mathscr{A}^\circ$, equivalent to $\mathscr{A}$, which is a direct sum of any two proper ideals.

It is well-known that every closed subring of $C^*(X)$, the ring of all bounded functions in $C(X)$, that contains all the rational constants is a lattice. But this is not true for arbitrary closed subrings of $C(X)$. We give an example of a uniformly closed Wallman ring on a space Y which is not a sublattice of $C(Y)$. This corrects an assertion stated in ([1], p. 27).

II. **Definitions and basic results.** All topological spaces under consideration will be completely regular and Hausdorff. A nonempty collection $\mathscr{F}$ of subsets of a nonempty set X is said to be a ring of sets if it is closed under the formation of finite unions and finite intersections. The collection $\mathscr{F}$ is said to be disjunctive if for each closed set G in X and point $x \in X \sim G$ there is a set $F \in \mathscr{F}$ satisfying $x \in F$ and $F \cap G = \varnothing$. It is said to be normal if for F_1 and F_2 in $\mathscr{F}$ with empty intersection there exist G_1 and G_2 which are complements of members of $\mathscr{F}$ satisfying $F_1 \subset G_1$, $F_2 \subset G_2$ and $G_1 \cap G_2 = \varnothing$. The collection $\mathscr{F}$ is a normal base for the topological space X in case it is a normal, disjunctive, ring of sets that is a base for the closed sets of X.

Throughout this section $\mathscr{D}$ will denote a disjunctive ring of closed sets in a topological space X that is a base for the closed sets of X. Let $\omega(X, \mathscr{D})$ denote the collection of all $\mathscr{D}$-ultrafilters, and topologize them with a topology having as a base for the closed sets, sets of the form $D^* = \{\mathscr{U} \in \omega(X, \mathscr{D}): D \in \mathscr{U}\}$ where $D \in \mathscr{D}$. Then X can be embedded in $\omega(X, \mathscr{D})$ as a dense subspace when it carries the relative topology. The embedding map takes each $x \in X$ into the unique $\mathscr{D}$-ultrafilter of supersets of x in $\mathscr{D}$. The space $\omega(X, \mathscr{D})$ is a T_1-compactification of X ([3], p. 122).

We now state some facts concerning the space $\omega(X, \mathscr{D})$ which will be needed. For a proof see ([3], p. 119, p. 123).

PROPOSITION 2.1. *The space $\omega(X, \mathscr{D})$ is Hausdorff if and only if $\mathscr{D}$ is a normal base on X.*

The following result is an interesting characterization of $\omega(X, \mathscr{D})$ due to Sanin.

THEOREM 2.2. *The space $S = \omega(X, \mathscr{D})$ is uniquely determined (in the usual sense) among T_1-compactifications of X by its properties*

(a) $\{\mathrm{cl}_S D: D \in \mathscr{D}\}$ *is a base for the closed sets of* $\omega(X, \mathscr{D})$.

(b) *For* F_1, F_2 *in* $\mathscr{D}$, $\mathrm{cl}_S F_1 \cap \mathrm{cl}_S F_2 = \mathrm{cl}_S(F_1 \cap F_2)$.

According to the Proposition 2.1 if any Hausdorff compactification of X satisfies (a) and (b), then $\mathscr{D}$ is a normal base on X.

III. Disconnectedness of $\omega(X, Z(\mathscr{A}))$. The next result is a necessary and sufficient condition for the disconnectedness of $\omega(X, Z(\mathscr{A}))$ being $\mathscr{A}$ a Wallman ring on X.

