Pacific Journal of Mathematics

ON THE BEHAVIOR OF A CAPILLARY SURFACE AT A RE-ENTRANT CORNER

Nicholas Jacob Korevaar

Vol. 88, No. 2 April 1980

PACIFIC JOURNAL OF MATHEMATICS
Vol. 88, No. 2, 1980

ON THE BEHAVIOR OF A CAPILLARY SURFACE AT A RE-ENTRANT CORNER

NICHOLAS J. KOREVAAR

Changes in a domain's geometry can force striking changes in the capillary surface lying above it. Concus and Finn [1] first studied capillary surfaces above domains with corners, in the presence of gravity. Above a corner with interior angle θ satisfying $\theta < \pi - 2\gamma$, they showed that a capillary surface making contact angle γ with the bounding wall must approach infinity as the vertex is approached. In contrast, they showed that for $\theta \geqq \pi - 2\gamma$ the solution $u(x, y)$ is bounded, uniformly in θ as the corner is closed. Since their paper appeared, the continuity of u at the vertex has been an open problem in the bounded case. In this note we show by example that for any $\theta > \pi$ and any $\gamma \neq \pi/2$ there are domains for which u does not extend continuously to the vertex. This is in contrast to the case $\pi > \theta > \pi - 2\gamma$; here independent results of Simon [5] show that u actually must extend to be C^1 at the vertex.

We consider bounded domains Ω in $\boldsymbol{R}^2$ with piecewise smooth boundaries $\partial\Omega$, and functions $u(x, y)$ satisfying

(i) $\operatorname{div} Tu = 2H(u) = \kappa u$ in Ω; $Tu = Du/\sqrt{1 + Du^2}$, $H(u) =$ mean curvature of the surface $z = u(x, y)$, $\kappa > 0$.

(ii) $Tu \cdot n = \cos\gamma$ on the smooth part of $\partial\Omega$; $0 \leqq \gamma \leqq \pi$, $n =$ exterior normal to $\partial\Omega$.

Physically u describes the capillary surface formed when a vertical cylinder with horizontal cross section Ω is placed in an infinite reservoir of liquid having rest height $z = 0$. Then

$$\kappa = \frac{\rho g}{\sigma} ,$$

where

$\rho =$ density of liquid
$g =$ (downward) acceleration of gravity
$\sigma =$ surface tension between liquid and air.

$$\cos\gamma = \frac{\sigma_1}{\sigma} ,$$

where

$\sigma_1 =$ surface attraction between liquid and cylinder.

Geometrically γ is the contact angle between the capillary surface and the bounding cylinder; it is the angle between the downward normal of the surface $z = u(x, y)$, and the exterior normal of the cylinder $\partial\Omega \times \boldsymbol{R}$.

If $\gamma = \pi/2$, the only solution to (i) and (ii) is $u \equiv 0$. If $\gamma \neq \pi/2$, by considering either u or $-u$, we make the usual assumption that $0 \leqq \gamma < \pi/2$. This is the case in which the surface rises to meet the cylinder, or "wets" it.

Let θ and γ satisfy

$$\pi < \theta \leqq 2\pi , \qquad 0 < \gamma < \pi/2 .$$

We will construct a domain for which a bounded solution u to (i) and (ii) exists, but having a corner of interior angle θ at which there is a jump discontinuity in u. (The arguments can be modified to include the case $\gamma = 0$.)

Determine the domain scale by fixing $R > 0$ (Fig. 1). Since $\theta > \pi$, we can pick θ_1 and θ_2, satisfying

$$\theta_1 > \pi - \gamma , \qquad \pi > \theta_2 > \gamma , \qquad \theta_1 + \theta_2 = \theta .$$

