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The concept of genus of an invariant, closed set A in a paracompact free G-space E is introduced for any compact Lie group G and the general result that G-genus $A = \text{cat}_B A^*$ is proven where $B = E/G$, $A^* = E/G$ and cat is short for Ljusternick-Schnirelman category. As a special case, the concept of genus (Krasnoselskii) coincides with the notion of category (Ljusternik-Schnirelman) as employed in a real or complex Banach space.

1. Introduction. The Min-Max principle in critical point theory as introduced by Ljusternik-Schnirelman [6] is based on the concept of category of a set X in an ambient space B. Krasnoselskii [5] and others [9], [1], employed the concept of genus instead of category. For example, consider the following setting. Let E denote a Banach space and observe that $\boldsymbol{Z}_2 = \{-1, 1\}$ acts freely on $E - 0$ by scalar multiplication. Let Σ denote the closed invariant (symmetric) subsets of $E - 0$. Furthermore, let $B = E - 0/\boldsymbol{Z}_2$ and for $A \in \Sigma$, set $A^* = A/\boldsymbol{Z}_2$. Then,

$$\text{cat}_B A^* = k$$

is defined to mean that there exist k sets $A_1, \cdots, A_k$ in Σ such that $A = \cup A_i$ and for each i, A_i^* is contractible to a point in B and k is minimal with this property ($k = \infty$, allowed). Thus the function γ given by

$$\gamma(A) = \text{cat}_B A^*$$

classifies the elements of Σ.

Alternatively, following Krasnoselskii [2], the statement

$$\text{genus}\, A = k$$

is defined to mean that there exists an equivariant (odd) map $f: A \to \boldsymbol{R}^k - 0$ and k is minimal with this property ($k = \infty$ means that there is no equivariant map $f: A \to \boldsymbol{R}^k - 0$, for any finite k and, as usual, $\boldsymbol{R}^k$ is Euclidean k-space).

REMARK 1.1. Actually this concept of "genus" was introduced and studied earlier by Yang [11] under the name "B-index". In fact, genus $A = B$-index $+ 1$.

The function γ' given by

$$\gamma'(A) = \text{genus } A$$

also classifies the sets in Σ. Our objective in this note is to verify that these classifications are identical in general, i.e.,

$$(1) \qquad \gamma(A) = \text{cat}_B A^* = \text{genus } A = \gamma'(A)\,, \qquad A \in \Sigma\,.$$

A special case of (1) for compact A's is contained in Rabinowitz [9]. We will verify (1) in a very general setting as follows.

Let E denote any contractible paracompact free G-space where G is a compact Lie group. Let Σ denote the closed, invariant subsets of E and set $B = E/G$. Then for $A \in \Sigma$, $\text{cat}_B A^*$ is defined as before, where A^* is the orbit space A/G. Now, set G-genus $A = k$ if there is a G-equivariant map

$$(2) \qquad f\colon A \longrightarrow \overbrace{G \circ G \circ \cdots \circ G}^{k}\,, \qquad (k\text{-fold join [7]})$$

and k is minimal with this property.

THEOREM. *For $A \in \Sigma$ we have*

$$(3) \qquad \text{cat}_B A^* = G\text{-genus } A\,.$$

Note that (1) is (in the case of infinite dimensional Banach spaces) a corollary of (3) by taking $G = \boldsymbol{Z}_2$ and observing that the k-fold join of the 0-sphere S^0 is just S^{k-1} which is the unit sphere in $\boldsymbol{R}^k$. The corresponding result to (1) for complex Banach spaces is obtained by taking $G = S^1$, unit circle of complex numbers of norm 1. We should also remark that the idea of using (2) for an "index theory" appears briefly in [2].

2. Preliminaries. Throughout G will denote a compact Lie group and $\mathscr{F}$ will denote the category of free paracompact G-spaces. An object $X \in \mathscr{F}$ may be identified with the principal bundle $p\colon X \to X/G$, where p is the natural projection to the orbit space X/G. Hence, the general theory of principal bundles over a paracompact base applies (see [4]). We will also find the following definitions convenient.

DEFINITION 2.1. A free G-space $Y \in \mathscr{F}$ is called a G-ENR (Euclidean Neighborhood Retract G-space) if

(a) there is a real representation $\varphi\colon G \to O(n)$ of G as orthogonal matrices for some n;

(b) there is an equivariant imbedding $h: Y \to \boldsymbol{R}^n$ of Y in $\boldsymbol{R}^n$, i.e., $h(gy) = \varphi(g)h(y)$;

(c) there is an invariant neighborhood U of $f(Y) \subseteqq \boldsymbol{R}^n$ and an equivariant retraction of U onto $f(Y)$, i.e., there is a map $r: U \to h(Y)$ such that $r(u) = u$ when $u \in f(Y)$ and $r(\varphi(g)u) = \varphi(g)r(u)$.