THEOREM 3.1. *Let $\mathscr{A}$ be a Wallman ring on a space X. Then $\omega(X, Z(\mathscr{A})$ is disconnected if and only if there is a Wallman ring $\mathscr{A}^\circ$, equivalent to $\mathscr{A}$* (i.e., $\omega(X, Z(\mathscr{A})) = \omega(X, Z(\mathscr{A}^\circ))$), *which is the direct sum of any two proper ideals.*

Proof. The sufficiency has been proved in ([1], Theorem 3.14) with $\mathscr{A} = \mathscr{A}^\circ$. Necessity. Suppose that $S = \omega(X, Z(\mathscr{A}))$ is disconnected. Then there exist nonempty disjoint closed subsets A and B of S whose union is S. Since A is a closed set of S,

$$A = \bigcap \{\mathrm{cl}_S Z: A \subset \mathrm{cl}_S Z,\ Z \in Z(\mathscr{A})\}\ .$$

It follows from $A \cap B = \emptyset$ that $\{B, \mathrm{cl}_S Z: A \subset \mathrm{cl}_S Z,\ Z \in Z(\mathscr{A})\}$ does not have the finite intersection property. Therefore $B \cap \mathrm{cl}_S Z_1 \cap \cdots \cap \mathrm{cl}_S Z_n = \emptyset$, for some $Z_i \in Z(\mathscr{A})$, $A \subset \mathrm{cl}_S Z_i$, $1 \leqq i \leqq n$. This implies $A = \bigcap \{\mathrm{cl}_S Z_i: 1 \leqq i \leqq n\} = \mathrm{cl}_S \cap \{Z_i: 1 \leqq i \leqq n\}$. So $A = \mathrm{cl}_S Z(f)$ where

$f \in \mathscr{A}$. In the same way we find that $B = \mathrm{cl}_S Z(g)$, $g \in \mathscr{A}$.

The set $\mathscr{A}^\circ = \{h/s\colon h, s \in \mathscr{A}, Z(s) = \varnothing\}$ is a subring of $C(X)$ such that $Z(\mathscr{A}) = Z(\mathscr{A}^\circ)$. So $\mathscr{A}^\circ$ is a Wallman ring on X equivalent to $\mathscr{A}$. The functions $h_1 = f^2/(f^2 + g^2)$, $h_2 = g^2/(f^2 + g^2)$ belong to $\mathscr{A}^\circ$ and they are the characteristic functions of the zero-sets $Z(g)$ and $Z(f)$, respectively. Since $Z(f) \cap Z(g) = \varnothing$, the ideal (h_i) of $\mathscr{A}^\circ$ generated by h_i is proper, $1 \leqq i \leqq 2$. On the other hand, $1 = h_1 + h_2$ implies that $\mathscr{A}^\circ = (h_1) \oplus (h_2)$.

The following is an example of Wallman ring which cannot be expressed as the direct sum of nontrivial ideals.

EXAMPLE 3.2. Let $X = [0, 1) \cup [2, 3)$, $\mathscr{B} = \{f \in C(X)\colon$ for some compact set $K \subset X$, f is an integer constant on $X \sim K\}$.

Since X is locally compact, $Z(\mathscr{B})$ is a disjunctive base for the closed sets of X.

Consider the following functions in $C(X)$

$$\varphi_1(x) = e\,, \quad x \in [0, 1) \qquad \varphi_1(x) = 0\,, \quad x \in [2, 3)$$
$$\varphi_2(x) = 0\,, \quad x \in [0, 1) \qquad \varphi_2(x) = e\,, \quad x \in [2, 3)\,.$$

Let $\mathscr{A}$ be the subring of $C(X)$ generated by $\mathscr{B} \cup \{\varphi_1, \varphi_2\}$. Since $\varphi_1\varphi_2 = 0$, a function of $\mathscr{A}$ will be of the form

$$f = g_{00} + g_{10}\varphi_1 + g_{20}\varphi_1^2 + \cdots + g_{m0}\varphi_1^m + g_{01}\varphi_2 + \cdots + g_{0j}\varphi_2^j$$

where g_{ik} belong to $\mathscr{B}$ and m, j are nonnegative integers.