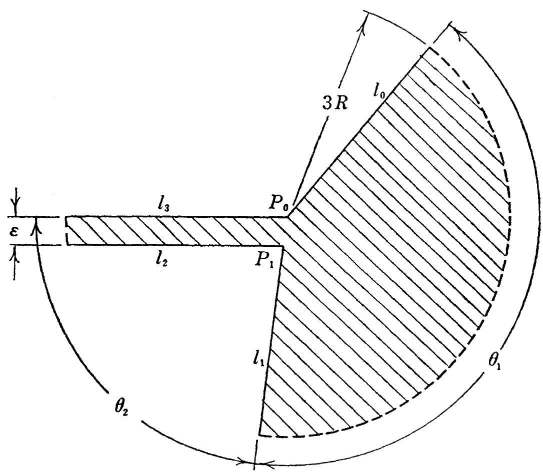

FIGURE 1. The intersection of Ω_ε with the disc of radius $3R$

$\theta_1 > \pi - \gamma$	$P_0 = (0, 0)$	$l_0 = \{y \cos \theta = x \sin \theta\}$
$\pi > \theta_2 > \gamma$	$P_1 = (-\varepsilon \cot \theta_2, -\varepsilon)$	$l_1 = \{y \cos \theta_2 = x \sin \theta_2\}$
$\theta_1 + \theta_2 = \theta > \pi$		$l_2 = \{y = -\varepsilon\}$
		$l_3 = x$-axis

For positive ε less than $R \sin \theta_2$, let Ω_ε be a bounded domain, of which the intersection with $B_{3R}(0)$ is shown in Fig. 1, and which has C^4 boundary except at P_0 and P_1. ($B_{3R}(0)$ is the disc of radius $3R$ centered at the origin.)

LEMMA 1. *There exists a unique solution to* (i) *and* (ii) *in any* Ω_ε. *It is bounded above and nonnegative.*

Proof. Because Ω_ε is C^2, except for a finite number of re-entrant corners, it satisfies a uniform internal sphere condition with contact angle γ, for any γ. Therefore it is admissible in the sense of Finn and Gerhardt [4]. Thus there is a bounded, nonnegative, real analytic function $u_\varepsilon(x, y)$ in Ω_ε, satisfying (i). Because u is energy minimizing in the sense of Emmer [3], the regularity theory of Simon and Spruck [6] implies that everywhere the boundary is C^4, u_ε extends to be at least C^2, and satisfies (ii). Uniqueness follows from a maximum principle of Concus and Finn [2].

We are interested in the behavior of u_ε near P_0, as ε approaches 0. Lemma 2 will show that u_ε stays uniformly bounded in one sector near P_0, and Lemma 3 show that in another sector it gets uniformly large. It follows that u_ε eventually has a jump discontinuity at P_0.

Let I_ε be the subdomain of Ω_ε shown in Fig. 2. Then we have

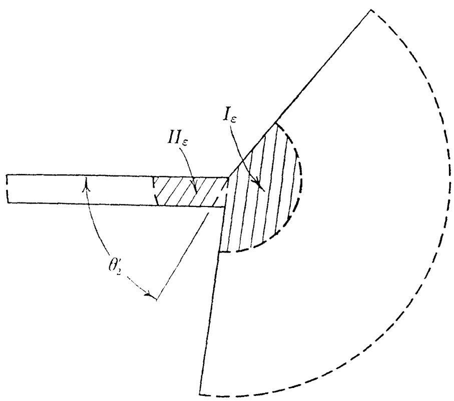

FIGURE 2. The subdomains I_ε and II_ε

$\theta_2 > \theta_2' > \gamma$

$$B_R(0) = \{x^2 + y^2 < R^2\}$$
$$I_\varepsilon = B_R(0) \cap \{y \cos\theta > x \sin\theta\} \cap \{y \cos\theta_2 < x \sin\theta_2\}$$
$$II_\varepsilon = B_R(0) \cap \{y < 0\} \cap \{y > -\varepsilon\} \cap \{y \cos\theta_2' > x \sin\theta_2'\}$$

LEMMA 2. *u_ε is uniformly bounded in I_ε, independently of ε.*

Proof. In this and the following lemma the basic tool is a comparison method of Concus and Finn [2] for surfaces of known mean curvature and contact angle.

Consider circles of radius R which either lie entirely in Ω_ε or contact $\partial\Omega_\varepsilon$ only at a point of tangency. (In particular, do not allow them to have contact at P_0 or P_1.) If $\theta_1 < \pi$, also allow circles which intersect $\partial\Omega_\varepsilon$ at two points on $l_0 - P_0$, making an angle of no more than $\pi - \theta_1$ with l_0 at these intersections. Every point in I_ε lies interior to at least one of these circles (see Fig. 3).

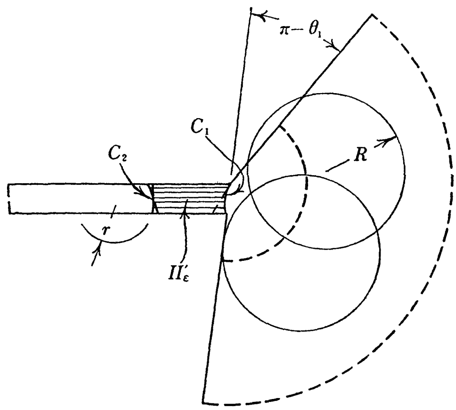

FIGURE 3. Equatorial circles near I_ε
The region II'_ε above which v is defined.

