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Let Ω be a strictly pseudoconvex smoothly bounded domain in C^n and let $M \subset \Omega$ be a complex hypersurface in Ω. In this paper I develop a condition that is sufficient to ensure that $M = f^{-1}(0)$ for some $f \in H^p(\Omega)$ (i.e., some f belonging to some Hardy class of Ω). That condition refers to the growth of the $2n-2$ dimensional volume of M as it approaches the boundary $\partial\Omega$.

Introduction. Quite recently G. M. Henkin and H. Skoda, independently, in two remarkable papers [7] [15] have obtained a complete characterization of the zero sets M of functions in the Nevanlinna class of a strictly pseudoconvex domain $\Omega \subset \boldsymbol{C}^n$. The characterization is very simple and is just the Blaschke condition on the growth of the volume of M as it approaches the boundary (cf. § 1 for the exact definitions).

The question naturally arises as to what can be said about the zero sets of functions in the Hardy H^p-classes ($p > 0$) of the same domains. When n, the complex dimension, is one it is of course well known that the same Blaschke condition is necessary and sufficient to characterize the zero sets of all these classes.

When $n \geqq 2$ the situation is considerably more complicated. Indeed it is known [8], [14] that for two different values of $p > 0$ the zero sets of functions in the corresponding H^p-classes differ (this fact is essentially proved in § 9 below). In view of this fact a complete characterization of the zero sets of functions in H^p becomes much more difficult and to, my mind, not even very feasible.

In this paper I give a general condition on M an analytic set in Ω to be the zero set of some function in some $H^p(\Omega)$-class (i.e., for some value of $p > 0$). The exact statement will be given in § 1. If I were to attempt to describe the condition in general terms, I would say that it is just the Blaschke condition again, except that it makes sure that no point at infinity (i.e., on $\partial\Omega$) takes more limiting mass of M than its due. In other words the Blaschke condition holds in a uniform fashion as we approach the boundary (hence the terminology "uniform Blaschke").

The above uniformity can be expressed in terms of a Carleson condition at the boundary. In the case of the complex ball which is a domain that admits a transitive group of holomorphic self-mappings, the above uniformity can be expressed by the fact that the Blaschke nature of the divisor M stays stable by all holomorphic self-mappings of Ω. For $n = 1$ the above "uniform Blaschke condition" reduces to the classical Carleson interpolation condition for sequences in the unit disc [1][2].

The way the material is organized is as follows:

§1. Exact definitions and statements of the theorems are given.

§2. The uniform Blaschke condition is examined in details and the basic real variable Theorem 2.1 is stated.

§3. The geometric condition needed for the proof of Theorem 2.1 is given.

§4. The Theorem 2.1 is proved modulo the geometric estimates that are postponed untill the next paragraph.

§5. The geometric estimates are given.

§6. The proof of the main theorem is completed.

§7. The passage to the general strictly pseudoconvex domains is examined.

§8. Special kind of divisors that are obtained from complex lines are examined.

§9. The previous results are illustrated by examples.

§10. The best possible nature (in some sense) of the uniform Blaschke condition is exhibited.

The heart of the matter is §3 and §4. The" power-house" (i.e., the sordid computations that are needed to make everything else work) is §5.

All the theorems are stated and hold for general strictly pseudoconvex domains. Some of the more tedious "local" estimates and geometric computations, however, are only given for B, the complex ball, and dimension $n=2$ for that matter. The situation is perfectly typical and the reader who possesses some technique (and is sufficietly perverse) can, I am sure, carry these details out in general domains for himself.

1. Notations, definitions, and statement of the main theorem. Let $\Omega = \{z \in C^n; \rho < 0\}$ be a bounded strictly pseudoconvex domain of C^n and let us suppose that Ω is defined by some function ρ which is C^4 and strictly plurisubharmonic in some neighborhood of $\bar{\Omega}$ and such that $d\rho \neq 0$ on $\partial\Omega$.

We shall say that μ a Radon measure on Ω is a Carleson measure if:

$$|\mu|(\tilde{B}_t(\zeta_0)) \leqq C|B_t(\zeta_0)|; \quad 0 < t < t_0, \quad \zeta_0 \in \partial\Omega \tag{1.1}$$

where C is a constant that depends only on μ and not on ζ_0 or t. The definition of the demains $B_t(\zeta_0)$ and $\tilde{B}_t(\zeta_0)$ has been given in [18] 2.1 where also the notion of a Carleson measure has been elaborated at length. We shall refer the reader there for the details.

Let now

$$\omega = \sum_{I.J} \omega_{I,J} dz_I \wedge d\bar{z}_J$$

be a current of order zero in Ω (i.e, such that its coefficients $\omega_{I,J}$ can be identified with Radon measures in Ω). We shall denote then by

$$|\omega| = \sum_{I,J} |\omega_{I,J}|$$

which is a positive measure in Ω and by

$$||\omega|| = \sum_{I,J} ||\omega_{I,J}|| = || \, |\omega| \, ||$$

the total mass of ω when that mass is finite. ($|\omega_{I,J}|$ and $||\omega_{I,J}||$ denotes, of course, the absolute value and the total mass of the measure $\omega_{I,J}$). We shall also say that a current ω of order zero satisfies the Carleson condition, or that it is Carleson, if $|\omega|$ is a Carleson measure in Ω. (Observe that this definition is *not* consistant with the one given in [18] 3.1.)

Let $X \subset \Omega$ be a p-dimensional orientable submanifold regularly embedded in Ω, we shall adopt then the standard notation $[X]$ for the integration current on X, provided of course that the integration current is well defined. More explicitely let us assume that for all compact subset $K \subset\subset \Omega$ the p-dimensional volume $|X \cap K|_p < +\infty$, we can then define

$$[X]\varphi = \int_X \varphi$$

for all C^∞ compactly supported differential form φ in Ω. $[X]$ when defined is clearly a current of order zero.

Let now $\tilde{M}$ be a divisor in Ω given by the Cousin data $\{f_i, U_i, i = 1, 2, \cdots\}$ where $\{U_i\}$ is a covering of Ω. Concerning divisors we shall follow all the notations and definitions of P. Lelong [10] (especially Ch. VII). In particular we shall denote by M the underlying analytic set of $\tilde{M}$, by $M^* \subset M$ the subset of regular points of M, and by

$$M^* = \bigcup_{k=1}^{\infty} M_k^*$$

the decomposition of M^* into its topological components. We shall denote by $m(x)$ ($x \in M^*$) the multiplicity function of the divisor defined on M^* ($m(x) = m_k$ is constant on each component M_k^*, $k = 1, 2, \cdots$).

We shall denote by

$$t = \frac{i}{\pi}\partial\bar{\partial}\log|f_i| \text{ on } U_i \; i = 1, 2, \cdots$$

the Lelong current associated with the divisor $\tilde{M}$ in Ω [cf. [10] Ch. VII]. P. Lelong's basic theorem says then that t is a positive current in Ω and that

$$t = \sum_{k=1}^{\infty} m_k[M_k^*] . \tag{1.2}$$

In particular therefore t is a current of order zero in Ω.

Let us now denote by $\delta(x)$ the Euclidean distance of $x \in \Omega$ from $\partial\Omega$, $\delta(x)$ is then comparable with $-\rho(x)$ when x is near $\partial\Omega$ in the sense that there exists $C > 0$ some positive constant such that

$$C^{-1}(-\rho(x)) \leqq \delta(x) \leqq C(-\rho(x)) . \tag{1.3}$$

t being a current of order zero we can define $b = b_{\tilde{M}} = \delta(x)t$ a new current which is also a current of order zero. We shall then say that the devisor $\tilde{M}$ satisfies the Blaschke condition, or that it is a Blaschke divisor if $||b_{\tilde{M}}|| < +\infty$.

Let us denote by $d\sigma_\beta$ the $2n - 2$ Euclidean volume element on M^* ($d\sigma_\beta$ as a measure is given by the volume current $1/(n-1)![M^*]\wedge \beta^{n-1}$ where $\beta = i/2\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$, but this fact is irrelevant for us). In terms of $d\sigma_\beta$ we can then express the Blaschke condition as

$$||\tilde{M}||_\beta = \int_{M^*} m(x)\delta(x)d\sigma_\beta(x) < +\infty .$$

Indeed the two "norms" $||\tilde{M}||_\beta$ and $||b_{\tilde{M}}||$ are easily seen to be equivalent.

Let us now consider

$$\Phi = -i\partial\bar{\partial}\log(-\rho) = i\frac{\partial\rho \wedge \bar{\partial}\rho}{\rho^2} + i\frac{\partial\bar{\partial}\rho}{-\rho} \tag{1.4}$$

which is a positive (1, 1) form on Ω and let us consider the positive measure on M^* defined by

$$d\sigma_\Phi = \frac{1}{(n-1)!}[M^*] \wedge \Phi^{n-1} \tag{1.5}$$

$d\sigma_\Phi$ is then the $2n - 2$ volume element on M^* for the Hermitian metric on Ω defined by the fundamental form Φ (but this again, for our purposes, is irrelevant).

Let us also denote by

$$||\tilde{M}||_{\Phi} = \int_{M^*} m(x)\delta^n(x)d\sigma_{\Phi}(x) .$$

We shall then say that $\tilde{M}$ satisfies the Malliavin condition if $||\tilde{M}_{\Phi}|| < +\infty$. We shall also denote in what follows by

$$(1.6) \qquad d\nu = d\nu_{\tilde{M}} = m(x)\delta^n(x)d\sigma_{\Phi}(x)$$

the Malliavin measure of the divisor. The above measure and the above condition were introduced and studied for the first time by P. Malliavin (cf. [12]). It has since been proved that there exists $C > 0$ a positive constant that only depends on Ω such that

$$(1.7) \qquad C^{-1}||\tilde{M}||_{\Phi} \leqq ||\tilde{M}||_{\beta} \leqq C||\tilde{M}||_{\Phi}$$

for all divisor $\tilde{M}$ in Ω (cf. [15], II § 2).

We are now in a position to introduce the main notion of our paper.

DEFINITION. We shall say that a divisor $\tilde{M} \subset \Omega$ satisfies the uniform Blaschke condition (U. B. in short) if the Malliavin measure of the divisor $d\nu_{\tilde{M}}$ is a Carleson measure in Ω.

The above definition is quite general. If we specialize however Ω, to be the unit ball in C^n

$$\Omega = B = \{z \in C^n; ||z||^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}$$

we can give an equivalent definition that does not involve Carleson measures. Towards that let us denote by G the group of all holomorphic automorphisms of B (i.e., all holomorphic injective mappings of B onto B), for all $g \in G$ let us also denote by $\tilde{M}_g$ the image of the divisor $\tilde{M}$ by the mapping $g: z \to g.z$, $\tilde{M}_g$ is then (for all $g \in G$) a new divisor in B. We have then

PROPOSITION 1.1. *Let $\tilde{M}$ be a divisor in B. The following conditions on $\tilde{M}$ are then equivalent*:

(i) *$\tilde{M}$ satisfies the* U.B. *condition.*

(ii) $\sup_{g \in G} ||\tilde{M}_g||_{\beta} < +\infty$.

(iii) $\sup_{g \in G} ||\tilde{M}_g||_{\Phi} < +\infty$.

The fact that (ii) and (iii) are equivalent follows from (1.7), the equivalence with (i) will be proved in the next paragraph.

We are finally in a position to state the main theorem of this paper.

THEOREM 1.1. *Let Ω be a bounded strictly pseudoconvex domain in C^n of class C^4 as above, and let $\tilde{M}$ be a divisor in Ω that satisfies the* U.B. *condition and whose canonical cohomology class in $H^2(\Omega; Z)$ is zero. There exists then $p > 0$ some positive number and $F \in H^p(\Omega)$ a holomorphic function belonging to the Hardy p-class such that $\tilde{M} = F^{-1}(0)$* (i.e., *such that $\tilde{M}$ is exactly the divisor of zeros of F, with multiplicity counted*).

We can further choose our function F as above such that the admissible boundary values of F on $\partial\Omega$ (which exist almost everywhere on $\partial\Omega$) define a function $F^*(x)$ $(x \in \partial\Omega)$ such that

$$\log |F^*(x)| \in \mathrm{BMO}(\partial\Omega) .$$

The definition of the classes $H^p(\Omega)$ is classical (cf. [14]). The definition of BMO is also classical (cf. [4]), for the adaptation of BMO to the boundary of complex domains cf. [18].

In some sense, but in some sense *only*, the above theorem is best possible. Indeed we have

THEOREM 1.2. *For all $p_0 > 0$ there exists $\tilde{M}$ a divisor in $B \subset C^2$ that satisfies the* U.B. *condition and such lhat*

$$F \in H^{p_0}(\Omega);\ F^{-1}(0) \supseteqq \tilde{M} \Longrightarrow F \equiv 0 .$$

2. The uniform Blaschke condition for currents. In this paragraph I shall lay down the basic ground work and I shall state the key real variable theorem needed for the proof of Theorem 1.1. That theorem is, I believe, of some independent interest. For the definition of positive currents and the necessary background that is needed in this paragraph, we shall refer the reader to [10] Ch. VII and also [11].

Let Ω be as in § 1 and let

$$T = \sum_{i,j} T_{ij} dz_i \wedge d\bar{z}_j \tag{2.1}$$

be a (1, 1) current of order zero in Ω. We shall then say that T satisfies the U. B. condition (uniformly Blaschke) if the current

$$\delta T + \delta^{1/2} T \wedge \partial\rho + \delta^{1/2} T \wedge \bar{\partial}\rho + T \wedge \partial\rho \wedge \bar{\partial}\rho$$

is a Carleson current in Ω (ρ and δ are as in (1.2)).

We have then the following:

PROPOSITION 2.1. *Let T be as in* (2.1) *and let us assume that T is a positive current, then T satisfies the* U.B. *cohdition if and* *only if the current*

$$\delta T + T \wedge \partial\rho \wedge \bar{\partial}\rho$$

is a Carleson current.

PROPOSITION 2.2. *Let T be as in* (2.1) *and let us assume that T is positive and closed* (i.e., $dT = 0$), *then T satisfies the* U.B. *condition if and only if the current*

$$T \wedge \partial\rho \wedge \bar{\partial}\rho$$

is a Carleson current.

PROPOSITION 2.3. *Let $\tilde{M} \subset \Omega$ be a divisor in Ω and let t be the Lelong current associated to $\tilde{M}$. Then the divisor $\tilde{M}$ satisfies the* U.B. *condition if and only if the current t satisfies the* U.B. *condition.*

We shall now state the following basic

THEOREM 2.1. *Let T be a* (1, 1) *closed current in Ω satisfying the* U.B. *condition, and let us suppose that the canonical cohomology class of T in $H^2(\Omega; \boldsymbol{Z})$ is zero. There exists then H a current of degree* 1 *and order zero such that*

$$dH = T$$

and such that

$$H + \delta^{-1/2} H_{0,1} \wedge \bar{\partial}\rho$$

is a Carleson current. ($H_{0,1}$ *denotes of course the* (0, 1) *component of H). Furthermore if T is a real current, H can also be chosen real.*

We say of course, in general, that a current S is real if $\bar{S} = S$ where the bar indicates the complex conjugation operator that can be extended, naturally, to the space of currents.

The above theorem is the "power-house" for the rest of this paper, and its proof will be our main task in § 3, § 4 and § 5. For the rest of this paragraph however, I propose to give the proofs of Propositions 2.1, 2.2 and 2.3 as well as of Proposition 1.1.

To simplify notations I shall assume for the rest of this paragraph that $n = 2$ and $\Omega = B$ is the unit ball in $\boldsymbol{C}^2$. The situation is perfectly typical. We have then

$$\rho = ||z||^2 - 1 = |z_1|^2 + |z_2|^2 - 1$$

and

$$\partial\rho = \bar{z}_1 dz_1 + \bar{z}_2 dz_2\,, \quad \bar{\partial}\rho = z_1 d\bar{z}_1 + z_2 d\bar{z}_2\,.$$

We shall also set

$$\omega = z_2 dz_1 - z_1 dz_2\,; \quad \bar{\omega} = \bar{z}_2 d\bar{z}_1 - \bar{z}_1 d\bar{z}_2\,.$$

It is clear then that the four forms

$$\partial\rho\,, \quad \bar{\partial}\rho\,, \quad \omega\,, \quad \bar{\omega} \tag{2.2}$$

are orthogonal and they are of Euclidean norm $\sqrt{2}\,||z||^2$ at every point $z \in B$, they form thus a basis of the cotangent space in some neighborhood of ∂B. We shall in fact use that basis systematically in the rest of this paper.

Let us also define for all $t > 0$

$$C_t = \{z = (z_1, z_2) \in B;\ |z_1 - 1| \leqq t\}$$

which is a neighborhood basis of 1 in B that is equivalent to $\widetilde{B_t(1)}$ in the sense of [18] 2.2 (i.e., we can use C_t and their complex rotations to verify the Carleson condition (1.1)).

LEMMA 2.1. *Let T be as in Proposition* 2.1 *and let us assume in addition that T is C^∞ up to the boundary in B* (i.e., *that the coefficients T_{ij} are C^∞ functions up to the boundary*) *we have then for all* $0 < t < 1/2$

$$\int_{C_t} \delta^{1/2} |T \wedge \partial\rho| = \int_{C_t} \delta^{1/2} |T \wedge \bar{\partial}\rho| \leqq C_0 \left\{ \left(\int_{C_t} \delta\, |T| \right)^{1/2} \cdot \left(\int_{C_t} |T \wedge \bar{\partial}\rho \wedge \partial\rho| \right)^{1/2} + \int_{C_t} \delta^{1/2} |T \wedge \partial\rho \wedge \bar{\partial}\rho| \right\} \tag{2.3}$$

where C_0 is a numerical constant ($C_0 = 10^{10}$ *say, in fact* $C_0 = 10$ *will also do!*).

LEMMA 2.2. *Let T be as in Lemma* 2.1 *and let us assume in addition that $dT = 0$, we have then for all* $0 < t < 1/2$

$$\int_{C_{t/2}} \delta\, |T| \leqq C_0 \left\{ \int_{C_t} |T \wedge \partial\rho \wedge \bar{\partial}\rho| + \left(\int_{C_t} \delta\, |T| \right)^{1/2} \cdot \left(\int_{C_t} |T \wedge \partial\rho \wedge \bar{\partial}\rho| \right)^{1/2} \right\} \tag{2.4}$$

where again C_0 is a numerical constant.