PROPOSITION 2.2. *Let $X \in \mathscr{F}$, A a closed invariant subspace of X and Y a G-ENR. Then any equivariant map $f: A \to Y$ has an equivariant extension $\bar{f}: V \to Y$, where V is an invariant neighborhood of A in X.*

Proof. We assume without loss ttha $Y \subset \boldsymbol{R}^n$ and $G \subseteqq O(n)$. Then, employing the Tietze-Gleason Extension Theorem [8], there is an equivariant extension $F: X \to \boldsymbol{R}^n$. Let U denote the invariant neighborhood of Y which admits an equivariant retraction $r: U \to Y$. Then, if $V = r^{-1}(U)$, $f = r \circ (F|V)$ is the required extension: $V \to Y$.

REMARK 2.3. The compact Lie group G is a G-ENR [8]. In fact, every compact smooth G-manifold is a G-ENR [8]. Hence, the neighborhood extension theorem (Proposition 2.2) applies for maps into these spaces. Palais [8] defines a G-ANR as a space Y which satisfies Proposition 2.2 for normal spaces X, so that every G-ENR is a G-ANR.

We also recall the notion of join. Let $Y_1, Y_2, \cdots, Y_k$ denote G-spaces and consider the space

$$(I \times Y_1) \times (I \times Y_2) \times \cdots \times (I \times Y_k) \tag{4}$$

a point of which is designated by

$$(t_1 y_1, t_2 y_2, \cdots, t_k y_k) . \tag{5}$$

Let J denote the subset of (4) consisting of points (5) with the added condition that $\Sigma t_j = 1$. Define an equivalence relation $\sim$ by setting

$$(t_1 y_1, t_2 y_2, \cdots, t_k y_k) = (t'_1 y'_1, t'_2 y'_2, \cdots, y'_k y'_k)$$

if $t_j = t'_j$ for all j and $y_j = y'_j$ whenever $t_j \neq 0$. Then we set

$$Y_1 \circ Y_2 \circ \cdots \circ Y_k = J/\sim \tag{6}$$

employing the identification topology. The action

$$G \times (Y_1 \circ \cdots \circ Y_k) \longrightarrow Y_1 \circ \cdots \circ Y_k$$

given by

$$g[t_1y_1, \cdots, t_ky_k] = [t_1gy_1, \cdots, t_kgy_k]$$

is continuous whenever the Y_j's are compact [7].

LEMMA 2.4. *Suppose Y is a free G-space, with $Y \subset \boldsymbol{R}^n$ and $G \subset O(n)$. Then, there is an equivariant imbedding*

$$f\colon Y \longrightarrow \boldsymbol{R}^{n+1}$$

with the additional property that $y_1 \neq y_2$ implies $f(y_1)$ and $f(y_2)$ are independent, i.e., *they do not lie on a line thru the origin.*

Proof. Set $f(y) = (y, ||y||^2)$, $y \in \boldsymbol{R}^n$, $||y|| =$ norm y.

This lemma is used to prove the following proposition which is essentially Lemma 2.7.9 of [8].

PROPOSITION 2.5. *If $Y_1, \cdots, Y_k$ are compact G-*ENR's, *so is the k-fold join*

$$Y_1 \circ \cdots \circ Y_k .$$

Proof. We need only show this for $k = 2$. Clearly $Y_1 \circ Y_2$ is compact. We may assume without loss, that Y_1 is a closed G_1-subspace of $\boldsymbol{R}^p$, where $G_1 \subset O(p)$ and G_1 is isomorphic to G, say by $\varphi_1\colon G_1 \to G_1$. Similarly, we may assume that there is an isomorphism $\varphi_2\colon G \to G_2 \subset O(q)$ and Y_2 is a G_2-subspace of $\boldsymbol{R}^q$.

Then, there is a natural equivariant map $\eta\colon Y_1 \circ Y_2 \to \boldsymbol{R}^p \oplus \boldsymbol{R}^q$ given by

$$\eta\colon [t_1y_1, t_2y_2] \longrightarrow t_1y_1 \oplus t_2y_2$$

where G acts on $\boldsymbol{R}^p \oplus \boldsymbol{R}^q$ via the diagonal action

$$g(y_1, y_2) = (\varphi_1(g)y_1, \varphi_2(g)y_2) .$$

Now, if we use Lemma 2.4 we may also assume that distinct points y_1, y_1' of Y_1 are independent vectors and similarly for Y_2. Then, if