From the definition of $\mathscr{B}$, there exist compact sets $K_1 \subset [0, 1)$ and $K_2 \subset [2, 3)$ such that if $x \in X \sim (K_1 \cup K_2)$ then $g_{ik}(x) = \alpha_{ik} \in Z$ (the set of integer numbers). Therefore

$$(*) \qquad \begin{aligned} f(x) &= \alpha_{00} + \alpha_{10}e + \cdots + \alpha_{m0}e^m\,, &\quad x \in [0, 1) \sim K_1 \\ f(x) &= \alpha_{00} + \alpha_{01}e + \cdots + \alpha_{0j}e^j\,, &\quad x \in [2, 3) \sim K_2\,. \end{aligned}$$

Since $Z(\mathscr{B}) \subset Z(\mathscr{A})$ it follows that $Z(\mathscr{A})$ is a disjunctive base for the closed sets of X and a ring of sets.

Now, we will show that $K = [0, 1] \cup [2, 3]$ is a compactification of X equivalent to $\omega(X, Z(\mathscr{A}))$. According to Theorem 2.2 it suffices to show that: (a) The family $\{\mathrm{cl}_K Z\colon Z \in Z(\mathscr{A})\}$ is a base for the closed sets of K (b) For Z_1, Z_2 in $Z(\mathscr{A})$, $\mathrm{cl}_K(Z_1 \cap Z_2) = \mathrm{cl}_K Z_1 \cap \mathrm{cl}_K Z_2$.

(a) If C is a closed set in K and $1 \notin C$, then the set $C \cap [0, 1]$ is compact and $1 \notin C \cap [0, 1]$. Let β be a point in $[0, 1)$ such that $C \cap [\beta, 1] = \varnothing$. Then, there exists a function $f \in C(K)$ such that $f([\beta, 1] \cup [2, 3]) = \{1\}$ and $f(C \cap [0, 1]) = \{0\}$. If g is the restriction of f to X, then $g \in \mathscr{B}$, $h = \varphi_1 g \in \mathscr{A}$, $C \subset \mathrm{cl}_K Z(h)$ and $1 \notin \mathrm{cl}_K Z(h)$. With the point 3 a similar argument can be used (also in (b)).

(b) Let $f, g \in \mathscr{A}$ and suppose that $1 \in \mathrm{cl}_K Z(f) \cap \mathrm{cl}_K Z(g)$. From (*) there exists $\beta \in [0, 1)$ such that $f(x) = m_1$ and $g(x) = m_2$ for every $x \in [\beta, 1)$. By our assumption $m_1 = m_2 = 0$, therefore $1 \in \mathrm{cl}_K(Z(f) \cap Z(g))$.

Then $K = \omega(X, Z(\mathscr{A}))$, hence $Z(\mathscr{A})$ is a normal base on X and $\mathscr{A}$ is a Wallman ring.

Now, we will show that the characteristic function of the interval $[0, 1)$ is not in $\mathscr{A}$. Let $h \in \mathscr{A}$. From (*), there exist $\beta \in [0, 1)$, $\gamma \in [2, 3)$ and $\alpha_{ik} \in Z$, $0 \leqq i \leqq m$, $0 \leqq k \leqq j$ such that

$$h(x) = \alpha_{00} + \alpha_{01}e + \cdots + \alpha_{0j}e^j\,, \quad x \in [\gamma, 3)$$
$$h(x) = \alpha_{00} + \alpha_{10}e + \cdots + \alpha_{m0}e^m\,, \quad x \in [\beta, 1)\,.$$

If $[2, 3) \subset Z(h)$, then $\alpha_{00} = \alpha_{01} = \cdots = \alpha_{0j} = 0$ because e is a transcendental number. Therefore $h(x) = \alpha_{10}e + \cdots + \alpha_{m0}e^m \neq 1$ if $x \in [\beta, 1)$.