Proofs of the lemmas. If we use the basis (2.2) we can write

$$T = i(S_{11}\partial\rho \wedge \bar{\partial}\rho + S_{12}\partial\rho \wedge \bar{\omega} + S_{21}\omega \wedge \bar{\partial}\rho + S_{22}\omega \wedge \bar{\omega})$$

and by the positivity of T we see that

$$S_{21} = \bar{S}_{12};\ S_{11},\ S_{22} \geqq 0;\ |S_{12}|^2 \leqq S_{11}S_{22} \tag{2.5}$$

at every point of some neighborhood of 1. It is also clear that

$$\begin{cases} T \wedge \partial\rho = i(S_{21}\omega \wedge \bar{\partial}\rho \wedge \partial\rho + S_{22}\omega \wedge \bar{\omega} \wedge \partial\rho) \\ T \wedge \partial\rho \wedge \bar{\partial}\rho = iS_{22}\omega \wedge \bar{\omega} \wedge \partial\rho \wedge \bar{\partial}\rho\ . \end{cases} \tag{2.6}$$

Using then Schwartz's inequality and (2.5) we obtain

$$\int_{C_t} \delta^{1/2} |S_{12}| \leqq \Big(\int_{C_t} \delta\, |S_{11}|\Big)^{1/2} \Big(\int_{C_t} |S_{22}|\Big)^{1/2}\ . \tag{2.7}$$

But (2.6) implies that

$$\begin{aligned} \int_{C_t} \delta^{1/2} |T \wedge \partial\rho| &= \int_{C_t} \delta^{1/2} |T \wedge \bar{\partial}\rho| \\ &\leqq C_0\Big\{\int_{C_t} \delta^{1/2} |S_{21}| + \int_{C_t} \delta^{1/2} |T \wedge \partial\rho \wedge \bar{\partial}\rho|\Big\} \end{aligned}$$

and this together with (2.6) and (2.7) completes the proof of (2.3).

The proof of (2.4) is a little more subtle. We shall immitate the proof in [15] II. 2. Towards that let us fix $0 < t < 1/2$ and let us define $0 \leqq \varphi(z_1) \leqq 1$ a C^∞ function in C^2 that depends only on the variable z_1 and which satisfies

$$\begin{cases} \varphi(z_1) \equiv 1 & \forall z_1 \quad |1 - z_1| \leqq t/2 \\ \varphi(z_1) = 0 & \forall z_1 \quad |1 - z_1| \geqq t \\ |\nabla \varphi| \leqq \dfrac{10}{t} & \forall z_1\ . \end{cases} \tag{2.8}$$

Such a function can clearly be constructed. We shall now apply Stoke's formula to the form

$$\Omega = \rho\varphi\partial\rho \wedge T$$

in the ball B. So that

$$\int_B d\Omega = 0\ . \tag{2.9}$$

We also have

$$d\Omega = \varphi\bar{\partial}\rho \wedge \partial\rho \wedge T + \varphi\rho\bar{\partial}\partial\rho \wedge T + \rho\bar{\partial}\varphi \wedge \partial\rho \wedge T\ ; \tag{2.10}$$

$$\bar{\partial}\varphi = \frac{\partial\varphi}{\partial\bar{z}_1} d\bar{z}_1 = \varphi_1\bar{\partial}\rho + \varphi_2\bar{\omega}\ ,$$

where, as an easy computation shows, we have:

$$|\varphi_1| \leqq 100\,|\nabla\varphi|;\ |\varphi_2| \leqq 100\,|z_2|\,|\nabla\varphi|\ .$$

We conclude therefore that

$$(2.11)\qquad |\rho\bar{\partial}\varphi \wedge \partial\rho \wedge T| \leqq 100\delta\,|\nabla\varphi||S_{22}| + 100\,\delta\,|z_2||\nabla\varphi||S_{12}|\,.$$

Integrating then (2.10) and using (2.9), (1.3) and (2.11) we finally obtain

$$(2.12)\qquad \begin{cases} \displaystyle\int_{C_{t/2}} \delta\,|T| \leqq C_0 \int_B \varphi\,|\rho|\,|T \wedge \bar{\partial}\partial\rho| \\ \displaystyle\qquad \leqq C_0\left\{\int_{C_t} |T\wedge\bar{\partial}\rho\wedge\partial\rho| + \int_{C_t} \delta|\nabla\varphi||S_{22}| + \int_{C_t}\delta|z_2||\nabla\varphi||S_{12}|\right\} \end{cases}$$

and if we use the fact that

$$\delta \leqq t\,;\quad |z_2| \leqq 10\sqrt{t}\quad \forall z \in C_t$$

we obtain by (2.6)(2.8) and (2.7) that:

$$(2.13)\qquad \begin{cases} \displaystyle\int_{C_t} \delta\,|\nabla\varphi|\,|S_{22}| \leqq C_0 \int_{C_t} |T \wedge \partial\rho \wedge \bar{\partial}\rho| \\ \displaystyle\int_{C_t} \delta\,|z_2|\,|\nabla\varphi|\,|S_{12}| \leqq C_0 \int_{C_t} \delta^{1/2}\,|S_{12}| \\ \displaystyle\qquad \leqq \left(\int_{C_t} \delta\,|S_{11}|\right)^{1/2}\left(\int_{C_t} |S_{22}|\right)^{1/2}. \end{cases}$$

From (2.12), (2.13) and (2.6) our inequality (2.4) follows.

We can now give the

Proof of Proposition 2.1. Let T be as in Proposition 2.1, if T is C^∞ up to the boundary our proposition is an immediate consequence of (2.3). If T is not C^∞, by an obvious regularization process we can prove that (2.3) also holds for arbitrary positive currents and our proposition again follows.

By a regularization process, that is perhaps slightly less obvious, because now we have to preserve the d-closure of the form, we can also prove that (2.4) also holds for arbitrary d-closed positive currents, from this we can deduce the

Proof of Proposition 2.2. Let T be as in Proposition 2.2 and let us denote for $0 < t < 1/2$

$$\alpha(t) = \frac{1}{t^2}\int_{C_t} |T \wedge \partial\rho \wedge \bar{\partial}\rho|$$

$$\beta(t) = \frac{1}{t^2}\int_{C_t} \delta\,|T|\,.$$

By multiplying then T by an appropriately small constant and by using Proposition 2.1 of II, [15] we see that we can suppose that

$$\alpha(t)\leqq 10^{-10}(C_0+1)^{-2},\ \forall 0<t<1/2;\ \beta(t)\leqq 1,\ \forall 10^{-10}\leqq t\leqq 1/2\ . \tag{2.14}$$

Inequality (2.4), which is valid for T, as we just pointed out, gives then

$$\beta(t/2)\leqq\frac{1}{10}+\frac{1}{10}(\beta(t))^{1/2}\quad\forall 0<t<1 \tag{2.15}$$

and (2.14) together with (2.15) implies that

$$\beta(t)\leqq 1\quad 0<t<1/2$$

and completes the proof of Proposition 2.2.

Proof of Proposition 2.3. Here again for simplicity I shall assume that $n=2$. A simple computation for the Malliavin measure $d\nu$ of the divisor $\tilde{M}$ [cf. (1.2) (1.4) (1.5) (1.6)] gives

$$d\nu=\frac{\delta^2}{\rho^2}t\wedge(i\partial\rho\wedge\bar{\partial}\rho)+\frac{\delta^2}{|\rho|}t\wedge(i\partial\bar{\partial}\rho)=\nu_1+\nu_2 \tag{2.16}$$

where ν_1 and ν_2 are two positive measures. And using (1.3) we see that there exists some constant C such that in some neighborhood of $\partial\Omega$ we have

$$C^{-1}\tilde{\nu}_i\leqq\nu_i\leqq C\tilde{\nu}_i\qquad i=1,2$$

where

$$\begin{aligned}\tilde{\nu}_1&=|t\wedge\partial\rho\wedge\bar{\partial}\rho|\\ \tilde{\nu}_2&=\delta\,|t|\ .\end{aligned} \tag{2.17}$$

From the above it follows at once that $\tilde{M}$ is a U. B. divisor if and only if $\tilde{\nu}_1+\tilde{\nu}_2$ is a Carleson measure i.e., if and only if t satisfies the U. B. condition.

The geometric meaning of $|t|$ is of course clear, it is just the 2-dimensional Euclidean volume of M^* counted with multiplicity. The geometric meaning of $|t\wedge\partial\rho\wedge\bar{\partial}\rho|$ is just as obvious. It is just the projection of that volume on the complex normal line that passes from each point. It is worth nothing that in view of Proposition 2.2 $\tilde{M}$ satisfies the U. B. condition if and only if the measure $\tilde{\nu}_1$ is a Carleson measure. In § 9 we shall exhibit a divisor $\tilde{M}$ that does not satisfy the U. B. condition but for which nevertheless the corresponding measure $\tilde{\nu}_2$ is Carleson. The above considerations suggest that, contrary to what may appear at first sight, the measure $\tilde{\nu}_1$ is a more significant invariant of the divisor than the measure $\tilde{\nu}_2$.

Let us now observe that when $\Omega=B$ the fundamental form

(1.4) becomes

$$\Phi = -i\partial\bar{\partial}\log(1 - \|z\|^2) . \tag{2.18}$$

So that the induced Hermitian metric is then the Bergman metric of B, cf. [10]. That metric is of course invariant under all the analytic automorphisms of B.

From the above we deduce that if $\tilde{M} \subset B$ is a divisor of B and if $\tilde{M}_g$ $(g \in G)$ is the divisor that we obtain by the action of the automorphism $g \in G$ on M as in Proposition 1.1 than

$$d\sigma^g = \check{g}(d\sigma_\Phi) \tag{2.19}$$

where $d\sigma_\Phi$ and $d\sigma_\Phi^g$ are the measures obtained in (1.5) from the divisors $\tilde{M}$ and $\tilde{M}_g$ respectively, and $\check{g}$ is the mapping on the space of Radon measures on B induced by $g: z \to g.z$. We are now in a position to give the

Proof of Proposition 1.1. For the proof of that proposition I shall use a devise that was suggested to me by John Garnett and which makes the original proof much clearer. We shall need two lemmas.

LEMMA 2.3. (*J. Garnett* [5]). *A positive measure* λ *in* $B \subset C^n$ *is a Carleson measure if and only if*

$$\sup_{z_0 \in B}\int\frac{(1 - \|z_0\|^2)^n}{|1 - z.\bar{z}_0|^{2n}}d\lambda(z) = M(\lambda) < \infty ,$$

and $M(\lambda)$ *defines a norm which is equivalent to the "Carleson norm" (which is implicit in the definition* (1.1)).

LEMMA 2.4. *Let* $g: B \to B$ $(g \in G)$ *be an analytic automorphism of B. We have then*

$$1 - \|g(z)\|^2 = \frac{(1 - \|g^{-1}(0)\|^2)(1 - \|z\|^2)}{|1 - \bar{z}.g^{-1}(0)|^2} \quad \forall z \in B . \tag{2.20}$$

Before giving the proof of these two lemmas we shall complete the proof of Proposition 1.1.

Towards that let $\tilde{M} \subset B$ be a divisor in B. Using then the fact that G acts transitively on B and Lemma 2.3 we see that $\tilde{M}$ is U. B. if and only if

$$\sup_{g \in G}\int\frac{(1 - \|g^{-1}(0)\|^2)^n}{|1 - \bar{z}.g^{-1}(0)|^{2n}}(1 - \|z\|^2)^n m(z) d\sigma_\Phi(z) < +\infty$$

which by Lemma 2.4 is equivalent to the fact that

$$(2.21) \qquad \sup_{g \in G} \int (1 - \|g(z)\|^2)^n m(z) d\sigma(z) < +\infty .$$

If we use the fact that $m^g(g(z)) = m(z)$ where m^g denotes the multiplicity function on $\tilde{M}_g$ and the definition of the mapping $\check{g}$ (of (2.19)) we see that (2.21) is equivalent to

$$(2.22) \qquad \sup_{g \in G} \int (1 - \|z\|^2)^n m^g(z) d\sigma_\Phi^g(z) < +\infty .$$

Here we have of course used (2.19). But that last relation (2.22) is just a reformulation of condition (iii) in Proposition 1.1, and our proof is complete.

Proof of Lemma 2.3. The proof of that lemma for $n = 1$ is contained in [5]. That proof is elementary and it amounts to analyzing the level lines, for $n = 1$, of the function

$$(2.23) \qquad f_n(z) = \left(\frac{1 - \|z_0\|^2}{|1 - z.\bar{z}_0|^2}\right)^n$$

inside the unit disc $D \subset C$. The case $n \geqq 1$ is just as simple. Indeed the function $f_n(z)$ in (2.23) only depends on $z.\bar{z}_0 = u \in C$ and therefore the level surfaces of $f_n(z)$ as $z \in B \subset C^n$ are determined by the level lines of $f_1(z)$ as $z \in D \subset C$. That devise allows us to reduce the general case to the one dimensional one and completes the proof. (The details are left to the reader.)

Proof of Lemma 2.4. The proof is again elementary and is also done by reducing the problem to the one dimensional case, where (2.20) is a very well known identity for Möbius transformations. To show how this reduction is done let us assume for simplicity that $n = 2$, and let $g_1 \in G$ be a general automorphism of B.

Let us then denote by D the complex disc

$$D = \{z g_1^{-1}(0);\ z \in C;\ \|z g_1^{-1}(0)\| < 1\} .$$

There exists then $g_0 \in G$ a particular automorphism of B such that

$$(2.24) \qquad g_0(0) = g_1^{-1}(0);\ g_0(D) \subset D$$

and which act on D as a Möbius transformation. (g_0 determined as above is, modulo complex rotations, in fact unique.) Using the explicit form of g_0 we can also verify directly that Lemma 2.4 holds for $g = g_0^{-1}$.

Let us then denote by $\gamma = g_1 g_0 \in C$. We then have $\gamma(0) = 0$ and this implies that $\gamma \in U(C)$ is a unitary transformation in C^2. It follows therefore from the above that

$$1-\|g_1(z)\|^2=1-\|\gamma g_0^{-1}(z)\|^2=1-\|g_0^{-1}(z)\|^2 = \frac{(1-\|g_0(0)\|^2)(1-\|z\|^2)}{|1-\bar{z}g_0(0)|^2}$$

which together with (2.24) gives (2.20) and proves our lemma.

3. The main construction. To make the central idea come through as clearly as possible and to avoid irrelevant technical complications we shall concentrate, once more, in this paragraph on the unit ball $B = \{\|z\| < 1\} \subset C^2$.

Let us denote by $T_a(\partial B)$ the tangent space of ∂B at $a \in \partial B$, and let us also denote by $n = n(a)$ the inwards unit normal at a. One thing that simplifies matters in $B \subset C^2$ is that the manifold ∂B can be identified canonically with $SU(C; 2)$ which acts on ∂B by complex rotation. That identification induces then a natural parallelization on ∂B. In other words it is possible to choose continuously an orthonormal basis on $T_a(\partial B)$ as a runs through ∂B.

More explicitely let us denote by $1 = (1, 0) \in \partial B$ the north pole of ∂B and let:

$$\begin{aligned} i &= i(1) = (0, 1, 0, 0) \\ j &= j(1) = (0, 0, 1, 0) \\ k &= k(1) = (0, 0, 0, 1) \end{aligned}$$

be the standard basis of $T_1(\partial B)$ for the real coordinates (x_1, y_1, x_2, y_2) of C^2 where $z = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Let now $a \in \partial B$ be given, there exists then a *unique* $g = g_a \in SU(C; 2)$ such that $g1 = a$, we shall set then

$$L_a = \{i(a) = g.i;\ j(a) = g.j;\ k(a) = g.k\}$$

which will then be a basis of $T_a(\partial B)$, that depends smoothly on a.

Let us standardize further some more notations.

For any $z \in B$ $z \neq (0, 0) = 0$ we shall denote by $z^* \in \partial B$ the radial projection of z onto ∂B, i.e., z^* is the unique point on ∂B such that

$$\text{dist}\,(z, z^*) = 1 - \|z\| = \delta(z) = t\,.$$

Let us now fix three real numbers

$$\theta = (\theta_1, \theta_2, \theta_3);\ |\theta_i| \leqq c \quad i = 1, 2, 3\,, \tag{3.1}$$

where c is a small numerical constant ($c = 10^{-10}$ say). For all $z \in B\setminus\{0\}$ we shall then denote by $p_\theta(z)$ the unique point on $T_{z^*}(\partial B)$ whose coordinates with respect to the basis L_{z^*} are

$$p_\theta(z) = (\theta_1 t, \theta_2\sqrt{t}, \theta_3\sqrt{t});\ t = 1 - \|z\|\,. \tag{3.2}$$

If we identify $T_{z^*}(\partial B)$ with a hyperplane in C^2 we can consider $p_\theta(z)$ as a point in C^2. The real line $p_\theta(z) + \lambda n(z^*)$ $\lambda \in \boldsymbol{R}$ intersects then ∂B at two points we shall denote by $\varphi_\theta(z)$ the one that is nearest to $T_{z^*}(\partial B)$.

A more intrinsic definition for $\varphi_\theta(z)$ (and one that is just as good for our construction) would have been to set $\varphi_\theta(z) = \operatorname{Exp}(p_\theta(z))$ where Exp is the exponential mapping from $T_{z^*}(\partial B)$ on ∂B. We defined $\varphi_\theta(z)$ as we did because we can then perform all the explicit computations that will be needed much more easily.

Let us finally denote by $l_\theta(z)$ the directed line segment in C^2

$$l_\theta(z) = \overline{[z, \varphi_\theta(z)]} \quad z \in B\backslash 0 . \tag{3.3}$$

The family of segments $l_\theta(z)(z \in B\backslash 0)$ will now be used to define a smooth homotopy in $B\backslash 0$

$$H_\theta(z, s) \in \bar{B}\backslash 0; \quad z \in \bar{B}\backslash 0 \quad s \in [0, 1]$$

such that $H_\theta(z, 0) = z$, $H_\theta(z, 1) = \varphi_\theta(z)$, $H(z, u) \in B$ $(u < 1)$ and

$$H_\theta(z, s) \in l_\theta(z) \quad \forall z \in B\backslash 0 \quad s \in [0, 1] .$$

That homotopy gives us in fact a retractation of $\bar{B}\backslash 0$ on ∂B that is smooth in the interior.

For each fixed θ the homotopy defined above can then be used in a standard and canonical way to solve the Poincaré equation $d\omega = \Omega$ [cf. [3] specially § 14].