$$t_1y_1 \oplus t_2y_2 = t_1'y_1' \oplus t_2'y_2'$$

we have $t_1y_1 = t_1'y_1'$ and $t_2y_2 = t_2'y_2'$. This forces

$$[t_1y_1, t_2y_2] = [t_1'y_1', t_2'y_2']$$

and η is injective, hence an imbedding. Now, suppose

$$\rho_i\colon U_i \longrightarrow Y , \qquad i = 1, 2$$

are invariant retractions where U_1, U_2 are invariant neighborhoods of Y_1 and Y_2 in $\boldsymbol{R}^p$, $\boldsymbol{R}^q$, respectively. Now, let U denote the union of all lines $L(u_1, u_2)$, $u_i \in U_i$. Thus a point $u \in U$ has the form

$$(1 - t)u_1 + tu_2 , \qquad -\infty < t < \infty .$$

Set

$$\rho((1 - t)u_1 + tu_2) = \begin{cases} \rho_1(u_1) , & \text{if} \quad t \leqq 0 \\ (1 - t)\rho_1(u_1) + t\rho_2(u_2) , & \text{if} \quad 0 \leqq t \leqq 1 \\ \rho_2(u_2) , & \text{if} \quad t \geqq 1 . \end{cases}$$

ρ: $U \to \eta(Y_1 \circ Y_2)$ is an equivariant retraction and hence $Y_1 \circ Y_2$ is a G-ENR.

The following proposition uses the obvious fact that L-S category is subadditive, i.e., if $Y = Y_1 \cup Y_2 \subset M$, where Y_i are closed in M, $i = 1, 2$, then

$$\operatorname{cat}_M Y \leqq \operatorname{cat}_M Y_1 + \operatorname{cat}_M Y_2 .$$

PROPOSITION 2.6. *Suppose Y_1, Y_2 are compact invariant subspaces contained in a free G-space E, and let $Y = Y_1 \circ Y_2$. Then,*

$$\operatorname{cat}_{Y^*} Y^* \leqq \operatorname{cat}_{Y_1^*} Y_1^* + \operatorname{cat}_{Y_2^*} Y_2^*$$

where $A^* = A/G$.

Proof. $Y_1 \circ Y_2$ splits into two pieces

$$X_1 = \left\{[y_1, t, y_2], t \leqq \frac{1}{2}\right\}$$

$$X_2 = \left\{[y_1, t, y_2], t \leqq \frac{1}{2}\right\}$$

with Y_i a strong deformation retract of X_i (equivalently). Thus Y_i^* is a strong deformation of X_i^* and since

$$\operatorname{cat}_{Y^*} Y^* \leqq \operatorname{cat}_{X_1^*} X_1^* + \operatorname{cat}_{X_2^*} X_2^*$$

we have the desired result.

COROLLARY 2.7. *If $Y = \overbrace{G \circ \cdots \circ G}^{k}$, then $\operatorname{cat}_{Y^*} Y^* \leqq k$.*

The next proposition establishes that G-genus is also subadditive.

PROPOSITION 2.8. *If $Y \in \mathscr{F}$ and $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are closed invariant subspaces, then*

$$G\text{-genus } Y \leqq G\text{-genus } Y_1 + G\text{-genus } Y_2 .$$

Proof. Suppose G-genus $Y_1 = k_1$ and G-genus $Y_2 = k_2$. Let

$$H_1 = \overbrace{G \circ \cdots \circ G}^{k_1}, \qquad H_2 = \overbrace{G \circ \cdots \circ G}^{k_2}$$

and observe that H_1 and H_2 are compact G-ENR's (Proposition 2.5). Suppose

$$f_i \colon Y_i \longrightarrow H_i, \qquad i = 1, 2$$

are equivariant maps. Then f_i extends to an equivariant map

$$f_i' \colon U_i \longrightarrow H_i, \qquad i = 1, 2$$

where U_i is an invariant open set containing Y_i. Select an equivariant partition of unity $\varphi_i \colon Y \to [0, 1]$ so that

$$Y_i \subset \varphi_i^{-1}((0, 1]) \subset U_i, \qquad i = 1, 2 .$$

Then, define an equivariant map

$$f \colon Y \longrightarrow H_1 \circ H_2$$

by setting

$$f(y) = [\varphi_1(y) f_1'(y), \varphi_2(y) f_2'(y)]$$

as the result follows.

REMARK 2.9. Let us recall that if we set $Y_k = \overbrace{G \circ \cdots \circ G}^{k}$ and $Y_k^* = Y_k/G$, we have natural imbeddings

$$\begin{array}{ccccccccc} G & \longrightarrow & \cdots & \longrightarrow & Y_k & \longrightarrow & Y_{k+1} & \longrightarrow & \cdots \\ \downarrow & & & & \downarrow & & \downarrow & & \\ * & \longrightarrow & \cdots & \longrightarrow & Y_k^* & \longrightarrow & Y_{k+1}^* & \longrightarrow & \cdots \end{array}$$

and the direct limit yields the Milnor universal bundle [7] (E_G, p_G, B_G) for G. Now, if E is a contractible, paracompact free G-space, and if $E/G = B$, then (E, p, B) is also a universal boundle for G-bundles over paracompact spaces [3].