In $\boldsymbol{R}^3$ consider a closed lower hemisphere L with equatorial circle E, so that the projection $\pi(E)$ of E onto $\boldsymbol{R}^2$ is one of the above circles (see Fig. 4). If L contacts $l_0 \times \boldsymbol{R}$, then along the arc of intersection A the contact angle γ_L equals the angle between $\pi(E)$ and l_0. Thus $\gamma_L \leqq \pi - \theta_1 < \gamma$. Because P_0 and P_1 are the only two boundary points at which u_ε may not be C^2, u_ε is C^2 on $\overline{\pi(L) \cap \Omega_\varepsilon}$.

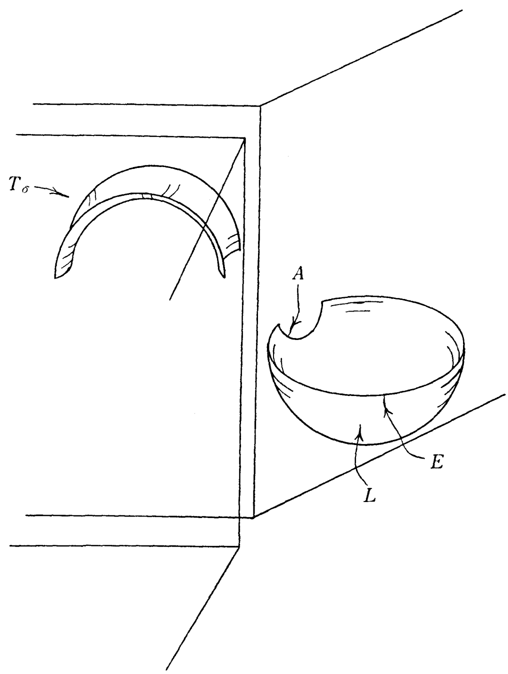

FIGURE 4. A lower hemisphere L contacting $\partial\Omega_\varepsilon \times \boldsymbol{R}$ along A, with contact angle less than γ. The "undeside" T_δ of a torus, contacting $\partial\Omega_\varepsilon \times \boldsymbol{R}$ with contact angle greater than γ.

Raise L until it lies above the bounded surface $\{z = u_\varepsilon(x, y)\}$. Lower L until the two surfaces first contact each other. Let $Q_0 = (x_0, y_0, u_\varepsilon(x_0, y_0))$ be a point of first contact.

Q_0 is not on E. This is because L is vertical along E whereas u_ε is C^2.

Q_0 is not on A: The end points of A are on E and are already excluded. If Q_0 was not an end point, the traces of the two surfaces on $l_0 \times \boldsymbol{R}$ would be tangent there. Since L contacts $l_0 \times \boldsymbol{R}$ at a steeper angle than the capillary surface, it would follow that L was actually below the surface in the interior normal direction from Q_0. Thus Q_0 would not be a point of first contact.

Thus (x_0, y_0) lies in the interior of $\pi(L) \cap \Omega_\varepsilon$. Since Q_0 is an interior point of first contact, the two surfaces are tangent there, and since L is nowhere below $\{z = u_\varepsilon(x, y)\}$, it follows that

$$H(u_\varepsilon)(x_0, y_0) \leqq \frac{1}{R} \quad \left(\text{since } \frac{1}{R} \text{ is the mean curvature of } L\right).$$

Using (i) gives:

$$u_\varepsilon(x_0, y_0) \leqq \frac{2}{\kappa R}.$$

Since L varies in height by R,

$$u_\varepsilon(x, y) \leqq \frac{2}{\kappa R} + R \quad \text{for all} \quad (x, y) \in \pi(L) \cap \Omega_\varepsilon .$$

By our previous comments this estimate holds in all of I_ε.

Fix θ_2' with $\gamma < \theta_2' < \theta_2$ and let II_ε be the subregion of Ω_ε as described in Fig. 2. Then we have

LEMMA 3. *$u_\varepsilon(x, y)$ approaches ∞ uniformly in II_ε, as ε approaches* 0.