More precisely let us denote by Λ the space of all currents T in B such that $0 \notin \operatorname{supp} T$ we can then define a linear operator (cf. Appendix)

$$H_\theta \colon \Lambda \longrightarrow \Lambda \tag{3.4}$$

(de Rham uses the letter M and M^*) that has the following properties

(i) H_θ is real, i.e.,

$$H_\theta(\bar{T}) = \overline{H_\theta(T)} \quad \forall T \in \Lambda .$$

(ii) H_θ is a homotopy operator for the d-complex

$$H_\theta \circ d(T) + d \circ H_\theta(T) = T; \ \forall T \in \Lambda . \tag{3.5}$$

Furthermore the operator H_θ is continuous for the natural topology on Λ and depends continuously on θ in an obvious manner.

Let us finally denote by

$$H(T) = (2c)^{-3} \int_{-c}^{c} \int_{-c}^{c} \int_{-c}^{c} H_\theta(T) d\theta_1 d\theta_2 d\theta_3 ; \quad \forall T \in \Lambda$$

where the above integral has to be interpreted as a weak integral in the linear topological space Λ (one readily verifies that the convergence of the above integral gives no problems). I shall give no more details on the operator H_θ and their average H. The reader should consult the above reference of de Rham and reconstruct all the details.

We can state now the following basic proposition (the proof will have to wait until the next paragraph).

PROPOSITION 3.1. *Let T be a* (1, 1) *current in B that satisfies the* U.B. *condition. Let us assume that $T \in \Lambda$, and let us denote by $H = H(T)$ and by $K = H_{0,1}$ the* (0, 1) *component of H. The current*

$$H + \delta^{-1/2} K \wedge \bar{\partial}\rho$$

is then a Carleson current. ($\delta = 1 - \|z\|$.)

The above proposition essentially contains Theorem 2.1 in the case $\Omega = B$.

Indeed if we ignore the assumption $0 \notin \operatorname{supp} T$ for the moment we see that for T is as in Theorem 2.1 with $dT = 0$ we have by (3.5) $dH = T$, and if in addition T is real then H is also real.

It is quite clear also that the assumption $0 \notin \operatorname{Supp} T$ is not essential for the above construction, indeed we can replace 0 by any other point, it is enough therefore to assume that $\operatorname{Supp} T \neq B$.

If $\operatorname{Supp} T = B$ we have to use a smooth partition of unity and decompose $T = T_1 + T_2$ where $\operatorname{Supp} T_1$, $\operatorname{Supp} T_2 \neq B$, and then use the procedure that will be developed in § 7 for the general strictly pseudoconvex domains. Observe however that in the applications that we have in mind T is the Lelong current associated with a divisor $\tilde{M}$ in B, and that there the assumption $\operatorname{Supp} T \neq B$ is generously satisfied.

We can generalize the above construction quite easily for $B \subset \boldsymbol{C}^n$ the unit ball in $\boldsymbol{C}^n$ ($n \geqq 2$) and even more generally for any $\Omega \subset \boldsymbol{C}^n$ strictly convex domain in $\boldsymbol{C}^n$ with smooth (say C^4) boundary.

The only new problem here is the smooth choice of an orthonormal basis

$$L_a = \{i, j_1, j_2, \cdots, j_{2n-2}\} \tag{3.6}$$

in $T_a(\partial B)$. Indeed this cannot be done globally in general (except for $n = 1, 2, 4$).

A global choice of i in (3.6) is of course always possible. We simply set $i = Jn(a)$ where $n(a)$ denotes as before the inwards normal and J is the almost complex structure in C^n (i.e., J denotes multiplication by $\sqrt{-1}$ in C^n). What is also possible is to give local definitions of $j_1, j_2, \cdots, j_{2n-2}$ to complement Jn in $T_a(\partial B)$.

More precisely for every point $a \in \partial\Omega$ we can find Θ an open neighborhood of a in $\partial\Omega$ in which a smooth choice of a basis L_b $(b \in \Theta)$ is possible. We can then construct as above an operator

$$H_\Theta : \Lambda_\Theta \longrightarrow \Lambda_\Theta$$

where

$$\Lambda_\Theta = \{T; 0 \notin \operatorname{Supp} T; z \in \operatorname{Supp} T \Longrightarrow z^* \in \Theta\}$$

such that

$$H_\Theta(\bar{T}) = \overline{H_\Theta(T)};\ H_\Theta \circ d(T) + d \circ H_\Theta(T) = T;\ \forall T \in \Lambda_\Theta$$

and such that H_Θ satisfies the conditions of Proposition 3.1.

The problem is then to globalize the above construction. But that is exactly the problem that will be faced and solved in § 7 where the general strictly pseudoconvex domains will be examined.

4. The geometry of currents that satisfy the uniform Blaschke condition. The aim of this paragraph is to give a proof of Proposition 3.1. Once more I shall work entirely with the unit ball $\Omega = B \subset C^2$, and I shall preserve all the notations already introduced.

Let $\rho = ||z||^2 - 1$ and let $\partial\rho, \bar{\partial}\rho, \omega, \bar{\omega}$ be the orthogonal basis of the cotangent space of $B\backslash 0$ introduced in (2.2).

For every $z \in \Omega$ let us also denote by

$$(4.1)\qquad \begin{cases} \alpha = \alpha(z) = \delta_z \partial\rho \wedge \bar{\partial}\rho;\ \beta_1 = \beta_1(z) = \delta_z \partial\rho \wedge \bar{\omega} \\ \beta_2 = \beta_2(z) = \delta_z \omega \wedge \bar{\partial}\rho;\ \gamma = \gamma(z) = \delta_z \omega \wedge \bar{\omega} \end{cases}$$

where δ_z denotes the Dirac δ-mass at the point $z \in \Omega$. α, β_1, β_2, and γ are then currents of order zero supported by the point z. The above currents generate in an obvious way all the (1, 1) currents of order zero in the space Λ (Λ was defined just before equation (3.4)).

Indeed let

$$T = a\partial\rho \wedge \bar{\partial}\rho + b_1\partial\rho \wedge \bar{\omega} + b_2\omega \wedge \bar{\partial}\rho + c\omega \wedge \bar{\omega}$$

be such a current and let us identify a, b_1, b_2 and c with Radon measures in Ω. We can then write T in the form

$$T = \int_{\Omega} \alpha(z)da(z) + \int_{\Omega} \beta_1(z)db_1(z) + \int_{\Omega} \beta_2(z)db_2(z) + \int_{\Omega} \gamma(z)\,dc(z) \tag{4.2}$$

where the above integrals are weak integrals in the space of currents of order zero (which is a dual space). Furthermore the above decomposition is obviously unique. We have then

PROPOSITION 4.1. *Let T be a* (1, 1) *current of order zero in Λ and let a, b_1, b_2, c be the Radon measures associated to the decomposition* (4.2). *Then the current T satisfies the* U.B. *condition if and only if the measure*

$$\tau = |c| + \delta^{1/2}(|b_1| + |b_2|) + \delta\,|a|$$

is a Carleson measure in B.

This is, of course, just an alternative formulation of the definition. We shall introduce now some more notations. Let θ be as in (3.1) and let us denote by

$$\alpha^\theta = H_\theta(\alpha);\ \beta_i^\theta = H_\theta(\beta_i),\ i = 1, 2;\ \gamma^\theta = H_\theta(\gamma)$$
$$p^\theta = (\alpha^\theta)_{0,1};\ q_i^\theta = (\beta_i^\theta)_{0,1},\ i = 1, 2;\ r^\theta = (\gamma^\theta)_{0,1}\,.$$

All these are currents of order zero. α^θ, β_i^θ and γ^θ are obtained from α, β_i and γ by the homotopy operator H_θ and p^θ, q_i^θ, r^θ are the (0, 1) components of the currents α^θ, β_i^θ, γ^θ. All these currents depend of course on the point z on which α, β, γ are supported (so a more comprehensive notation would have been $\alpha^\theta(z) = H_\theta(\alpha(z))$ etc . . .).

Let us finally denote by

$$R_h = \{z \in B;\ ||z|| > 1 - h\} \qquad h > 0 \tag{4.3}$$

and by χ_{R_h} its characteristic function. We have then

PROPOSITION 4.2. *For all $\varepsilon < 1$ there exists a constant $C = C_\varepsilon$ that depends only on ε such that for all $z \in R_\varepsilon$ we have the following estimates*

$$\begin{cases} ||\alpha^\theta|| \leqq Ct \\ ||\chi_{R_h}\alpha^\theta|| \leqq C\dfrac{h}{t}t \\ ||\delta^{-1/2}p^\theta \wedge \bar\partial\rho|| \leqq Ct \\ ||\chi_{R_h}\delta^{-1/2}p^\theta \wedge \bar\partial\rho|| \leqq C\sqrt{\dfrac{h}{t}}t \end{cases} \tag{4.4}$$

$$(4.5)\qquad \begin{cases} \|\beta_i^\theta\| \leqq Ct^{1/2} \\ \|\chi_{R_h}\beta_i^\theta\| \leqq C\dfrac{h}{t}t^{1/2} \\ \|\delta^{-1/2}q_i^\theta \wedge \bar\partial\rho\| \leqq Ct^{1/2} \\ \|\chi_{R_h}\delta^{-1/2}q_i^\theta \wedge \bar\partial\rho\| \leqq C\sqrt{\dfrac{h}{t}}t^{1/2} \end{cases} \qquad \textit{for } i = 1, 2$$

$$(4.6)\qquad \begin{cases} \|\gamma^\theta\| \leqq C \\ \|\chi_{R_h}\gamma^\theta\| \leqq C\dfrac{h}{t} \\ \|\delta^{-1/2}r^\theta \wedge \bar\partial\rho\| \leqq C \\ \|\chi_{R_h}\delta^{-1/2}r^\theta \wedge \bar\partial\rho\| \leqq C\sqrt{\dfrac{h}{t}} \end{cases}$$

where $t = 1 - \|z\| = \delta(z)$.

The above proposition contains the essential set of estimates on which the proof of Proposition 3.1 will be based. The proof is elementary but very lengthy. It will be postpone until the next paragraph.

Let us now define

$$(4.7)\qquad A = H(\alpha) = (2c)^{-3}\int_{-c}^{c}\int_{-c}^{c}\int_{-c}^{c}\alpha^\theta d\theta_1 d\theta_2 d\theta_3 .$$

A which is then a current of order zero that depends on z. We denote analogously $B_i = H(\beta_i)$ $i = 1, 2$; $C = H(\gamma)$. We also denote by P, $Q_i (i = 1, 2)$ and R the (0, 1) components of A, $B_i (i = 1, 2)$ and C respectively. An integral representation of the form (4.7) holds then for all these currents with α^θ replaced by β_i^θ, γ^θ, p^θ, q_i^θ and r^θ as the case may be. We have then

PROPOSITION 4.3. *For all* $\varepsilon < 1$ *there exists* $C = C_\varepsilon$ *a constant that depends only on* ε *such that for all* $z \in R_\varepsilon$ *we have*

$$\begin{cases} \|A\| \leqq C_\varepsilon t \\ \|\delta^{-1/2}P \wedge \bar\partial\rho\| \leqq C_\varepsilon t \end{cases}$$

$$\begin{cases} \|B_i\| \leqq C_\varepsilon t^{1/2} \\ \|\delta^{-1/2}Q_i \wedge \bar\partial\rho\| \leqq C_\varepsilon t^{1/2} \end{cases} \quad i = 1, 2$$

$$\begin{cases} \|C\| \leqq C_\varepsilon \\ \|\delta^{-1/2}R \wedge \bar\partial\rho\| \leqq C_\varepsilon \end{cases}$$

where $t = 1 - \|z\| = \delta(z)$.

Proof. The above estimates are obtained immediately by integrating over θ the corresponding estimates in Proposition 4.2.

Let now $\widetilde{B_h(1)}$ be the region in (1.1) that defines the Carleson condition centered at the point 1, and for any $z \in B$ let us denote by $h(z) = \inf\{h > 0; z \in \widetilde{B_h(1)}\}$. With $l_\theta(z)$ defined as in (3.3) we have the following

LEMMA 4.1. *Let $h > 0$ and $z \in B\backslash\{0 \cup \widetilde{B_{ch}(1)}\}$ suppose that for some $\theta, |\theta| < c^{-1}$, we have $l_\theta(z) \cap \widetilde{B_h(1)} \neq \phi$. Then $t = 1 - ||z|| \geqq c^{-1}h(z)$. c denotes here a large numerical constant* ($c = 10^{10}$ say).

LEMMA 4.2. *Let h and z be as in Lemma* 4.1 *and let us denote by*

$$X_z = \{\theta = (\theta_1, \theta_2, \theta_3)\, ; l_\theta(z) \cap \widetilde{B_h(1)} \neq \phi\} \subset \boldsymbol{R}^3 \,. \tag{4.8}$$

The three dimensional measure of X_z satisfies then

$$|X_z|_3 \leqq C\left(\frac{h}{t}\right)^2; \quad |X_z|_3 \leqq C\left(\frac{h}{t}\right)^{3/2}$$

where C is a numerical constant.

Proof of Lemma 4.1. The fact that $l_\theta(z) \cap \widetilde{B_h(1)} \neq \phi$ for some $\theta\ |\theta| < c^{-1}$ implies clearly that $\widetilde{B_{c_1t}(z^*)} \cap \widetilde{B_h(1)} \neq \phi$ for some constant c_1 ($c_1 = 10^5$ say, $z^* = z/||z||$), but this implies that

$$B_{c_1t}(z^*) \cap B_h(1) \neq \phi \,. \tag{4.9}$$

Our hypothesis on z on the other hand implies that

$$t = 1 - ||z|| \geqq ch \text{ or } z^* \notin B_{ch}(1) \tag{4.10}$$

or both. In either case we conclude from (4.9) and (4.10) that $1 \in B_{ct}(z^*)$. But then we are done. Indeed

$$1 \in B_{ct}(z^*) \Longleftrightarrow z^* \in B_{ct}(1) \Longrightarrow z \in \widetilde{B_{ct}(1)} \Longrightarrow t > c^{-1}h(z) \,.$$

Proof of the Lemma 4.2. Let z be as in the lemma. By changing the role of z^* and 1 we see that the set X_z is obtained by a rotation on ∂B of the set

$$Y_t = \{\theta \in \boldsymbol{R}^3; l_\theta(\xi) \cap \widetilde{B_h(\zeta_0)} \neq 0\}$$

where $\xi = (1 - t, 0)$, $0 < t = 1 - ||z|| < 1$, and where ζ_0 is some fixed point on ∂B. We also have that

$$Y_t \subset \{\theta \in \boldsymbol{R}^3; \varphi_\theta(\xi) \in B_{ch}(\zeta_0)\} = Z_t \tag{4.11}$$

where c is some large numerical constant and $\varphi_\theta(\xi)$ is the point defined in § 3 (it is the end point of $l_\theta(\xi)$). The verification of (4.11) will be left for the reader (observe that verification becomes much easier for the Siegel upper half space).

We have now

$$Z_t = \{\theta = (\theta_1, \theta_2, \theta_3);\ p_\theta(\xi) \in S_h(\zeta_0)\}$$

where $S_h(\zeta_0)$ is the region in $T_1(\partial B)$ obtained by projecting $B_{ch}(1)$ along the $n(1)$ direction on the tangent plane $T_1(\partial B)$.

The 3-dimensional volume of that region clearly satisfies

$$|S_h(\zeta_0)|_3 \leqq |B_{ch}(\zeta_0)|_3 \leqq Ch^2$$

and since the point $p_\theta(\xi)$ is given on $T_1(\partial B)$ by the coordinates

$$p_\theta(\xi) = (\theta_1 t, \theta_2\sqrt{t}, \theta_3\sqrt{t}),$$

we deduce that $|X_z|_3 = |Y_t|_3 \leqq |Z_t|_3 \leqq C(h/t)^2$.

But from Lemma 4.1 we also deduce that $X_z = \phi$ unless $t > C^{-1}h(z) \geqq h$ and from that last estimate it readily follows that

$$|X_z|_3 \leqq C\left(\frac{h}{t}\right)^{3/2}$$

and the proof of the lemma is complete.

Let us now denote by χ_h the characteristic function of the set $\widetilde{B_h}(1)$. We can state then the following

PROPOSITION 4.4. *For all* $\varepsilon < 1$ *there exists* $C = C_\varepsilon > 0$ *such that for all* $h > 0$ *and all* $z \in R_\varepsilon \backslash \widetilde{B_{ch}}(1)$ ($c = 10^{10}$ *say*) *we have*

$$\text{(4.12)} \qquad \begin{cases} \|\chi_h A\| \leqq C_\varepsilon\left(\dfrac{h}{t}\right)^{5/2} t \\ \|\chi_h \delta^{-1/2} P \wedge \bar\partial\rho\| \leqq C_\varepsilon\left(\dfrac{h}{t}\right)^{5/2} t \end{cases}$$

$$\text{(4.13)} \qquad \begin{cases} \|\chi_h B_i\| \leqq C_\varepsilon\left(\dfrac{h}{t}\right)^{5/2} t^{1/2} \\ \|\chi_h \delta^{-1/2} Q_i \wedge \bar\partial\rho\| \leqq C_\varepsilon\left(\dfrac{h}{t}\right)^{5/2} t^{1/2} \end{cases} \qquad i = 1, 2$$

$$\text{(4.14)} \qquad \begin{cases} \|\chi_h C\| \leqq C_\varepsilon\left(\dfrac{h}{t}\right)^{5/2} \\ \|\chi_h \delta^{-1/2} R \wedge \bar\partial\rho\| \leqq C_\varepsilon\left(\dfrac{h}{t}\right)^{5/2} \end{cases}$$

where $t = 1 - \|z\| = \delta(z)$ and $i = 1, 2$.

Proof. We have

$$\|\chi_h A\| \leqq \int_{X_z} \|\chi_{R_h}\alpha^\theta(z)\|\, d\theta_1 d\theta_2 d\theta_3$$
$$\leqq \sup_z \|\chi_{R_h}\alpha^\theta(z)\|\, |X_z|_3 \leqq C\frac{h}{t}\, t\left(\frac{h}{t}\right)^{3/2} = C\left(\frac{h}{t}\right)^{5/2} t$$

by (4.4) and by Lemma 4.2.

Similarly

$$\|\chi_h \delta^{-1/2} P \wedge \bar\partial\rho\| \leqq \int_{X_z} \|\chi_{R_h}\delta^{-1/2} p^\theta \wedge \bar\partial\rho\|\, d\theta_1 d\theta_2 d\theta_3$$
$$\leqq \sup_z \|\chi_{R_h}\delta^{-1/2} p^\theta \wedge \bar\partial\rho\| .\, |X_z|_3 \leqq C\sqrt{\frac{h}{t}}\, t.\left(\frac{h}{t}\right)^2 \leqq C\left(\frac{h}{t}\right)^{5/2} t$$

again by (4.4) and the Lemma 4.2. This proves (4.12).