Finally, we will show that $\mathscr{A}$ cannot be expressed as the direct sum of nontrivial ideals. Suppose that $\mathscr{A} = \mathscr{C} \oplus \mathscr{H}$ where $\mathscr{C}$ and $\mathscr{H}$ are proper ideals of $\mathscr{A}$. Then $1 \in \mathscr{A}$ implies that there exist $f \in \mathscr{C}$ and $g \in \mathscr{H}$ such that $1 = f + g$ and $fg = 0$. Hence $\{Z(g), Z(f)\}$ is a partition on X. On the other hand, since $\mathscr{C}$ and $\mathscr{H}$ are proper ideals, the zero-sets $Z(f)$ and $Z(g)$ are nonempty, so $[0, 1) = Z(f)$ and $[2, 3) = Z(g)$. Therefore $g \in \mathscr{A}$ is the characteristic function of the interval $[0, 1)$, which is a contradiction.

IV. An example of a closed Wallman ring which is not a lattice. Let N denote the set of natural numbers. By a sublattice of $C(X)$ we mean a subset of $C(X)$ which contains the supremum and infimum of each pair of its elements. By a closed subring of $C(X)$ we mean a subring of $C(X)$ which is closed in the uniform topology on $C(X)$.

EXAMPLE 4.1. Let $\mathscr{B}$ be the set $\{f \in C(N)$: for some finite subset $M \subset N$, f is an integer constant on $N \sim M\}$. Then $\mathscr{B}$ is a subring of $C(N)$ and $Z(\mathscr{B}) = \{B \subset N$: B or $N \sim B$ is finite$\}$. It is well-known that $\mathscr{B}$ is a Wallman ring on N such that $\omega(N, Z(\mathscr{B}))$ is the one-point compactification of N.

Let φ be the function defined $\varphi(2n) = n$, $\varphi(2n - 1) = -n$, $n = 1, 2, \cdots$. Let $\mathscr{A}$ be the subring of $C(N)$ generated by $\mathscr{B} \cup \{\varphi\}$. Obviously $Z(\mathscr{B}) \subset Z(\mathscr{A})$. To show that $Z(\mathscr{A}) \subset Z(\mathscr{B})$, let $f \in \mathscr{A}$. Then $f = g_0 + g_1\varphi + \cdots + g_m\varphi^m$, where $g_i \in \mathscr{B}$, $0 \leqq i \leqq m$. From the definition of $\mathscr{B}$, there exist $n_0 \in N$, $\alpha_i \in Z$, $0 \leqq i \leqq m$ such that $g_i(2n - 1) = g_i(2n) = \alpha_i$, $0 \leqq i \leqq m$ for every $n \geqq n_0$. If $\alpha_1 = \cdots = \alpha_m = 0$, then $f(2n - 1) = f(2n) = \alpha_0$ for every $n \geqq n_0$ and therefore

$f \in \mathscr{B}$. Suppose $\alpha_{i_0} \neq 0$ for some $i_0 \geqq 1$. Then, if $n \geqq n_0$, $f(2n) = \alpha_0 + n\alpha_1 + \cdots + n^m\alpha_m$ and $f(2n-1) = \alpha_0 - n\alpha_1 + \cdots + (-1)^m n^m \alpha_m$. So $Z(f)$ is finite and $Z(f) \in Z(\mathscr{B})$. Hence $\mathscr{A}$ is a Wallman ring on X.

If $\varphi^+ = \varphi \vee 0$, then $Z(\varphi^+) = \{1, 3, 5, \cdots\} \notin Z(\mathscr{A})$. Therefore $\varphi^+ \notin \mathscr{A}$ and $\mathscr{A}$ is not a lattice. Finally, since the functions of $\mathscr{A}$ are integer-valued, it follows that $\mathscr{A}$ is uniformly closed in $C(N)$.

References

1. H. L. Bentley and B. J. Taylor, *Wallman rings*, Pacific J. Math., **58** (1975), 15-35.

2. C. M. Biles, *Gelfand and Wallman-type compactifications*, Pacific J. Math., **35** (1970), 267-278.

3. J. Nagata, *Mordern General Topology*, North-Holland Publishing Co., Amsterdam-Gronigen, 1968.

Received May 2, 1979.

CATEDRA DE MATEMATICAS II
FACULTAD DE CIENCIAS
BURJASOT, VALENCIA
SPAIN

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 88, No. 1 March, 1980