Proof. Consider the unique circle C_1, containing P_0, making an angle θ_2' with l_3 and going through P_1 if $\theta_2 \leqq \pi/2$, or through $(0, -\varepsilon)$ if $\theta_2 > \pi/2$. Let C_2 be a circle of the same radius translated $2R$ units to the left.

There is a unique torus in $\boldsymbol{R}^3$ containing C_1 and C_2. It is generated by rotating C_1 about an axis parallel to the y-axis and going through Q_1, the point midway between C_1 and C_2. Let II_ε' be the part of $\bar{\Omega}_\varepsilon$ on or to the left of C_1, and on or to the right of C_2 (see Fig. 3). Then in II_ε', the "underside" T of the torus is given by

$$v(x, y) = [(R - \sqrt{r^2 - (y - y_1)^2})^2 - (x - x_1)^2]^{1/2} ,$$

where $(x_1, y_1) = Q_1$ (see Fig. 4). T contacts $l_3 \times \boldsymbol{R}$ with contact angle $\theta_2' > \gamma$, and contacts $l_2 \times \boldsymbol{R}$ with contact angle of at least θ_2'. It is vertical at C_1 and C_2.

Let any $\delta > 0$ be given. In order to avoid P_0 and P_1 translate T δ units to the left and call it T_δ, as in Fig. 4. Lower T_δ beneath $\{z = u_\varepsilon(x, y)\}$, and raise it until the first contact is made. By reasoning as in Lemma 2 it follows that if $(x_0, y_0, u_\varepsilon(x_0, y_0))$ is a point of first contact, then it does not occur on the boundary of T_δ. Thus it is a point of tangency and since T_δ is nowhere above $\{z = u_\varepsilon(x, y)\}$, the mean curvature of T_δ is no bigger than that of u_ε at $(x_0, y_0, u_\varepsilon(x_0, y_0))$. But by looking at the normal curvatures for a torus, one can calculate the following inequality:

$$H(v)(x, y) \geqq \frac{1}{2}\left(\frac{1}{r} - \frac{1}{R - r}\right) \quad (x, y) \in II_\varepsilon'$$

so that

$$\operatorname{div} Tu_\varepsilon(x_0, y_0) \geqq \left(\frac{1}{r} - \frac{1}{R - r}\right)$$

or

$$u_\varepsilon(x_0, y_0) \geqq \frac{1}{\kappa}\left(\frac{1}{r} - \frac{1}{R - r}\right).$$

Since T_δ varies in height by at most R, and since δ can be chosen arbitrarily small,

$$u_\varepsilon(x, y) \geqq \frac{1}{\kappa}\left(\frac{1}{r} - \frac{1}{R - r}\right) - R \quad \text{for } (x, y) \text{ in } II_\varepsilon'.$$

Since $II_\varepsilon \subset II_\varepsilon'$ for ε small enough, the last inequality eventually holds in II_ε. Noticing that r is proportional to ε and R is fixed, the result follows.

Combining the three lemmas yields the desired result:

THEOREM. *For ε sufficiently small, the solution $u_\varepsilon(x, y)$ to the capillary problem* (i) *and* (ii) *in Ω_ε cannot be extended continuously to the vertex of the re-entrant corner of angle θ.*

Although this theorem shows that u_ε need not extend nicely to the vertex, simple experiments with glass slides placed vertically in water indicate that the capillary surface itself still extends in a regular fashion to its boundary.

References

1. P. Concus and R. Finn, *On the behavior of a capillary surface in a wedge,* Proc. Nat. Acad. Sci., **63** (1969), 292-299.

2. ———, *On capillary free surfaces in a gravitational field,* Acta Math., **132** (1974), 207-223.

3. M. Emmer, *Esistenzá, unicitá e regolaritã nelle superfici di equilibrio nei capillari,* Ann. Univ. Ferrara Sez VII, **18** (1973), 79-94.

4. R. Finn and C. Gerhardt, *The internal sphere condition and the capillarity problem,* Ann. Mat. Pura Appl. IV, **112** (1977), 13-31.

5. L. Simon, *Regularity of capillary surjaces over domains with corners,* (to appear in Pacific J. Math.).

6. L. Simon and J. Spruck, *Existence and regularity of a capillary surface with prescribed contact angle,* Arch. Rational Mech. Anal., **61** (1976), 19-34.

Received November 16, 1979.

STANFORD UNIVERSITY
STANFORD, CA 94305

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $84.00 a year (6 Vols., 12 issues). Special rato: $42.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address shoud be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics
Vol. 88, No. 2 April, 1980