The other two estimates (4.13) and (4.14) are obtained analogously. We are finally in a position to complete the

Proof of Proposition 3.1. Let T be as in Proposition 3.1 and let a, b_1, b_2 and c the Radon measures obtained by the decomposition (4.2). Let us denote by $H = H(T)$ and by $K = (H(T))_{0,1}$ the $(0, 1)$ component of H. We have then

$$H = \int_\Omega A(z)da(z) + \int_\Omega B_1(z)db_1(z) + \int_\Omega B_2(z)db_2(z) + \int_\Omega C(z)dc(z)$$
$$K = \int_\Omega P(z)da(z) + \int_\Omega Q_1(z)db_1(z) + \int_\Omega Q_2(z)db_2(z) + \int_\Omega R(z)dc(z)\,.$$

If we denote by

$$\tau = |c| + \delta^{1/2}(|b_1| + |b_2|) + \delta|a|$$

which is a Carleson measure in Ω (by Proposition 4.1) and by

$$\mathfrak{S}(z) = |C(z)| + |\delta^{-1/2}R(z) \wedge \bar\partial\rho|$$
$$+ \frac{1}{\sqrt{\delta(z)}}\sum_{i=1}^{2}(|B_i(z)| + |\delta^{-1/2}Q_i(z) \wedge \bar\partial\rho|)$$
$$+ \frac{1}{\delta(z)}(|A(z)| + |\delta^{-1/2}P(z) \wedge \bar\partial\rho|)$$

which is a measure that depends on z. We have the following estimates

$$\mathfrak{X} = |H| + |\delta^{-1/2}K \wedge \bar\partial\rho| \leqq C\int_\Omega \mathfrak{S}(z)d\tau(z) \tag{4.15}$$

where C is a positive constant and the above inequality refers of course to the order relation in the space of positive measures.

From Proposition 4.3 we also obtain that

$$\|\mathfrak{S}(z)\| \leqq C \quad \forall z \in \operatorname{supp} T \tag{4.16}$$

and from Proposition 4.4 we obtain that

$$\|\chi_h \mathfrak{S}(z)\| \leqq C\left(\frac{h}{t}\right)^{5/2} = C\left(\frac{h}{\delta(z)}\right)^{5/2}; \quad \forall z \in \operatorname{supp} T \backslash \widetilde{B_{ch}(1)} . \tag{4.17}$$

From the above three relations (4.15) (4.16) and (4.17) we shall conclude that

$$\mathfrak{X}(\widetilde{B_h(1)}) = \|\chi_h \mathfrak{X}\| \leqq Ch^2 \tag{4.18}$$

for some constant C independent of h, and that, of course, will complete the proof of Proposition 3.1 since by an obvious rotation we can bring any test set $\widetilde{B_h(\zeta_0)}$ to the position $\widetilde{B_h(1)}$.

Towards that let us observe that by (4.15) we have

$$\|\chi_h \mathfrak{X}\| \leqq \mathfrak{A} + \mathfrak{B}$$

where

$$\mathfrak{A} = \int_{z \in \widetilde{B_{ch}(1)}} \|\mathfrak{S}(z)\| \, d\tau(z)$$

$$\mathfrak{B} = \int_{z \notin \widetilde{B_{ch}(1)}} \|\chi_h \mathfrak{S}(z)\| \, d\tau(z)$$

where c is an appropriately chosen large constant.

But by (4.16) we clearly have

$$\mathfrak{A} \leqq \sup_z \|\mathfrak{S}(z)\| \, \tau(\widetilde{B_{ch}(1)}) \leqq Ch^2 \tag{4.19}$$

because τ is a Carleson measure.

Using Lemma 4.1 and (4.17) we also obtain that

$$\mathfrak{B} \leqq C \int_{\delta(z) \geqq c_1 h(z) \geqq c_2 h} \left(\frac{h}{\delta(z)}\right)^{5/2} d\tau(z) \tag{4.20}$$

where c_1 and c_2 are appropriately chosen constants. If we then denote by

$$F(\lambda) = \tau(\widetilde{B_\lambda(1)})$$

which by the hypothesis on τ satisfies

$$F(\lambda) \leqq C\lambda^2 \tag{4.21}$$

and if we use the definition of $h(z)$ (cf. Lemma 4.1) in (4.20) we obtain that

$$(4.22) \qquad \mathfrak{B} \leqq C\int_{c_3 h}^{\infty}\left(\frac{h}{\lambda}\right)^{5/2} dF(\lambda) .$$

And by an easy integration by parts in (4.22) we obtain using (4.21) that

$$(4.23) \qquad \mathfrak{B} \leqq Ch^2 .$$

The two estimates (4.19) and (4.23) give then at once the required estimate (4.18) and complete the proof of our proposition.

As a final remark observe that the only reason that we insisted that $T \in A$ (i.e., that $0 \notin \operatorname{supp} T$) was that we had to get rid of the ε dependence of the constants of Proposition 4.4 for $z \in R_\varepsilon$.

5. Geometric lemmas. The aim of this paragraph is to give the proof of Proposition 4.2.

In the course of this paragraph we shall use the parameters $\theta_1, \theta_2, \theta_3$ introduced in (3.1) and we shall need to introduce a number of new parameters, namely $\xi \in B\backslash 0$ and $t = 1 - ||\xi||$ in (5.1) and $\delta_1, \delta_2, \delta_3, \delta_4$; λ, μ, ν in (5.2) (5.5) and (5.8). Concerning the above parameters and to avoid constant repetition I shall make once and for all the following convention:

I shall say that κ (resp. $\tilde{\kappa}$) is an admissible (resp. weakly admissible) constant if it does not depend on λ, μ and ν but may well depend on the other parameters

$$\kappa = \kappa(\xi, \theta, \delta) \qquad \tilde{\kappa} = \tilde{\kappa}(\xi, \theta, \delta)$$

and if furthermore for all $\varepsilon > 0$ we have

$$\sup\{|\kappa|;\ t < 1 - \varepsilon;\ \theta, \delta\} < +\infty$$
$$(\text{resp. } \sup\{|\tilde{\kappa}|;\ \theta, \delta\} = C(t) < +\infty)$$

i.e., κ stays bounded when ξ stays away from 0.

We shall say similarily that $\psi(\lambda, \mu)$ (resp. $\tilde{\psi}(\lambda, \mu)$) is an admissible function (resp. weakly admissible function) if it is a function of λ and μ that may also depend on the parameters ξ, t, θ, δ but not on ν and for which

$$||\psi||_\infty + ||\nabla_{\lambda,\mu}\psi||_\infty$$
$$(\text{resp. } ||\tilde{\psi}||_\infty + ||\nabla_{\lambda,\mu}\tilde{\psi}||_\infty)$$

is an admissible (resp. weakly admissible constant).

The letters $\kappa, \tilde{\kappa}, \psi, \tilde{\psi}$ (possibly with suffixes) will be reserved *exclusively and without any further notice* for the above creatures. The letter θ (possibly with suffixes) will denote quantities that do not depend on λ, μ, and ν and that stay bounded for *all values* of the other parameters. Finally when the above letters appear several times in a formula it will not in general mean that they represent the same function or the same constant.

We shall now proceed to give a series of geometric lemmas. The situation is clearly rotation invariant so we can suppose once and for all that our point $z \in B\backslash 0$ for which we shall estimate $\alpha(z)$ etc... lies on the x_1 axis. To avoid possible confusion with later considerations we shall denote such a point by

$$\xi = (1 - t, 0) \quad 0 < t < 1 . \tag{5.1}$$

We shall also systematically use in C^2 the coordinates

$$(x_1, y_1, x_2, y_2);\ z_1 = x_1 + iy_1;\ z_2 = x_2 + iy_2 .$$

The first thing to observe is that

$$\varphi_\theta(\xi) = (1 - \theta_0 t, \theta_1 t, \theta_2\sqrt{t}, \theta_3\sqrt{t})$$

and that therefore the coordinates of the vector $\overrightarrow{l_\theta(\xi)}$ are given by

$$\overrightarrow{l_\theta(\xi)} = ((1 - \theta_0)t, \theta_1 t, \theta_2\sqrt{t}, \theta_3\sqrt{t}) .$$

We shall parametrize $\overrightarrow{l_\theta(z)}$ linearly by a parameter $\nu \in [0, 1]$ so that the coordinates of the general point on $l_\theta(\xi)$ are given by

$$\begin{aligned} x_1 &= 1 - t + (1 - \theta_0)t\nu;\ y_1 = \theta_1 t\nu; \\ x_2 &= \theta_2\sqrt{t}\,\nu;\ y_2 = \theta_3\sqrt{t}\,\nu \quad 0 \leqq \nu \leqq 1 . \end{aligned} \tag{5.2}$$

We have then

LEMMA 5.1. *For all ξ and θ as above and all $z \in l_\theta(\xi)$ parametrized as above we have*

$$1 - \|z\| = \delta(z) \geqq (1 - t)t(1 - \nu) = \kappa^{-1}t(1 - \nu) . \tag{5.3}$$

Therefore if $z \in l_\theta(\xi) \cap R_h$ [cf. (4.3)] *we have*

$$1 - \nu \leqq \frac{1}{1 - t}\,\frac{h}{t} = \kappa\frac{h}{t} . \tag{5.4}$$

Proof.

(5.4) is of course an immediate consequence of (5.3). The proof of (5.3) is done by elementary geometry and it involves drawing a picture. The reader has to do that for himself.

Let us now fix small real numbers $\delta_1, \delta_2, \delta_3, \delta_4$ and consider the point:

(5.5) $\qquad \xi(\lambda, \mu) = (1 - t + \delta_1\lambda, \delta_2\mu, \delta_3\lambda, \delta_4\mu) \in C^2;\ 0 \leqq \lambda,\ \mu \leqq 1$

which is the generic point on a small rectangle Δ that is spanned by the two vectors

$$(\delta_1, 0, \delta_3, 0); \quad (0, \delta_2, 0, \delta_4)$$

that are translated so that the origin comes to the point ξ.

We shall denote by

$$\delta^2 = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2$$

and we shall prove a series of lemmas concerning 2nd order estimates with respect to δ.

LEMMA 5.2. *For all* $0 < t < 1$ *there exists* $\delta(t) > 0$ *such that*

$$\begin{aligned} &||\xi(\lambda, \mu)|| = ||\xi|| + \delta_1\lambda + \delta^2\psi(\lambda, \mu) \\ &t(\lambda, \mu) = 1 - ||\xi(\lambda, \mu)|| = t - \delta_1\lambda + \delta^2\psi(\lambda, \mu) \\ &\sqrt{t(\lambda, \mu)} = \sqrt{t} - \frac{\delta_1\lambda}{\sqrt{t}} + \delta^2\tilde{\psi}(\lambda, \mu) \end{aligned}$$

for all $\delta < \delta(t)$.

LEMMA 5.3. *The coordinates of* $\xi^*(\lambda, \mu) = \xi(\lambda, \mu)/||\xi(\lambda, \mu)||$ *satisfy*

$$\begin{aligned} x_1 &= 1 + \delta^2\psi(\lambda, \mu) \\ y_1 &= \kappa\delta_2\mu + \delta^2\psi(\lambda, \mu) \\ x_2 &= \delta\psi(\lambda, \mu) \\ y_2 &= \delta\psi(\lambda, \mu) \end{aligned}$$

$$\alpha = x_1 + iy_1 = 1 + \kappa\delta_2\mu + \delta^2\psi(\lambda, \mu); \ \beta = x_2 + iy_2 = \delta\psi(\lambda, \mu)$$

for all $\delta < \delta(t)$, *where* $\delta(t)$ *is as in Lemma* 5.2.

Let us now denote by

$$b_1(\lambda, \mu) = \theta_1 t(\lambda, \mu); \ a_2(\lambda, \mu) = \theta_2\sqrt{t(\lambda, \mu)}; \ b_2(\lambda, \mu) = \theta_3\sqrt{t(\lambda, \mu)}$$

where θ_1, θ_2 and θ_3 are as in (3.1). Let us also determine $a_1(\lambda, \mu) \geqq 0$ such that the point

$$X(\lambda, \mu) = (1 - a_1(\lambda, \mu), b_1(\lambda, \mu), a_2(\lambda, \mu), b_2(\lambda, \mu))$$

lies on ∂B and in some small neigaborhood of the point 1. $a_1(\lambda, \mu)$ is then uniquely determined by the equation

$$a_1^2 - 2a_1 + \theta_1^2 t^2(\lambda, \mu) + (\theta_2^2 + \theta_3^2)t(\lambda, \mu) = 0$$

and an easy computation involving power series gives

LEMMA 5.4.

$$a_1(\lambda, \mu) = \theta_4 t + \kappa\delta_1\lambda + \delta^2\psi(\lambda, \mu)$$

for all $\delta < \delta(t)$ ($\delta(t)$ *as in Lemma* 5.2).

If we denote by

$$X(\lambda, \mu) = (Z_1(\lambda, \mu), Z_2(\lambda, \mu))$$

the complex coordinates of the point $X(\lambda, \mu)$ we then have

$$Z_1(\lambda, \mu) = 1 + \theta t + \kappa\delta_1\lambda + \delta^2\psi(\lambda, \mu)$$
$$Z_2(\lambda, \mu) = \theta\sqrt{t} + \kappa\frac{\delta_1\lambda}{\sqrt{t}} + \delta^2\tilde{\psi}(\lambda, \mu) .$$

Let us now denote by U the unique complex rotation (special unitary transformation) that brings the point 1 to the point $\xi^*(\lambda, \mu) = (\alpha, \beta)$. By the definition of $\varphi_\theta(\xi)$ it is then clear that

$$\varphi_\theta(\xi(\lambda, \mu)) = U.X(\lambda, \mu) ;$$

as for U, it is given in matrix form by

$$U = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

where α and β are as in Lemma 5.3.

We conclude therefore that if the complex coordinates of $\varphi_\theta(\xi(\lambda, \mu))$ are given by

$$\varphi_\theta(\xi(\lambda, \mu)) = (U_1, U_2)$$

then

$$(5.6) \qquad \begin{cases} U_1(\lambda, \mu) = U_1 = \alpha Z_1 - \bar{\beta} Z_2 \\ U_2(\lambda, \mu) = U_2 = \beta Z_1 + \bar{\alpha} Z_2 \end{cases}$$

and to the first order we have

$$(5.7) \qquad \begin{cases} U_1(\lambda, \mu) = U_1(0, 0) + \delta\tilde{\psi}(\lambda, \mu) = 1 + \theta t + \delta\tilde{\psi}(\lambda, \mu) \\ U_2(\lambda, \mu) = U_2(0, 0) + \delta\tilde{\psi}(\lambda, \mu) = \theta\sqrt{t} + \delta\tilde{\psi}(\lambda, \mu) . \end{cases}$$

Taking differentials in (5.6) with respect to λ and μ we obtain

LEMMA 5.5.

$$dU_1 = \kappa\delta_1 d\lambda + \kappa\delta_2 d\mu + \delta\sqrt{t}\, d[\psi(\lambda, \mu)] + \delta^2 d[\tilde{\psi}(\lambda, \mu)]$$
$$dU_2 = \delta d[\psi(\lambda, \mu)] + \kappa\frac{\delta_1}{\sqrt{t}} d\lambda + \delta^2 d[\tilde{\psi}(\lambda, \mu)] .$$

Let us now consider the 3-dimensional chain element (in the sense of G. de Rham [3] § 6; except that here our chain element is not compact in B) that is obtained by the following parametrization

$$(5.8)\qquad \begin{cases} W_1 = (1 - t + \delta_1\lambda + i\delta_2\mu)(1 - \nu) + U_1(\lambda, \mu)\nu \\ W_2 = (\delta_3\lambda + i\delta_4\mu)(1 - \nu) + U_2(\lambda, \mu)\nu \end{cases}$$

where $0 \leqq \lambda, \mu \leqq 1$ $0 \leqq \nu < 1$, and where W_1 W_2 are the complex coordinates in C^2 of the generic point on that chain element.

To the first order we have by (5.7)

$$(5.9)\qquad \begin{cases} W_1 = (1 - t)(1 - \nu) + U_1(0, 0)\nu + \delta[\tilde{\psi}(\lambda, \mu) + \nu\tilde{\psi}(\lambda, \mu)] \\ W_2 = U_2(0, 0)\nu + \delta[\tilde{\psi}(\lambda, \mu) + \nu\tilde{\psi}(\lambda, \mu)] . \end{cases}$$

The support of that chain is of course the set

$$\bigcup_{z \in \Delta} l_\theta(z)$$

and the integration current on that chain is just the current $H_\theta([\Delta])$ i.e., the current that we obtain from the integration current $[\Delta]$ on Δ by applying the homotopy operator H_θ. (Cf. Appendix.)

Differentiating the equations (5.8) in λ, μ and ν and using Lemma 5.5 we obtain the following expression on the differentials on that chain element.

$$(5.10)\qquad dW_1 = a\delta_1 d\lambda + a\delta_2 d\mu + \delta\sqrt{t}\,D(\lambda, \mu) + \theta t d\nu + R$$

$$(5.11)\qquad \sqrt{t}\,dW_2 = a\delta_1 d\lambda + a\delta_2 d\mu + \delta\sqrt{t}\,D(\lambda, \mu) + \theta t d\nu + R$$

where a, $D(\lambda, \mu)$ and R are of the form

$$\begin{aligned} a &= \kappa + \nu\kappa \\ D(\lambda, \mu) &= d[\psi(\lambda, \mu)] + \nu d[\psi(\lambda, \mu)] \\ R &= \delta\tilde{\psi}(\lambda, \mu)d\nu + \delta^2 d[\tilde{\psi}(\lambda, \mu)\nu] . \end{aligned}$$

We conclude that:

$$\begin{aligned} dW_1 &= X_1 d\lambda + X_2 d\mu + X_3 d\nu \\ dW_2 &= Y_1 d\lambda + Y_2 d\mu + Y_3 d\nu \end{aligned}$$

where

$$(5.12)\qquad \begin{cases} \|X_1\|_\infty \leqq \kappa(\delta_1 + \delta\sqrt{t}) + \tilde{\kappa}\delta^2 \\ \|X_2\|_\infty \leqq \kappa(\delta_2 + \delta\sqrt{t}) + \tilde{\kappa}\delta^2 \\ \|X_3\|_\infty \leqq \kappa t \qquad\qquad\quad + \tilde{\kappa}\delta \end{cases}$$

$$(5.13)\quad \begin{cases} \|Y_1\|_\infty \leqq \dfrac{\kappa}{\sqrt{t}}(\delta_1 + \delta\sqrt{t}) + \tilde{\kappa}\delta^2 \\ \|Y_2\|_\infty \leqq \dfrac{\kappa}{\sqrt{t}}(\delta_2 + \delta\sqrt{t}) + \tilde{\kappa}\delta^2 \\ \|Y_3\|_\infty \leqq \kappa\sqrt{t} \qquad\qquad + \tilde{\kappa}\delta\,. \end{cases}$$

Let us finally set

$$dW_1 \wedge d\bar{W}_1 \wedge dW_2 = Ad\lambda \wedge d\mu \wedge d\nu$$
$$dW_2 \wedge d\bar{W}_2 \wedge dW_1 = Bd\lambda \wedge d\mu \wedge d\nu$$

and let us denote by

$$S = S(t, \delta_1, \delta_2, \delta_3, \delta_4) = \delta_1\delta_2 + (\delta_1 + \delta_2)\delta\sqrt{t} + \delta^2 t\,.$$

Using then the estimates (5.12) and (5.13) we obtain the following key lemma.

LEMMA 5.6. *For all $0 < t < 1$ there exists $\delta_0 = \delta_0(t)$ such that*

$$\|A\|_\infty \leqq \kappa S\sqrt{t} + \tilde{\kappa}\delta^3; \ \|B\|_\infty \leqq \kappa S + \tilde{\kappa}\delta^3$$

for $\delta \leqq \delta_0$.

From the above proposition we can already obtain an estimate for $\|H_\theta([\Delta])\|$. Indeed we have

$$(5.14)\qquad \|H_\theta([\Delta])\| \leqq 100\int(|A| + |B|)d\lambda d\mu d\nu \leqq S + \tilde{\kappa}\delta^3\,.$$

Similarly by Lemma 5.1 we see that there exists $\delta_0 = \delta_0(t, h)$ such that for all $\delta < \delta_0$ we have

$$\|\chi_{R_h} H_\theta([\Delta])\| \leqq 100\int_{1-\nu\leqq\kappa h/t}(|A| + |B|)d\lambda d\mu d\nu \leqq \kappa\frac{h}{t}S + \tilde{\kappa}\delta^3\,.$$

Let us now denote by

$$K(\Delta) = K = [H_\theta([\Delta])]_{0,1}$$

the (0, 1) component of the current $H_\theta([\Delta])$. To obtain estimates on K we shall need some ground work first.

Let us denote by

$$H_\theta([\Delta]) = a\bar{\omega} + b\bar{\partial}\rho + c\omega + e\partial\rho$$
$$K = a\bar{\omega} + b\bar{\partial}\rho$$

where a, b, c, e can be canonically identified with Radon measures in Ω. We have then

$$K \wedge \bar{\partial}\rho = a\bar{\omega} \wedge \bar{\partial}\rho \tag{5.15}$$

$$H_\theta([\varDelta]) \wedge \bar{\partial}\rho \wedge \partial\rho \wedge \omega = a\bar{\omega} \wedge \bar{\partial}\rho \wedge \partial\rho \wedge \omega = V \tag{5.16}$$

where V now is just a volume form and can therefore be identified with a Radon measure in Ω. From (5.15) and (5.16) we deduce that

$$|K \wedge \bar{\partial}\rho| \leqq \kappa\, |V| \tag{5.17}$$

(observe that the length of the vectors $\partial\rho$, $\bar{\partial}\rho$, ω and $\bar{\omega}$ is $\sqrt{2}\,||z||$). Let us now denote by

$$V_1 = H_\theta([\varDelta]) \wedge dz_2 \wedge dz_1 \wedge d\bar{z}_1$$
$$V_2 = H_\theta([\varDelta]) \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_1$$

and let us expand

$$\begin{aligned}\bar{\partial}\rho \wedge \partial\rho \wedge \omega &= z_1(|z_1|^2 + |z_2|^2)dz_2 \wedge dz_1 \wedge d\bar{z}_1 \\ &\quad - z_2(|z_1|^2 + |z_2|^2)dz_2 \wedge d\bar{z}_2 \wedge dz_1\,.\end{aligned} \tag{5.18}$$

From (5.18) we deduce then that

$$|V| \leqq C(|V_1| + |z_2 V_2|)$$

and if we also have in mind that $|z_2| \leqq C\sqrt{t}$ in the support of $H_\theta([\varDelta])$ we finally conclude that

$$|V| \leqq C(|V_1| + \sqrt{t}\,|V_2|) \tag{5.19}$$

where C is a numerical constant. We shall now use Lemma 5.6 to make estimates on V_1 and V_2.

Towards that observe first, that by Lemma 5.1 and (5.9), we have

$$1 - ||z||^2 = -\rho(z) \geqq |\kappa^{-1}(t(\nu - 1) + \nu\tilde{\psi}_1(\lambda, \mu)\delta + \tilde{\psi}_2(\lambda, \mu)\delta)|$$

$$\sqrt{\frac{t}{|\rho(z)|}} \leqq \kappa\,|(1 + \delta\tilde{\psi}_1(\lambda, \mu))\nu - (1 + \delta\tilde{\psi}_2(\lambda, \mu))|^{-1/2}\,. \tag{5.20}$$

From (5.20) and from Lemma 5.6 we obtain

$$\begin{cases} ||\,|\rho|^{-1/2}V_1|| \leqq 100\displaystyle\int |\rho|^{-1/2}\,|A|\,d\lambda d\mu d\nu \\ \leqq (\kappa S + \tilde{\kappa}\delta^3)\displaystyle\int\sqrt{\frac{t}{|\rho|}}\,d\lambda d\mu d\nu \\ \leqq \kappa S + \tilde{\kappa}\delta^3 \end{cases} \tag{5.21}$$

provided that $\delta < \delta_0$ where $\delta_0 = \delta_0(t)$ may depend on t.

Similarly we have

$$\|\chi_{R_h}|\rho|^{-1/2}V_1\| \leqq (\kappa S + \tilde{\kappa}\delta^3)\int_{z\in R_h}\sqrt{\frac{t}{|\rho(z)|}}d\lambda d\mu d\nu$$

z in the above integral is of course the generic point of the chain (5.8). Notice now that by Lemma 5.1 when $z \in R_h$ lies on the chain (5.8) and when $\delta < \delta_0$ (where $\delta_0 = \delta_0(t, h) > 0$ depends on t and on h) then

$$1 - \nu \leqq \kappa\frac{h}{t} .$$

We conclude therefore from (5.20) that

$$\int_{z\in R_h}\sqrt{\frac{t}{\rho(z)}} \leqq \kappa\sqrt{\frac{h}{t}}$$

provided again that $\delta < \delta_0 = \delta_0(t, h)$.

We obtain therefore that

$$\|\chi_{R_h}|\rho|^{-1/2}V_1\| \leqq \kappa\sqrt{\frac{h}{t}}S + \tilde{\kappa}\delta^3 . \tag{5.22}$$

We have similarly

$$\begin{cases} \| |\rho|^{-1/2}V_2\| \leqq 100\int |\rho|^{-1/2}|B|\, d\lambda d\mu d\nu \\ \leqq \frac{1}{\sqrt{t}}(\kappa S + \tilde{\kappa}\delta^3)\int\sqrt{\frac{t}{|\rho|}}d\lambda d\mu d\nu \leqq \frac{\kappa}{\sqrt{t}}S + \tilde{\kappa}\delta^3 \end{cases} \tag{5.23}$$

$$\begin{cases} \|\chi_{R_h}|\rho|^{-1/2}V_2\| \leqq \frac{1}{\sqrt{t}}(\kappa S + \tilde{\kappa}\delta^3)\int_{z\in R_h}\sqrt{\frac{t}{|\rho(z)|}}d\lambda d\mu d\nu \\ \leqq \sqrt{\frac{h}{t}}\frac{\kappa}{\sqrt{t}}S + \tilde{\kappa}\delta^3 . \end{cases} \tag{5.24}$$

If we combine now the estimates (5.21) (5.22) (5.23) (5.24) together with (5.17) and (5.19) we finally obtain the following

LEMMA 5.7. *For all $0 < t < 1$ and $h > 0$ there exists $\delta_0 = \delta_0(t, h)$ such that*

$$\| |\rho|^{-1/2}K(\Delta) \wedge \bar{\partial}\rho\| \leqq \kappa S + \tilde{\kappa}\delta^3$$

$$\|\chi_{R_h}|\rho|^{-1/2}K(\Delta) \wedge \bar{\partial}\rho\| \leqq \kappa\sqrt{\frac{h}{t}}S + \tilde{\kappa}\delta^3$$

for all $\delta < \delta_0$.

Let us now consider the point

$$\xi'(\lambda, \mu) = (1 - t + \delta_1\lambda, \delta_2\mu, \delta_3'\mu, \delta_4'\lambda) \quad 0 \leqq \lambda, \mu \leqq 1$$

which is the generic point on a small rectangle Δ' that is spanned by the two vectors

$$(\delta_1, 0, 0, \delta_4'); \ (0, \delta_2, \delta_3', 0)$$

Δ' can be obtained from our previous rectangle Δ by interchanging the axes x_2 and y_2. From this fact, or from reworking out all the estimates afresh for $\xi'(\lambda, \mu)$ and Δ' we see that all the estimates that we have obtained up to now for $H_\theta([\Delta])$ and $K(\Delta)$ also hold for $H_\theta([\Delta'])$ and $K(\Delta') = [H_\theta([\Delta'])]_{0,1}$, where of course now $\delta = \delta_1^2 + \delta_2^2 + \delta_3'^2 + \delta_4'^2$.

It is time now to make an assumption on $\delta_1, \delta_2, \delta_3, \delta_4, \delta_3', \delta_4'$. We shall distinguish three cases.

Case 1. We consider the rectangle Δ and set $\delta_1 = \delta_2 = 0$, $\delta_3 = \delta_4 \neq 0$ in that case

$$S = \delta^2 t \, . \tag{5.25}$$

Case 2. We consider the rectangle Δ and set: either

$$\delta_1 = \delta_4 = 0, \quad \delta_2 = \delta_3 \neq 0$$

or

$$\delta_2 = \delta_3 = 0, \quad \delta_1 = \delta_4 \neq 0 \, .$$

We consider the rectangle Δ' and set: either

$$\delta_1 = \delta_3' = 0, \quad \delta_2 = \delta_4' \neq 0$$

or

$$\delta_2 = \delta_4' = 0, \quad \delta_1 = \delta_3' \neq 0$$

in that case

$$S \leqq 10\delta^2 \sqrt{t} \, .$$

Case 3. We consider the rectangle Δ and set $\delta_3 = \delta_4 = 0$, $\delta_1 = \delta_2 \neq 0$, in that case

$$S \leqq 100\delta^2 \, .$$

We are finally in a position to give the

Proof of Proposition 4.2.

Case 1. Let us suppose that $\delta_1, \delta_2, \delta_3, \delta_4$ are as in Case 1 above. In that case we have

$$\frac{1}{\delta^2}[\varDelta] \xrightarrow[\delta\to 0]{} \delta_\xi dx_1 \wedge dy_1 \tag{5.26}$$

the convergence takes place for the weak topology of currents of order zero. From this it follows that for all θ as in (3.1)

$$H_\theta\left(\frac{1}{\delta^2}[\varDelta]\right) \xrightarrow[\delta\to 0]{} H_\theta(\delta_\xi dx_1 \wedge dy_1) \tag{5.27}$$

for the weak topology of currents of order zero, because H_θ is continuous for the convergence in (5.26) (observe that the fact that $1/\delta^2[\varDelta]$ stays bounded in norm and its support stays in some compact subset of Ω makes the verification of that fact very easy).

From (5.27) it follows therefore that

$$\|H_\theta(\delta_\xi dx_1 \wedge dy_1)\| \leqq \overline{\lim_{\delta\to 0}} \frac{1}{\delta^2} \|H_\theta([\varDelta])\| .$$

By (5.14) and (5.25) we deduce therefore that

$$\|H_\theta(\delta_\xi dx_1 \wedge dy_1)\| \leqq \kappa t$$

but by definition (4.1) we have

$$\alpha(\xi) = 2i(1-t)^2\delta_\xi dx_1 \wedge dy_1$$

and from this our estimate

$$\|\alpha^\theta\| = \|H_\theta(\alpha)\| \leqq \kappa t$$

follows at the point $\xi = (1-t, 0)$.

Since the situation is clearly rotation invariant we have the same estimate for $\alpha(z)$ at every point $z \in B\backslash 0$ this proves the first estimate in (4.4). All the other estimates are proved in an identical manner. Case 1 gives estimates (4.4). Case 2 gives estimates (4.5) and Case 3 gives estimates (4.6). The varification of these final details will be left to the reader.

6. Proof of Theorem 1.1 when $\Omega = \{\|z\| < 1\}$. In this paragraph we shall give the proof of Theorem 1.1 when Ω is the unit ball $\{\|z\| < 1\}$ (or even more generally an arbitrary convex set with smooth boundary). The passage to this theorem is not very simple, fortunately however all the extra work needed to obtain it has been done elsewhere so we shall be brief and follow very closely H. Skoda [15] II. The reader who wishes to understand the following few lines is strongly advised to study first Ch. VII of [10] and pt II of [15].

Let $\widetilde{M} \subset B$ be a divisor as in Theorem 1.1. Let us suppose as we may that $0 \notin M$ and let us denote by t the corresponding Lelong current which is then a real current that satisfies the U. B. condition. Using then Theorem 2.1 (or even Proposition 3.1, observe that $t \in \Lambda$) we can find a real current v of degree one such that

$$dv = t$$

and such that $v + \delta^{-1/2} v_{0,1} \wedge \bar{\partial}\rho$ is a Carleson current. Let us then set $w = -iv$ and let

$$w = -w_{1,0} + w_{0,1}$$

be the decomposition of w into its (1, 0) and (0, 1) components. The current $w_{0,1}$ is then $\bar{\partial}$-closed and the current $w_{0,1} + \delta^{-1/2} w_{0,1} \wedge \bar{\partial}\rho$ is Carleson, in other words $w_{0,1}$ satisfies all the conditions of Theorem 3.1.1 in [18] (observe that what we call a Carleson condition for a current μ there, differs from our present terminology and corresponds to the fact that $|\mu| + \delta^{-1/2}|\mu \wedge \bar{\partial}\rho|$ is a Carleson measure). From Theorem 3.1.1 in [18] then, and from Proposition 2.2 in [15] it follows that we can find two functions:

$$\begin{cases} u \in \mathrm{BM}\,0(\partial\Omega) \\ U \in L^1(\Omega) \text{ for the volume measure} \end{cases} \tag{6.1}$$

such that

$$-\int_\Omega U \wedge \bar{\partial}\varphi = \int_\Omega w_{0,1} \wedge \varphi - \int_{\partial\Omega} u \wedge \varphi \tag{6.2}$$

where (6.2) is valid for all φ a (2, 1) from that is C^1 in some neighborhood of Ω. From (6.2) and from the fact that v is real we deduce at once that

$$\bar{\partial} U = w_{0,1} \text{ in } \Omega$$
$$w_{1,0} = \overline{w_{0,1}}$$

and from these two facts it follows that if we set $W = \pi(U + \bar{U})$ we have

$$\frac{i}{\pi}\partial\bar{\partial} W = t\,.$$

But then by the Lelong theory (as developed say in [10] Ch. VII) and the work done in [15] II (or [7]) it follows that there exists $F(z)$ a holomorphic function in Ω belonging to the Nevanlinna $N^*(\Omega)$ (sometimes denoted $N^+(\Omega)$) class such that

$$\log |F(z)| = W(z)\,. \tag{6.3}$$

The $N^*(\Omega)$ class is the subclass of the Nevanlinna class $N(\Omega)$

that consists of these functions $f \in N(\Omega)$ that satisfy

$$\text{(6.4)} \qquad \lim_{r\to 1} \int_{\partial\Omega} |\log|f(rz)||\, d\sigma(z) \longrightarrow \int_{\partial\Omega} |\log|f^*(z)||\, d\sigma(z) .$$

In (6.4) Ω denotes the unit ball and $d\sigma(z)$ denotes the Lebesgue measure element on $\partial\Omega$, we also denote by $f^*(z)$ the radial limits of the function $f(z)$ that exist for almost all $z \in \partial\Omega$.

The fact that we are in $N^*(\Omega)$ rather than the general class $N(\Omega)$ is proved in [15] Appendix I (H. Skoda does not explicitely state that fact but he proves it. G. M. Henkin actually explicitely states it.)

In our case if we denote by $F^*(z)$ the boundary values of the function F that satisfies (6.3) and if we set

$$w^*(z) = \log|F^*(z)|$$

we have $w^*(z) = \pi(u(z) + \overline{u(z)})$ (cf. [15] Appendice I).

It follows therefore from (6.1) that

$$\log|F^*(z)| \in \mathrm{BMO}(\partial\Omega) .$$

This by the John-Nirenberg theorem [9] implies that there exists some $p > 0$ such that

$$F^*(z) \in L^p(\partial\Omega) .$$

But the above fact and the fact that $F \in N^*(\Omega)$ (cf. [20] Ch. 7, Th. 7.50) implies that

$$F(z) \in H^p(\Omega) .$$

The proof of Theorem 1.1 is complete.

7. The general strictly pseudoconvex domains and the Poincaré equation for Carleson currents. In this paragraph we shall prove Theorem 2.1 in its full generality, i.e., when Ω is a general bounded strictly pseudoconvex domain with smooth boundary in $\boldsymbol{C}^n$. Theorem 1.1 in its full generality can be dealt by the same method as in paragraph 6.

In the passage from Theorem 2.1 to Theorem 1.1 there arises a slight problem in applying the Lelong theory when $H^1(\Omega; \boldsymbol{R}) \neq 0$ that problem can be delt with by a method due to R. Harvey (cf. [6]).

In this paragraph we shall follow very closely, once more, H. Skoda in [15] II § 4.

By the assumption on Ω it follows that there exists a finite open covering of $\bar{\Omega}$ in $\boldsymbol{C}^n$

$$\bar{\Omega} \subset \bigcup_{j=1}^{m} \Omega_j$$

and for each Ω_j there exists an operator H_j defined in the space of all currents T of Ω that satisfy

$$\operatorname{supp} T \subset \Omega_j$$

which satisfies

$$d \circ H_j + H_j \circ d = Id$$

and which is such that the current $\chi_{\Omega_j}(H_j(T) + \delta^{-1/2}(H_j(T))_{0,1} \wedge \bar{\partial}\rho)$ is Carleson for each T that satisfies the U. B. condition.

The construction of H_j is obvious when $\Omega_j \cap \partial\Omega = \phi$, when $\bar{\Omega}_j \cap \partial\Omega \neq \phi$ we just have to use the fact that $\partial\Omega$ is locally biholomorphically equivalent with a piece of a strictly convex hypersurface in $\boldsymbol{C}^n$ and then apply the results of § 3, § 4 and § 5.

The problem now is to "glue" back all these operators.

Let $\psi_j \in C_0^\infty(\boldsymbol{C}^n)$ $1 \leqq j \leqq m$ be functions that satisfy

$$\operatorname{supp} \psi_j \subset \Omega_j \ \ 1 \leqq j \leqq m; \ \sum_{j=1}^{m} \psi_j^2 = 1 \text{ on } \bar{\Omega}\ .$$

For all T as in Theorem 2.1 let us set then:

$$\begin{aligned}\theta &= T - d\Big(\sum_{j=1}^{m} \psi_j H_j(\psi_j T)\Big)\\ &= \sum_{j=1}^{n} [\psi_j H_j(d\psi_j \wedge T) - d\psi_j \wedge H_j(\psi_j T)]\ .\end{aligned}$$

It follows that θ is a closed current of degree 2 and order zero that satisfies the Carleson condition. It also follows that the canonical cohomology class of θ in $H^2(\Omega; \boldsymbol{Z})$ (cf. [6] 1.5) is the same as the class of T i.e., it is zero.

To complete the proof of Theorem 2.1 it suffices therefore to prove the following

LEMMA 7.1. *Let θ be a closed Carleson current of degree 2 in Ω whose cohomology class in $H^2(\Omega; \boldsymbol{Z})$ is zero. There exists then a current w of order zero and degree 1 that satisfies*

$$dw = \theta$$

and which is such that $\delta^{-1/2}w$ is a Carleson current.

Observe that the above situation is self adjoined and therefore when θ is real w can also be chosen real.

The above lemma is what replaces H. Skoda's Lemma 4.1 in

[15] pt II.

For the proof of this lemma we shall follow Skoda's method in ([15], II § 4) and we shall use the three sheaves $\mathscr{F}_0$, $\mathscr{F}_1$, and $\mathscr{F}_2$ that are defined there. The only alteration that we shall make is that we shall replace (II. 4.4), (II. 4.5) and (II. 4.6) of [15] by the following

(II. 4.4)′ $\quad \chi_K\theta$ is Carleson .

(II. 4.5)′ $\quad \chi_K(|w| + \delta^{-1/2}|dw|)$ is Carleson .

(II. 4.6)′ $\quad \chi_K\delta^{-1/2}(|g| + |dg|)$ is Carleson.

From these onwards our proof follows Skoda's proof word by word.

To make the sheave theoretic machine in that proof work, however, we shall need to give a proof of the local version of Lemma 7.1.

This is contained in the following proposition which holds for all convex sets in C^n with smooth boundary but which, for simplicity, we shall state and prove only for the unit ball $B \subset C^n$. We shall again denote by Λ the space of all currents T in B such that $0 \notin \operatorname{Supp} T$. We have then

PROPOSITION 7.1. *There exists*

$$H^*: \Lambda \longrightarrow \Lambda$$

a linear operator such that

$$d\circ H^* + H^*\circ d = Id$$

and such that if $T \in \Lambda$ is a Carleson current then $\delta^{-1/2}H^*(T)$ is a Carleson current also.

In other words H^* gains an exponent 1/2 on δ. The above proposition is of the same nature as Proposition 3.1. Its proof however is considerably simpler, the reason for that is that the situation now is isotropic. Indeed the complex structure of C^n plays no role either in the statement or in the proof of Proposition 7.1. Observe also that the condition $0 \notin \operatorname{Supp} T$ is purely technical and can easily be eliminated.

The construction of H^*.

To simplify notations and to avoid repetition we shall suppose that $n = 2$ and we shall repeat all the constructions of § 3 making only one modification.

In (3.2) instead of defining $p_\theta(z)$ as we did, which was designed to bring out the anisotropic structure of $\partial\Omega$, we shall simply set:

$$p_\theta(z) = (\theta_1 t, \theta_2 t, \theta_3 t)$$

all the rest remains unchanged. We obtain then an operator H^* which is the average of the operators H_θ given by

$$H^*(T) = (2c)^{-3}\int_{|\theta_i|\leqq c} H_\theta(T)d\theta_1 d\theta_2 d\theta_3 \,.$$

We claim that H^* satisfies all the conditions of Proposition 7.1. The fact that H^* is a chain homotopy is of course obvious. What has to be verified is the estimate on $\delta^{-1/2}H^*(T)$. The strategy to prove that estimate is identical with the one in § 4 and § 5 and the details are much simpler. We shall be brief.

Let I be a multiindex and let us denote by

$$\eta = \eta(z) = \delta_z dx_I$$

which is of course a current of order zero supported by the point z, let us also denote by

$$\eta^\theta = \eta^\theta(z) = H_\theta(\eta);\ H = H(z) = (2c)^{-3}\int_{|\theta_i|\leqq c} \eta^\theta(z)d\theta_1 d\theta_2 d\theta_3 \,.$$

PROPOSITION 7.2. *For all $\varepsilon > 0$ there exists C_ε a constant that depends only on $\varepsilon > 0$ such that for all $z \in B$ with $\|z\| > \varepsilon$ and all θ we have*

$$\|\delta^{-1/2}\eta^\theta\| \leqq C_\varepsilon\sqrt{t} \tag{7.1}$$

$$\|\chi_{R_h}\delta^{-1/2}\eta^\theta\| \leqq C_\varepsilon\sqrt{h} \tag{7.2}$$

(R_h is as in (4.3)).

PROPOSITION 7.3. *For all $\varepsilon > 0$ there exists C_ε a constant that depends only on ε such that for all $z \in B$ with $\|z\| > \varepsilon$ and all θ we have*

$$\|\delta^{-1/2}H\| \leqq C_\varepsilon\sqrt{t} \,. \tag{7.3}$$

PROPOSITION 7.4. *For all $\varepsilon > 0$ there exists C_ε a constant that depends only on ε such that for all θ, all $h > 0$ and all $z \in B$ such that $z \notin \widetilde{B_{ch}}(1)$ ($c = 10^{10}$ say) and $\|z\| > \varepsilon$ we have*:

$$\|\chi_h\delta^{-1/2}H\| \leqq C_\varepsilon\frac{h^{3/2}}{t} \tag{7.4}$$

If we suppose in addition that

$$t = \delta(z) \geqq c\sqrt{h}$$

then we have

$$\|\chi_h \delta^{-1/2} H\| \leqq C_\varepsilon \frac{h^{5/2}}{t^3} \tag{7.5}$$

(χ_h *is as in Proposition* 4.4).

Proof of Proposition 7.2. Proposition 7.2 is the analogue of Proposition 4.2. To prove it we consider, as in § 5, a "cubical" chain element of size δ and dimension $n - |I|$ (which will play the role of an infinitesimal chain element since we shall let $\delta \to 0$) at ξ and we shall make estimates for $H_\theta([\Delta])$. We shall then let $\delta \to 0$ and observe that provided that Δ is properly oriented we have

$$\delta^{|I|-n}[\Delta] \xrightarrow[\delta \to 0]{} \eta(\xi)$$

just as in (5.26). The proof is then concluded as in § 5.

The estimates that we need on $H_\theta[\Delta]$ are the following

$$\|\delta^{-1/2} H_\theta([\Delta])\| \leqq \kappa \delta^{n-|I|} \sqrt{t} \quad \forall \delta < \delta_0(t)$$
$$\|\chi_{B_h} \delta^{-1/2} H_\theta([\Delta])\| \leqq \kappa \delta^{n-|I|} \sqrt{h} \quad \forall \delta < \delta_0(t)$$

where $\delta_0(t) > 0$ depends on t. These estimates are very easy to obtain here and no development up to the 2nd order in δ, as in §5, is needed.

Indeed the chain element that represents $H_\theta([\Delta])$ is a "spike" based on Δ with a long edge of length comparable with t along the vector $l_\theta(\xi)$. The estimates above follow immediately from that.

Proof of Proposition 7.3. Proposition 7.3 is of course the analogue of Proposition 4.3. To prove it we just have to integrate the estimate (7.1) over θ.

Proof of Proposition 7.4. Proposition 7.4 is the analogue of Proposition 4.4. To prove it we just have to integrate the estimate (7.2) over $\theta \in X_z$ where X_z is defined as in (4.8) but for our new definition of $p(z)$. Concerning that new set X_z Lemma 4.2 is no longer valid what replaces it is the following.

LEMMA 7.1. *Let* $h > 0$ *and* $0 \neq z \in \widetilde{B_{ch}}(1)$ (*where* $c = 10^{10}$ *say*) *then*

$$|X_z|_3 \leqq c \frac{h}{t}.$$

If in addition we assume that

$$t = \delta(z) > c\sqrt{h}$$

then

$$|X_z|_3 \leqq c\frac{h^2}{t^3} .$$

The proof of Lemma 7.1 follows exactly the same lines as the proof of Lemma 4.2. It will therefore be omitted.

Observe that the analogue of Lemma 4.1 does hold and that therefore X_z is empty unless $\delta(z) \geqq c^{-1}h$.

We can now complete the

Proof of Proposition 7.1. Let T be a Carleson current as in Proposition 7.1. We have then

$$|\delta^{-1/2}H^*(T)|\,(\widetilde{B_h(1)}) \leqq \int_B |\delta^{-1/2}H(z)|\,(\widetilde{B_h(1)})d\,|T|\,(z)$$
$$\leqq \int_{z\in \widetilde{B_{ch}}(1)} + \int_{z\notin \widetilde{B_{ch}}(1)\,;\,h(z)\leqq c\sqrt{h}} + \int_{h(z)\geqq c\sqrt{h}} = \mathfrak{A} + \mathfrak{B} + \mathfrak{C}$$

(where $h(z)$ is defined as in Lemma 4.1) .

We have then from (7.3):

$$\mathfrak{A} \leqq C\sqrt{h}\,|T|\,(\widetilde{B_{ch}(1)}) = Ch^{5/2} . \tag{7.6}$$

We also have from (7.4) and (7.5) that

$$\mathfrak{B} \leqq Ch^{3/2}\int \chi[z\in B; c\sqrt{h} \geqq h(z) \geqq h;\, \delta(z) \geqq c^{-1}h(z)]\frac{d\,|T|\,(z)}{\delta(z)} \tag{7.7}$$

$$\mathfrak{C} \leqq Ch^{5/2}\int \chi[z\in B;\, h(z) \geqq c\sqrt{h};\, \delta(z) \geqq c^{-1}h(z)]\,\frac{d\,|T|\,(z)}{\delta^3(z)} \tag{7.8}$$

where χ in the above integrals indicates the characteristic function of the corresponding sets. The inequality $\delta(z) \geqq c^{-1}h(z)$ inside these characteristic functions follows by the analogue of Lemma 4.1 which as we already pointed out does hold for the new definition of X_z.

Let us now denote by

$$F(\lambda) = |T|\,(\widetilde{B_\lambda(1)}) \leqq c\lambda^2 \tag{7.9}$$

by (7.7) and (7.8) we deduce then that

$$\mathfrak{B} \leqq Ch^{3/2}\int_{ch}^{c\sqrt{h}} \frac{dF(\lambda)}{\lambda} \tag{7.10}$$

$$\mathfrak{C} \leqq Ch^{5/2}\int_{c\sqrt{h}}^{\infty} \frac{dF(\lambda)}{\lambda^3} . \tag{7.11}$$

An easy integration by parts in (7.10) and (7.11) together with (7.9) gives then

$$\mathfrak{B} \leqq Ch^2 \tag{7.12}$$

$$\mathfrak{C} \leqq Ch^2 . \tag{7.13}$$

The estimates (7.6) (7.12) and (7.13) put together complete the required estimate on $|\delta^{-1/2}H^*(T)|$ and we are done.

8. Complex lines. The aim of this paragraph is to examine divisors that consist of a countable union of complex lines, and to give a necessary and sufficient condition for such a divisor to satisfy the U. B. condition.

We shall work exclusively in the unit ball $B \subset \boldsymbol{C}^2$ $(n = 2)$ and we shall also find it convenient to use the sets $C_h(\zeta_0)$ $(h > 0\colon \zeta_0 \in \partial B)$

$$C_h(\zeta_0) = \{\zeta \in B; |1 - \zeta.\bar{\zeta}_0| < h\} \tag{8.1}$$

to test the Carleson condition, this is certainly legitimate because these sets are equivalent to the sets $\widetilde{B_h(\zeta)}$ (cf. [18] § 2.2).

Let now $l \subset \boldsymbol{C}^2$ be a complex line represented parametrically by

$$l = \{z = (z_1, z_2) \mid z_1 = z_1^0 + \alpha z,\ z_2 = z_2^0 + \beta z,\ z \in \boldsymbol{C}\} \tag{8.2}$$

where $(z_1^0, z_2^0) \in \boldsymbol{C}$ and $\alpha \geqq 0$ $\alpha^2 + |\beta|^2 = 1$ and let us denote by $d\sigma$ the 2-dimensional Lebesgue (2-dimensional Hausdorff) measure element on l.

We have then

PROPOSITION 8.1. *There exists C a numerical constant (independent of l) such that if we denote by ν the Malliavin measure on $l \cap B$* (cf. (1.6)) *and by $d(l)$ the diameter of the disc $l \cap B$ we then have*

$$C^{-1}d(l)^2\sigma \leqq \nu \leqq Cd(l)^2\sigma \tag{8.3}$$

$$\nu(C_h(\zeta_0)) \leqq Ch^2 \quad \forall h > 0, \quad \forall \zeta_0 \in \partial B . \tag{8.4}$$

Proof. We shall use the decomposition

$$\nu = \nu_1 + \nu_2$$

and the two measures

$$\begin{cases} \tilde{\nu}_1 = |t \wedge \partial\rho \wedge \bar{\partial}\rho| \\ \tilde{\nu}_2 = \delta\,|t| \end{cases} \tag{8.5}$$

that were introduced in (5.16) and in (5.17), and we shall denote as usual by t the Lelong current associated to the divisor $l \cap B$, and by $\delta = \delta(z) = 1 - ||z||$.

For the proof of (8.3) we shall also suppose, as we may, that

$$l = \{z = (z_1, z_2) \mid z_1 = r\}$$

for some $0 < r < 1$.

We have then

$$C^{-1}(1 - r) \leqq d(l)^2 \leqq C(1 - r) \tag{8.6}$$

$$1 - (r^2 + |z_2|^2)^{1/2} = \delta(z) \leqq 1 - r \quad \forall z = (z_1, z_2) \in l \cap B \tag{8.7}$$

for some numerical constant C. We also have

$$t = \frac{i}{2}\sigma dz_1 \wedge d\bar{z}_1$$

$$\tilde{\nu}_1 = |t \wedge \partial\rho \wedge \bar{\partial}\rho| = \frac{1}{2}\,|z_2^2\sigma|\, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2\,| \tag{8.8}$$
$$= 2\,|z_2|^2\sigma\,.$$

But from (8.7) and (8.8) it then follows that

$$\tilde{\nu}_1 = 2[(1 - \delta(z))^2 - r^2]\sigma = (1 - r^2)\sigma - 2(\delta(z) - \delta^2(z))\sigma\,. \tag{8.9}$$

Since also by definition

$$\tilde{\nu}_2 = \delta\sigma \leqq C(1 - r)\sigma \tag{8.10}$$

we obtain at once that

$$\tilde{\nu}_1 + \tilde{\nu}_2 \leqq C(1 - r)\sigma$$

which together with (8.6) gives the right hand inequality in (8.3). To obtain the left hand inequality we observe that by (8.9)

$$\tilde{\nu}_1 \geqq C^{-1}(1 - r)\sigma \tag{8.11}$$

in the domain where $\delta(z) \leqq 10^{-10}(1 - r)$ but when $\delta(z) \geqq 10^{-10}(1 - r)$ we have by (8.5) that

$$\tilde{\nu}_2 \geqq C^{-1}(1 - r)\sigma \tag{8.12}$$

and the inequalities (8.11) and (8.12) put together complete the proof of (8.3).

For the proof of (8.4) we shall, as we may, assume that $\zeta_0 = 1$

and that l is a general line parametrized as in (8.2).

We first observe that

$$|z_2| \leqq C\sqrt{h} \quad \forall z = (z_1, z_2) \in C_h(1)$$

and that $l \cap C_h(1)$ is a convex subset of l whose diameter is bounded by the diameter of $C_h(1)$ which is $C\sqrt{h}$. From the above two observations it follows that

$$\sigma(C_h(1)) = |l \cap C_h(1)|_2 \leqq Ch \tag{8.13}$$

and that

$$\tilde{\nu}_2(C_h(1)) \leqq \sup_{z \in C_h(1)} \delta(z).\sigma(C_h(1)) \leqq Ch^2 . \tag{8.14}$$

It also follows that

$$J = \int_{C_h(1)} |z_2|^2 d\sigma \leqq Ch\sigma(C_h(1)) \leqq Ch^2 . \tag{8.15}$$

If we use the parametrization (8.2) of l we see that

$$dz_1 \wedge d\bar{z}_1 = |\alpha|^2 dz \wedge d\bar{z}$$

but that means that the integral

$$I = \int_{C_h(1)} |t \wedge dz_1 \wedge d\bar{z}_1| = \int_{C_h(1)} |[l] \wedge dz_1 \wedge d\bar{z}_1|$$

is just twice the 2-dimensional area on the line $\{z_2 = 0\}$ (which is the z_1 axis) of the orthogonal projection on that line of the set $l \cap C_h(1)$, that area is clearly bounded by the area $|C_h(1) \cap \{z_2 = 0\}|_2$ and this means that

$$I \leqq Ch^2 . \tag{8.16}$$

We can finally estimate

$$\begin{aligned} \tilde{\nu}_1(C_h(1)) &= \int_{C_h(1)} |t \wedge \partial\rho \wedge \bar{\partial}\rho| \\ &\leqq C\int_{C_h(1)} |t \wedge dz_1 \wedge d\bar{z}_1| + C\int_{C_h(1)} |z_2|\,|t \wedge dz_1 \wedge d\bar{z}_2| \\ &\quad + C\int_{C_h(1)} |z_2|^2\,|t \wedge dz_2 \wedge d\bar{z}_2| \leqq CI + CK + CJ \end{aligned}$$

where

$$K = \int_{C_h(1)} |z_2|\,|t \wedge dz_1 \wedge d\bar{z}_2| \leqq \sqrt{I.J} \tag{8.17}$$

by the positivity of the current t (cf. (2.5)). The estimates (8.15)

(8.16)(8.17) and (8.14) put together give (8.4) and complete the proof of Proposition 8.1.

We shall now examine divisors that are obtained by taking countable unions of complex lines.

More explicitely let $l_j \subset C^2$, $j = 1, 2, \cdots$ be a sequence of complex lines and let us denote by

$$\tilde{M} = \bigcup_{j=1}^{\infty} (l_j \cap B)$$

which is a divisor in B (we give multiplicity one on each line). Let us denote by

$$K_j = B \cap l_j, \quad d_j = \operatorname{diam}(K_j); \; j = 1, 2, \cdots .$$

The divisor $\tilde{M}$ then clearly satisfies the Blaschke condition if and only if $\sum_{j=1}^{\infty} d_j^4 < +\infty$ (this immediately follows from Proposition 8.1). We also have

PROPOSITION 8.2. *The divisor $\tilde{M}$ satisfies the* U.B. *condition if and only if there exists $c > 0$ some positive constant such that*

$$\sum_{K_j \subset C_h(\zeta_0)} d_j^4 \leqq ch^2 , \tag{8.18}$$

$$\operatorname{Card}\{j \mid K_j \cap C_h(\zeta_0) \neq \phi; \; K_j \not\subset C_{ch}(\zeta_0)\} \leqq c \tag{8.19}$$

for all $h > 0$ and all $\zeta_0 \in \partial B$.

Proof. The fact that the two conditions (8.18) and (8.19) put together (for some $c > 0$) are sufficient to ensure that $\tilde{M}$ is U. B. is an immediate consequence of Proposition 8.1. It is also clear from the same proposition that if $\tilde{M}$ is U. B. then (8.18) has to be verified for some $c > 0$. The proof that (8.19) also has to be verified when $\tilde{M}$ is U. B. is more delicate. That proof is based on the following

LEMMA 8.1. *Let l_m $m = 1, 2, \cdots$ be a sequence of distinct complex lines in C^2 and let us suppose that*

$$1 \in l_m \qquad m = 1, 2, \cdots$$

then the divisor

$$\tilde{M} = \bigcup_{m=1}^{\infty} (l_m \cap B)$$

is not U. B.

In fact, for the applications that we have in mind, the above lemma will be more useful than the actual Proposition 8.2. From the above lemma however we can easily give the

Conclusion of the proof of Proposition 8.2. Indeed let us suppose that $\tilde{M}$ is U. B. but that (8.19) fails to hold for every $c>0$. By letting $c \to \infty$ we can easily construct a new divisor $\tilde{N}$ that is U. B. and that consists of infinite many lines all going through a single point say 1 on ∂B. This together with Lemma 8.2 supplies a contradiction and completes the proof. To make this argument work it is best to work on the Siegel half space

$$S = \{u = (u_1, u_2) \in C^2;\ 2Im u_1 > |u_2|^2\} . \tag{8.20}$$

The reason is that S has a natural dilation structure

$$(u_1, u_2) \longrightarrow (\lambda^2 u_1, \lambda u_2), \quad \lambda > 0 \tag{8.21}$$

and by letting h fixed, $c \to \infty$ and $\lambda = 1/ch$ we can realize the above construction at the point $0 \in \partial S$ (which corresponds to the point 1 of the ball). We shall leave the details to the reader and proceed with the proof of the Lemma 8.1.

We shall need the following

LEMMA 8.2. *For all complex line $l \subset C^2$ such that $1 \in l$ $l \cap B \neq \phi$ there exists then $h_0 = h_0(l) > 0$ such that for all $h < h_0$ we have*

$$J_h(l) = \int_{C_h(1)} |[l] \wedge dz_1 \wedge d\bar{z}_1| \geqq \frac{1}{100} h^2 .$$

Proof. Indeed it is clear that there exists some $h_0 = h_0(l)$ depending only on l such that

$$\{z = (z_1, z_2) \in l;\ |z_1| < 1,\ |1 - z_1| \leqq 2(1 - |z_1|) \leqq h\} \subset C_h(1) \tag{8.22}$$

for all $h < h_0$. Let us denote by

$$D_h = \{z_1 \in C;\ |z_1| < 1,\ |1 - z_1| \leqq 2(1 - |z_1|) \leqq h\}$$

it then follows from (8.22) that

$$J_h(l) \geqq \int_{z_1 \in D_h} |[l] \wedge dz_1 \wedge d\bar{z}_1| = |D_h|_2 \geqq \frac{h^2}{100} \tag{8.23}$$

(where $|D_h|_2$ is the 2-dimensional area of the set D_h).

To see the second equation in (8.23) it suffices to parametrize l by

$$z_1 = z + 1;\ z_2 = \alpha z \tag{8.24}$$

and observe that then $dz_1 \wedge d\bar{z}_1 = dz \wedge d\bar{z}$.

Let now l be a complex line that goes through $\mathbf{1}$ and let us suppose that it is parametrized as in (8.24). It follows then that

$$
\begin{aligned}
L &= \int_{C_h(1)} |[l] \wedge (\partial\rho \wedge \bar{\partial}\rho - |z_1|^2 dz_1 \wedge d\bar{z}_1)| \\
&\leqq 2\int_{C_h(1)} |z_2|\,|[l] \wedge dz_1 \wedge d\bar{z}_2| + \int_{C_h(1)} |z_2|^2\,|[l] \wedge dz_2 \wedge d\bar{z}_2| \\
&\leqq \int_{C_h(1)} |z_2|^2\,|[l] \wedge dz_2 \wedge d\bar{z}_2| \\
&\quad + 2\Big(\int_{C_h(1)} |z_2|^2\,|[l] \wedge dz_2 \wedge d\bar{z}_2|\Big)^{1/2}\Big(\int_{C_h(1)} [l] \wedge dz_1 \wedge d\bar{z}_1\Big)^{1/2}
\end{aligned}
\tag{8.25}
$$

by the positivity of the current $[l]$ (cf. (2.5)).

But we clearly have

$$|z_2| \leqq \alpha h \quad \forall z = (z_1, z_2) \in l \cap C_h(\mathbf{1}) . \tag{8.26}$$

From (8.25) (8.26) and (8.13) we deduce therefore that

$$L \leqq C|\alpha|^2 h^3 + C|\alpha|\, h^{5/2} \tag{8.27}$$

where C is a numerical constant and α depends on l.

But if we combine (8.27) with Lemma 8.2 we deduce that for all l as in Lemma 8.2 there exists $h_0 = h_0(l) > 0$ such that

$$\int_{C_h(1)} |[l] \wedge \partial\rho \wedge \bar{\partial}\rho| \geqq \frac{1}{200} h^2 \tag{8.28}$$

for all $h < h_0$. From (8.28) Lemma 8.1 follows at once.

9. Examples. The aim of this paragraph will be to give a few examples and also to supply a proof of Theorem 1.2. Once again we shall work exclusively in B the unit ball in $\boldsymbol{C}^2$, we shall also preserve all the notations of the previous paragraph.

Example 1. Let $\{\alpha_j \in \boldsymbol{C}; |\alpha_j| < 1\ j = 1, 2, \cdots\}$ be a sequence of points in the unit disc that has the following property

$$
\begin{aligned}
&\operatorname{Card}\Big\{j \,\Big|\, |\alpha_j| \in [1 - 2^{-n}, 1 - 2^{-n-1}];\ \operatorname{Arg}\alpha_j \in \Big[\frac{p}{2^n}, \frac{p+1}{2^n}\Big]\Big\} \\
&\qquad \leqq C \quad \forall n \geqq 1;\ 1 \leqq p \leqq 2^n
\end{aligned}
\tag{9.1}
$$

where C is a fixed constant that depends on the sequence but is independent of n and p.

Observe that the above condition (9.1) implies that for all $\zeta_0 \in \boldsymbol{C}$, $|\zeta_0| = 1$ we have

$$(9.2)\qquad \sum_{|\alpha_j-\zeta_0|\leq C_h h^2}(1-|\alpha_j|)^2 \leq C_1 h^2$$

where C_1 only depends on C.

We have then

PROPOSITION 9.1. *Let $\{\alpha_j\}_{j=1}^{\infty}$ be a sequence as above that satisfies condition* (9.1) *for some C, the divisor*

$$(9.3)\qquad \tilde{M} = \bigcup_{j=1}^{\infty}(\{z_1 = \alpha_j\} \cap B)$$

then satisfies the U.B. *condition.*

We shall need the following

LEMMA 9.1. *Let $\zeta_0 = (\cos\theta,\ \sin\theta) \in \partial B$ $(-\pi/2 \leq \theta \leq \pi/2)$, and let us denote by $X(\theta, h)$ the orthogonal projection on the line $\{z_2 = 0\}$ (which is the z_1 axis) of the set $C_h(\zeta_0)$* [cf. (8.1)]. *$X(\theta, h)$ is then a convex set on that line and its diameter satisfies*

$$(9.4)\qquad \operatorname{diam} X(\theta, h) \leq C_0(h + \sqrt{h}\,|\sin\theta|)$$

where C_0 is a numerical constant.

We shall postpone the proof of that lemma until later and complete the

Proof of Proposition 9.1. We shall fix $\zeta_0 = (\cos\theta, \sin\theta)$ and $h > 0$ with $0 \leq \theta \leq \pi/2$ and we shall verify that the Malliavin measure ν of the divisor (1.6) satisfies the Carleson condition (1.1) for the set $C_h(\zeta_0)$. Clearly this is sufficient to complete the proof of the proposition because the configuration that we are considering is invariant by transformations of the form $(z_1, z_2) \rightarrow (e^{i\varphi_1}z_1, e^{i\varphi_2}z_2)$. We distinguish two cases.

Case 1. $\sqrt{h} \leq 1/2000(C_0 + 1)\,|\sin\theta|$.

It follows then from (9.4) that

$$(9.5)\qquad \operatorname{diam}\,(X(\theta, h)) \leq \frac{\sin^2\theta}{100}\,.$$

But we also have:

$$(9.6)\qquad 1 - 2\sin^2\frac{\theta}{2} = \cos\theta \in X(\theta, h)\,.$$

But then from conditions (9.1) (9.5) and (9.6) it follows that

$$\text{(9.7)} \qquad \operatorname{Card}\{j \mid \alpha_j \in X(\theta, h)\} \leqq C$$

where C depends only on the original sequence $\{\alpha_j\}_{j=1}^{\infty}$. From (9.7) and (8.4) it follows clearly then that

$$\nu(C_h(\zeta_0)) \leqq Ch^2$$

and we are done.

Case 2. $\sqrt{h} \geqq 1/2000(C_0 + 1)\sin\theta$.
In that case

$$\operatorname{diam}(X(\theta, h)) \leqq c_2 h$$

where c_2 is numerical. But this together with (9.6) implies then that

$$X(\theta, h) \subset \{z_1 \in \boldsymbol{C};\ |1 - z_1| \leqq C_3 h\}$$

and therefore that

$$C_h(\zeta_0) \subseteqq C_{c_3 h}(1)\ .$$

We deduce therefore that

$$\nu(C_h(\zeta_0)) \leqq \nu(C_{c_3 h}(1)) \leqq \sum_{|1-\alpha_j| \leqq c_3 h} \nu\{z_1 = \alpha_j\}$$

and if we use then (8.3) and (9.2) we obtain that

$$\nu(C_h(\zeta_0)) \leqq Ch^2$$

and we are done again.
It remains to give the

Proof of Lemma 9.1. Let $\zeta_0 = (\cos\theta, \sin\theta)$ and h be as in the lemma. For $a = (a_1, a_2) \in \boldsymbol{C}^2$ arbitrary let us denote by $L(a)$ the complex line that is represented parametrically by

$$z_1 = a_1 - z\sin\theta$$
$$z_2 = a_2 + z\cos\theta \qquad z \in \boldsymbol{C}\ .$$

$L(a)$ passes then through a and is perpendicular to the vector ζ_0 in $\boldsymbol{C}^2$. Using the lines $L(a)$ we can then fibrate $C_h(\zeta_0)$ as following

$$\text{(9.8)} \qquad C_h(\zeta_0) = \bigcup_{a \in \Delta_h(\zeta_0)} L'(a)$$

where we denote by

$$\Delta_h(\zeta_0) = \{a \in B;\ \|a - \zeta_0\| \leqq h\}, \quad L'(a) = L(a) \cap B\ .$$

We clearly have

$$\text{(9.9)} \qquad \begin{aligned} &\operatorname{diam}(\Delta_h(\zeta_0)) \leqq 2h \\ &\operatorname{diam}(L'(a)) \leqq 100\sqrt{h} \quad \forall a \in \Delta_h(\zeta_0)\,. \end{aligned}$$

Therefore

$$\text{(9.10)} \quad \operatorname{diam}(\text{projection of } L'(a) \text{ on } \{z_2 = 0\}) \leqq 100\sqrt{h}\,|\sin\theta|\,.$$

Our lemma now follows from (9.8) (9.9) and (9.10).

We are now in a position to give a proof of Theorem 1.2. To do that we shall need to recall first a few well known facts about $H^p(B)$ functions.

PROPOSITION. (*H.R.*) *Let* $p > 0$ *and let* $f(z_1, z_2) \in H^p(B)$, *then the function* $\varphi(z) = f(z, 0)$ *satisfies the following condition*:

$$(A_p) \qquad i\int_{|z|<1} |\varphi(z)|^p dz \wedge d\bar{z} < +\infty\,.$$

Conversely if $\varphi(z)$ *is a function of one variable* (*defined for* $|z|<1$) *that satisfies* (A_p) *for some* $p > 0$ *then the function*

$$f(z_1, z_2) = \varphi(z_1) \in H^p(B)\,.$$

Furthermore if $\varphi(z)$ *is as before and does not vanish at the origin then the sequence of its zeros* $\{\alpha_j \in C\}_{j=1}^{\infty}$ *satisfies*

$$\text{(9.11)} \qquad \sum_{j=1}^{N} (1 - |\alpha_j|) \leqq \frac{1}{p}\log(N+1) + C(\varphi);\ N \geqq 1$$

where $C(\varphi)$ *is a constant that depends on* φ *and where in* (9.11) *we can use any ordering of the sequence* $\{\alpha_j\}_{j=1}^{\infty}$ *that does not decrease the moduli* (i.e., $|\alpha_1| \leqq |\alpha_2| \leqq \cdots$).

The first part of this proposition is well known and easy to verify. Results of that kind were first brought to light by W. Rudin ([13] 3.4.4). The part about the zeros is a result of C. Horowitz and is an easy consequence of Jensen's formula (cf. [8] 3.9). We can now give the

Proof of Theorem 1.2. Let p_0 be as in Theorem 1.2. We first construct a sequence $\{\alpha_j \in C; |\alpha_j| < 1, j = 1, 2, \cdots\}$ that satisfies condition (9.1) for some C but for which (9.11) fails if $p = p_0$. This is very easy to do. It suffices then to set

$$\tilde{M} = \bigcup_{i=0}^{\infty} (\{z_1 = \alpha_i\} \cap B)$$

and apply Proposition 9.1 and Proposition (H.R) to obtain the required divisor.

EXAMPLE 2. Let $l_n \subset C^2$ $n = 1, 2, \cdots$ be a sequence of complex lines in C^2 and let us denote by $d_n = \operatorname{diam}(l_n \cap B)$ $(n \geqq 1)$ and by

$$\tilde{M} = \bigcup_{n=1}^{\infty} (l_n \cap B)$$

the corresponding divisor. Let us also denote by t the Lelong current associated to $\tilde{M}$ and by

$$\sigma = |t|$$

its absolute value which is a measure equivalent to the 2-dimensional Lebesgue measure on M. We have then

PROPOSITION 9.2. *Let $\tilde{M}$ be as above and let us assume that there exists $\varepsilon_0 > 0$ such that*

(9.12) $$d_n/d_{n+1} \geqq 1 + \varepsilon_0 \qquad n = 1, 2, \cdots .$$

The measure σ satisfies then

(9.13) $$\sigma(C_h(\zeta_0)) \leqq Ch; \ \forall \zeta_0 \in \partial B, \ h > 0$$

where C depends only on ε_0.

From (9.13) it follows in particular that the measure $\delta |T| = \tilde{\nu}_2$ is Carleson.

By combining the above proposition with, say, Lemma 8.1 we conclude

PROPOSITION 9.3. *There exists a divisor $\tilde{M}$ in B that does not satisfies the* U.B. *condition but for which never the less the measure $\tilde{\nu}_2 = \delta |t|$ (t being the associated Lelong current) is Carleson.*

Proof of Proposition 9.2. We shall suppose, as we may, that $\zeta_0 = 1$, and we shall denote by σ_n the 2-dimensional Lebesgue measure on $l_n \cap B$ $(n \geqq 1)$. We have then

$$\sigma(C_h(1)) = \sum_{n=1}^{\infty} \sigma_n(C_h(1)) = \sum_{d_n \leqq \sqrt{h}} + \sum_{d_n > \sqrt{h}} = A + B .$$

By our hypothesis we have

$$A \leqq \pi \sum_{d_n \leqq \sqrt{h}} d_n^2 \leqq Ch .$$

Similarly by our hypothesis and Proposition 8.1 we have

$$B \leqq C \sum_{d_n > \sqrt{h}} d_n^{-2} h^2 = h^2 \sum_{d_n > \sqrt{h}} d_n^{-2} \leqq Ch .$$

The above two estimates complete the proof of the proposition.

EXAMPLE 3. Let us denote here by d_Φ $(d_\Phi(a, b) a, b \in B)$ the Bergman metric in the ball B (cf. (1.4)).

Let $\mathfrak{Z}$ be a sequence of points

$$\mathfrak{Z} = \{z_i \in B;\ i = 1, 2, \cdots\} \tag{9.14}$$

concerning $\mathfrak{Z}$ we shall make two definitions.

We shall say that $\mathfrak{Z}$ satisfies the Carleson interpolation condition if the measure

$$\mu = \sum_{j=1}^{\infty} (1 - \|z_i\|)^2 \delta_{z_i}$$

is a Carleson measure in B (cf. [1], [19] for the motivation and the significance of the above definition).

We shall say that $\mathfrak{Z}$ is N-separated where $N \geqq 1$ is a positive integer if the following two conditions are verified

(α) $d_\Phi(z_i, z_j) \geqq N$, $\forall i \neq j$.

(β) There exists a fixed positive integer k depending on $\mathfrak{Z}$ such that

$$\mathfrak{Z} \subset \bigcup_{p=0}^{+\infty} \{z \in B;\ 2^{-(pN+k+1)} \leqq 1 - \|z\| \leqq 2^{-(pN+k)}\} .$$

In the above definition, if we denote by

$$\mathfrak{Z}_p = \{z \in \mathfrak{Z};\ 2^{-(pN+k+1)} \leqq 1 - \|z\| \leqq 2^{-(pN+k)}\}$$

it is clear that $\mathfrak{Z}_p$ is finite and that

$$\mathfrak{Z} = \bigcup_{p=0}^{\infty} \mathfrak{Z}_p . \tag{9.15}$$

The point of the second definition is the following

LEMMA 9.2. *Let $\mathfrak{Z}$ be a sequence of points in B that satisfies the Carleson interpolation condition, and let $N \geqq 1$ be a positive integer, we can decompose then $\mathfrak{Z}$ into finitely many sequences*

$$\mathfrak{Z} = \mathfrak{Z}^{(1)} \cup \mathfrak{Z}^{(2)} \cup \cdots \cup \mathfrak{Z}^{(s)}$$

such that each $\mathfrak{Z}^{(k)}$ $k = 1, 2, \cdots s$ is N-separated.

The proof is trivial and will be left to the reader.

We can state now

PROPOSITION 9.4. *Let $\mathfrak{Z}$ be a sequence of points in B as in* (9.14) *that satisfies the Carleson interpolation condition, there exists then $\{l_j \subset \boldsymbol{C}^2;\ j = 1, 2, \cdots\}$ a sequence of complex lines in $\boldsymbol{C}^2$ such that the divisor $\bigcup_{j=1}^{\infty}(l_j \cap B)$ satisfies the* U.B. *condition and such that $z_i \in l_i$ $(i = 1, 2, \cdots)$.*

From that proposition we have the following

COROLLARY. *Let $\mathfrak{Z}$ be as in the proposition, there exists then some $p > 0$ and some $0 \neq f \in H^p(B)$ such that $f^{-1}(0) \supset \mathfrak{Z}$.*

To prove the Proposition 9.4 we shall need the following

LEMMA 9.3. *There exists $c > 10$ a numerical constant such that for all $N > c$ and all $\mathfrak{Z}$ N-separated sequence there exists*

$$\{l_u \subset \boldsymbol{C}^2;\ u \in \mathfrak{Z}\}$$

a sequence of lines that satisfy the following conditions

(i) $u \in l_u$, $\forall u \in \mathfrak{Z}$

(ii) $l_u \cap B \subset C_{c(1-\|u\|)}(u/\|u\|)$, $\forall u \in \mathfrak{Z}$

(iii) *For all $\zeta_0 \in \partial B$ and $h > 0$ there exists at most one line l_u $(u \in \mathfrak{Z})$ that satisfies*

$$l_u \cap C_h(\zeta_0) \neq \phi;\ l_u \not\subset C_{ch}(\zeta_0)\ .$$

We shall postpone the proof of Lemma 9.3 until later and complete the

Proof of Proposition 9.4. Let $\mathfrak{Z}$ be as in Proposition 9.4 by decomposing then $\mathfrak{Z}$ into finitely many subsequences each satisfying the conditions of Lemma 9.3 (this can be done by Lemma 9.2) we can suppose without loss of generality that $\mathfrak{Z}$ itself satisfies the conditions of that lemma.

Let then

$$\{l_u;\ u \in \mathfrak{Z}\}$$

be the family of complex lines constructed in Lemma 9.3. We claim that the family of lines satisfies conditions (8.18) and (8.19) of Proposition 8.2 and that therefore the divisor $\bigcup_{u\in\mathfrak{Z}}(l_u \cap B)$ is a U. B. divisor.

Condition (8.19) follows trivially from (iii). Condition (ii) on the other hand implies that

$$\operatorname{diam}(l_u \cap B) \leqq c(1 - \|u\|)^{1/2} \quad \forall u \in \mathfrak{Z} \tag{9.16}$$

and (9.16) and the definition of the Carleson interpolation condition imply then that condition (8.18) is also verified. This completes the proof.

To give a proof of Lemma 9.3 we shall need the following

LEMMA 9.4. *For all $z\in B$ such that $t=1-\|z\|\leqq c^{-1}$ (c is a numerical constant $c=10^{1000}$ will certainly do) and all $l\subset C^2$ complex line in C^2 there exists a complex line $l_1\subset C^2$ such that*

$$z\in l_1;\ l_1\cap B\subset C_{ct}\Big(\frac{z}{\|z\|}\Big)\ ;$$

$$\inf\{|1-z;\ \bar{u}|;\ z\in l,\ u\in l_1,\ z, u\in\partial B\}\geqq c^{-1}t\ .$$

Proof. A direct proof of the above lemma can, no doubt, be given. The easiest way to prove it however is to use the generalized Cayley transformations and to pass to the Siegel upper half space S given by (8.20). Using then the natural dilation structure of S given by (8.21) we can assume that the point z in our lemma above becomes the point $I=(i,0)\in S$.

It is enough then to show that there exists a numerical constant c such that for all complex line l in C^2 there exists another complex line l_1 such that

$$I\in l_1;\ l_1\cap S\subset\{u=(u_1,u_2)\in C^2;\ |u_1|<c\}$$
$$d(l\cap\partial S,\ l_1\cap\partial S)>c^{-1}$$

where we denote by $d(a,b)$ the natural distance function on ∂S given by $d(a,b)=|a_1-b_1|+|a_2-b_2|^2$.

That fact follows by an easy compactness argument and no direct computation is needed. This completes the proof of Lemma 9.4.

Proof of Lemma 9.3. Let $\mathfrak{Z}$ be as in Lemma 9.3 were c is large ($c=10^{1000}$ will certainly do) and let

$$\mathfrak{Z}=\bigcup_{p=1}^{\infty}\mathfrak{Z}_p$$

the decomposition (9.15) of $\mathfrak{Z}$ into its successive "layers".

We shall then construct a sequence of lines $\{l_u;\ u\in\mathfrak{Z}\}$ such that

$$u\in l_u;\ l_u\cap B\subset C_{c(1-\|u\|)}\Big(\frac{u}{\|u\|}\Big);\ \forall u\in\mathfrak{Z} \tag{9.16}$$

and such that for two distinct $a, b\in\mathfrak{Z}$ we have

$$\begin{aligned}\text{(9.17)}\qquad \inf\{|1 - z.\bar{u}|;\ z \in l_a \cap \partial B,\ u \in l_b \cap \partial B\} \\ \geqq c^{-1} \min[(1 - \|a\|),\ (1 - \|b\|)]\,.\end{aligned}$$

The above lines, then, will clearly satisfy the conditions of Lemma 9.3 (possibly with a larger c).

The above construction of lines will be done inductively on the layers $\mathfrak{Z}_p$ of $\mathfrak{Z}$.

Indeed let us suppose that lines

$$\left\{l_u;\ u \in \bigcup_{q=0}^{p_0} \mathfrak{Z}_p\right\}$$

have been constructed for some $p_0 \geqq 1$ and that they satisfy (9.16) and (9.17). It is then very easy, using the hypothesis and Lemma 9.2 to construct finitely many more lines

$$\{l_u;\ u \in \mathfrak{Z}_{p_0+1}\}$$

such that the lines

$$\left\{l_u;\ u \in \bigcup_{p=0}^{p_0+1} \mathfrak{Z}_p\right\}$$

still satisfies conditions (9.16) and (9.17). This is the inductive step and it completes the construction.

10. The optimal nature of the uniform Blaschke condition. Let $\Omega \subseteqq C^n$ be as in §1 and let μ be a Radon measure in Ω. We shall say that μ satisfies the C_α $(\alpha > 0)$ condition if

$$|\mu|\,(\widetilde{B_t(\zeta_0)}) \leqq Ct^{n+\alpha};\ \zeta_0 \in \partial\Omega,\ 0 < t < t_0$$

where C is independent of t and ζ_0. We shall say that T a (1, 1) current as in (2.1) satisfies the U.B.$_\alpha$ condition if the measure

$$\delta\,|T| + \delta^{1/2}(|T \wedge \partial\rho| + |T \wedge \bar{\partial}\rho|) + |T \wedge \partial\rho \wedge \bar{\partial}\rho|$$

is a C_α-measure in Ω.

It is very easy to see that the condition C_α, if postulated at the beginning, it propagates in a very natural way right through the paper. In particular if T is a real d-closed current that satisfies the U. B.$_\alpha$ condition for some $\alpha > 0$ then there exists a real solution W of the equation

$$i\partial\bar{\partial} W = T$$

such that $W|_{\partial\Omega} \in \Lambda_\alpha(\partial\Omega)$.

The above facts may be of some mild interest, unfortunately however, they do not seem to have any significance in complex analysis because of the following

PROPOSITION 10.1. *Let $\tilde{M}\subset\Omega$ be a divisor in Ω (where Ω is as above) and let t be the Lelong current associated with the divisor. Let us also suppose that the measure $\tilde{\nu}_1=|t\wedge\partial\rho\wedge\bar{\partial}\rho|$ satisfies the condition*

$$\tilde{\nu}_1(\widetilde{B_t(\zeta_0)})=o(t^n)\ ; \tag{10.1}$$

as $t\to 0$, uniformly in $\zeta_0\in\partial\Omega$. Then the divisor $\tilde{M}$ is empty ($M=\phi$).

Proof (outline). To simplity life I shall suppose that $\Omega=B\subset C^2$ the unit ball in C^2. Let M satisfy condition (10.1) in B. Then by considerations analogous to the ones in the proofs of Propositions 2.1, 2.2 and 2.3 we conclude that ν the Malliavin measure of the divisor [cf. (1.6)] satisfies the condition

$$\nu(\widetilde{B_t(\zeta_0)})=o(t^2);\ \text{as } t\longrightarrow 0 \text{ uniformly in } \zeta_0\in\partial B\ .$$

But then by rerunning through the proof of Proposition 1.1 we conclude that

$$\lim_{G\ni g\to\infty}\|\tilde{M}_g\|_\beta=0 \tag{10.2}$$

where G is as in Proposition 1.1 and $g\to\infty$ means that we tend to the point at infinity of the locally compact (but not compact) space G (i.e., that we eventually leave every compact subset). This last fact (10.2) is however only possible if $M=\phi$.

Indeed suppose that $M\neq\phi$ and let us choose a sequence of points such that:

$$z_n\in M^*;\ n=1,2\cdots\ \|z_n\|\xrightarrow[n\to\infty]{}1$$

this clearly is always possible. Let also $g_n\in G$ ($n\geqq 1$) be a sequence of holomorphic automorphisms of G such that $g_n(z_n)=0$ for all $n\geqq 1$. Clearly then

$$0\in M^*_{g_n}\ n\geqq 1;\ g_n\xrightarrow[n\to\infty]{}\infty\ \text{in } G\ .$$

But by an easy application of Wirtinger's inequality [cf. [17] Theorem (B)] we see that the fact that $0\in M^*_{g_n}(n\geqq 1)$ implies that

$$\inf_{n\geqq 1}\|\tilde{M}_{g_n}\|_\beta\geqq 10^{-10}\ .$$

This completes the proof.

APPENDIX. In this Appendix I shall give a short guide of how to read the relevant passages of G. de Rham [3] so as to obtain formula (3.5).

Observe, first of all, that in the text I did not specify the exact value of $H_\theta(z, t) \in l_\theta(z)$ $(0 \leqq t \leqq 1)$ (i.e., I did not specify the speed with which the point z slides along $l_\theta(z)$ during the homotopy). The reason for that is the operator H_θ that is defined from that homotopy on the space Λ (cf. (3.4)) does not depend on that speed and is invariant by a change of parametrization along the different rays $l_\theta(z)$. This will, hopefully, become apparent in the next few lines. At any rate the reader who does not wish to overexert himself can think that the parametrization, that is fixed once and for all, is, say, the linear parametrization given in vector notations by:

$$H_\theta(z, t) = (1-t)z + t\varphi_\theta(z) \in \bar{B}\backslash\{0\};\ z \in B\backslash\{0\},\ 0 \leqq t \leqq 1\,.$$

Observe also that we called currents in this paper are "des courants impairs" in de Rham's terminology (cf. [3] § 8 p. 39, 2nd edition).

Let us now apply the construction of G. de Rham ([3] § 14 "FORMULES d'HOMOTOPIE" p. 68, 2nd edition) and let us set

$$W = V = B\backslash\{0\}$$
$$\mu(t, z) = \mu^{(\varepsilon)}(t, z) = H_\theta(z, (1-\varepsilon)t)\ \ 0 \leqq t \leqq 1$$

where $0 < \varepsilon < 1$.

Let us also suppose that T is a current with compact support in $B\backslash\{0\}$. G. de Rham's formula (3) reads then:

$$\mu_1^{(\varepsilon)}T - \mu_0^{(\varepsilon)}T = bM_\varepsilon T + M_\varepsilon bT$$

where M_ε is the M operator of G. de Rham that corresponds to the family of mappings $\mu^{(\varepsilon)}(t, z)$. We shall give an explicit formula for that operator at the end of this appendix.

Let us observe that $\mu_0^{(\varepsilon)}T = T$ and that $\mu_1^{(\varepsilon)}T \xrightarrow[\varepsilon\to 0]{} 0$ (this is because the support of $\mu_1^{(\varepsilon)}T$ is pushed to ∂B i.e., to infinity as $\varepsilon \to 0$). It is also easy to verify that $M_\varepsilon T$ converges as $\varepsilon \to 0$ for all T with compact support (the verification is done directly on the definition of the operator M). If we set then $\mathrm{MT} = \lim_{\varepsilon\to 0} M_\varepsilon T$ we get

$$-T = bMT + MbT \tag{11.1}$$

and this is just our formula (3.5) when we set $H_\theta = w \circ M$ where w is the operator defined in ([3] § 11, p. 54, 2nd edition). w is in fact just multiplication by ± 1 depending on the degree of the current.

Let us now consider the general case, i.e., the case when $T \in \Lambda$ is not assumed to have compact support. In that case our hypothesis on the support of T says that $\inf\{\|z\|;\ z \in \operatorname{supp} T\} > 0$, from this it follows that the conditions of [3] § 14 (second paragraph of p. 70, 2nd edition) are verified for all the mappings $\mu^{(\varepsilon)}$ so that the formula (3) of [3] § 14 still holds. The same passage to the limit as before still holds, again by the hypothesis on the support of T, and that completes the construction.

We shall finally give a completely explicit description of the operator M and therefore also of H_θ.

Let

$$T = \sum_I T_I dy_I$$

be a coordinate expression of $T \in \Lambda$ in some coordinate system, where the T_I's are identified with distributions, and let us denote by $X = X(t)$ $0 < t < 1$ the characteristic function of $I' = (0, 1)$ which we shall identify "qua distribution" to the Lebesgue measure on [0, 1]. We shall denote then by

$$I'T = \sum_I (X \otimes T_I) dy_I$$

where $X \otimes T_I$ is the tensor product of the two distributions. $I'T$ is then a current on the product manifold $I' \times (B \setminus \{0\})$ (observe that I follow very closely the notations of G. de Rham who denotes $IT = I(t)T(y)$ cf. [3] § 14 "FORMULES PRELIMINAIRES").

Let us also denote by

$$\mu: I' \times (B \setminus \{0\}) \longrightarrow B \setminus \{0\}$$

the mapping defined by $\mu(t, z) = H_\theta(z, t)$.

The operator M that satisfies (11.1) can then be defined by

$$MT = \mu(I'T)$$

i.e., the direct image of the current $I'T$ by μ (cf. [3] § 11 p. 55, 2nd edition).

To finish up let us suppose that T is the integration current on some chain element c regularily embedded in $B \setminus \{0\}$ [cf. [3] § 8, Example 1] sometimes such a current is denoted by $T = [c]$. It is clear that then $H_\theta(T)$ is also an integration current on a chain (element) that is no longer compact but goes all the way to the boundary ∂B. The support of that chain element is

$$\operatorname{supp} H_\theta(T) = \bigcup_{x \in \operatorname{supp} c} l_\theta(x) .$$

(Observe that what I called support of a chain element above is just the set $\pi(\Pi)$ in the notations of [3] § 6.)

It is worth observing also, that the easiest way to see that the operator H_θ does not depend on the particular parametrization along the lines $l_\theta(z)$, is to go through chain elements. Indeed by the above it follows that $H_\theta(T)$ is independent of the parametrization in question when $T = [c]$ for some chain element c. But the most general current can be thought as a weak integral of "infinitesimal chain elements", and that of course gives the result.

References

1. L. Carleson and J. Garnett, *Interpolation sequences and separation properties*, J. Anal. Math., **28** (1975), 273–299.

2. L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math., **80** (1958), 921–930.

3. G. de Rham, *Variétés Différentielles*, Paris, Hermann, Actualités Sci. Indust., 1222.

4. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math., **129** (1972), 137–193.

5. J. Garnett, *Bounded analytic functions*, (forthcoming book).

6. R. Harvey, *Holomorphic chains and their boundaries*, Proc. Symposia Pure Math. XXX, Amer. Math. Soc.

7. G. M. Henkin, *Solutions with estimates of the H. Levy and Poincaré-Lelong equations. Construction of functions of the Nevanlinna class with prescribed zeros in strictly pseudoconvex domains*, Dokl. Akad. Mauk SSSR, **224** n°4 (1975), 3–13.

8. C. Horowitz, *Zeros of functions in Bergman spaces*, Duke Math. J., **41** (1974), 693–710.

9. F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), 415–426.

10. P. Lelong, *Fonctionnelles analytiques et fonctions entières (n variables)*, Presses Univ. Montréal, (1968).

11. ———, *Fonctions Plurisousharmoniques et Formes Différentielles Positives*, Gordon and Breach, 1968.

12. P. Malliavin, *Fonctions de Green d'un ouvert strictement pseudoconvexe et classe de Nevanlinna*, C. R. Acad. Sci., Paris, **278** (1974), 141–144.

13. W. Rudin, *Function Theory in Polydiscs*, Benjamin (Math. Lecture Note series), 1969.

14. ———, *Zeros of holomorphic functions in balls*, Koninkl. Nederl. Akad. van Wetenschappen. Amsterdam Proc. (A) 79 n°1. Indag. Math., **38** n°1 (1976), 57–65.

15. H. Skoda, *Valeurs au bord pour les solutions de l' opérateur d' et caractérisation de zêros des fonctions de la classe de Nevanlinna*, Bull. Soc. Math. France, **104** (1976), 225–299.

16. E. M. Stein, *Boundary behavior of holomorphic functions of several complex variables*, Princeton Univ. Press, (1912), (Math. Notes).

17. G. Stolzenberg, *Volumes, limits and extensions of analytic varieties*, Springer Verlag, Lecture Notes in Math., **19**(1966).

18. N. Th. Varopoulos, *BMO functions and the $\bar\partial$-equation*, Pacific J. Math., **71** (1977), 221–273.

19. ———, *Sur un problème d'interpolation*, C. R. Acad. Sci., Paris, **274** (1972), 1539–1542.

20. A. Zygmund, *Trigonometric Series*, London, New York, Cambridge Univ. Press, (1959).

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