As we have seen, G-genus is subadditive but the proof was more substantial than the corresponding trivial result for L-S category. Just the opposite occurs for the "monotone" property. If $\varphi \colon X \to Y$ is an equivariant map (in $\mathscr{F}$), then it is immediate that

$$G\text{-genus } X \leqq G\text{-genus } Y .$$

However, the corresponding result for L-S category requires some details—and makes use of the classification theorem for G-bundles.

PROPOSITION 2.10. *Suppose X_1 and X_2 are closed invariant subspaces of paracompact free G-spaces E_1 and E_2, respectively. Then, if $\varphi: X_1 \to X_2$ is an equivariant map and if*

$$X_1^* = X_1/G \ , \qquad X_2^* = X_2/G \ , \qquad B_1 = E_1/G \ , \qquad B_2 = E_2/G \ ,$$

then

$$\mathrm{cat}_{B_1} X_1^* \leqq \mathrm{cat}_{B_2} X_2^* \ .$$

Proof. The bundles (E_i, p_i, B_i) $i = 1, 2$ are universal bundles and hence we have the following diagram of bundle maps

$$\begin{array}{ccccccc} X_1 & \xrightarrow{\varphi} & X_2 & \xrightarrow{i_2} & E_2 & \xrightarrow{\alpha} & E_1 \\ {\scriptstyle q_1}\downarrow & & {\scriptstyle q_2}\downarrow & & {\scriptstyle p_2}\downarrow & & {\scriptstyle p_1}\downarrow \\ X_1^* & \xrightarrow{\bar{\varphi}} & X_2^* & \xrightarrow{\bar{i}_2} & B_2 & \xrightarrow{\bar{\alpha}} & B_1 \end{array}$$

where φ is given, i_2 is inclusion and α exists via the universality of (E_1, p_1, B_1).

Now, suppose $\mathrm{cat}_{B_2} X_2^* = k < \infty$. There, X_2^* admits a closed cover $K_1^*, \cdots, K_k^*$ of sets contractible in B_2 to a point. If we set $A_l^* = \bar{\varphi}^{-1}(K_l^*)$, we have a closed cover $\{A_1^*, \cdots, A_k^*\}$ of X_1^* and

$$\bar{\alpha} \circ \bar{i}_2 \circ (\bar{\varphi} \,|\, A^*) \sim \text{constant (in } B_1) \ .$$

However, since (E_1, p_1, B_1) is universal, we have

$$\bar{\alpha} \circ \bar{i}_2 \circ \bar{\varphi} \sim \bar{i}_1$$

where $i_1: X_1 \to E_1$ is inclusion. Thus, each A_l^* is contractible to a point in B_1 and

$$\mathrm{cat}_{B_1} X_1^* \leqq \mathrm{cat}_{B_2} X_2^* \ .$$

3. Category vs genus.

THEOREM 3.1. *Let E denote a contractible, paracompact free G-space and let Σ denote the closed invariant subspaces of E. Then if $B = E/G$ and $A^* = A/G$, we have*

$$\mathrm{cat}_B A^* = G\text{-genus } A \ , \qquad A \in \Sigma \ .$$

Proof. (a) We show first that $\mathrm{cat}_B A^* \leqq G$-genus A. Suppose that G-genus $A = k < \infty$. Then, we have an equivariant map

$$f: A \longrightarrow Y = \overbrace{G \circ \cdots \circ G}^{k} \subset E_G \ .$$

But then, using Proposition 2.10 and Corollary 2.7

$$\operatorname{cat}_B A^* \leqq \operatorname{cat}_{B_G} Y^* \leqq \operatorname{cat}_{Y^*} Y^* \leqq k .$$

Thus,

$$\operatorname{cat}_B A^* \leqq G\text{-genus } A .$$

(b) Now, suppose $\operatorname{cat}_B A^* = k < \infty$. Then,

$$A^* = A_1^* \cup \cdots \cup A_k^*$$

where each A_l^* is closed and contractible in B. Now, since G-genus is subadditive (Proposition 2.8) we have

$$G\text{-genus } A \leqq \sum_{l=1}^{k} G\text{-genus } A_l$$

where $A_l = p_A^{-1}(A_l^*)$, $p_A: A \to A/G = A^*$ the natural projection. Since each A_l^* is contractible to a point in B, the bundle (A, p_A, A^*) is a trivial G-bundle and hence we have an equivariant map

$$f_l: A_l \longrightarrow G$$

so that G-genus $A_l = 1$, $l = 1, \cdots, k$. This proves that

$$G\text{-genus } A \leqq k = \operatorname{cat}_B A^*$$

and the proof is complete.
There are some noteworthy examples:

3.2. Let $\mathscr{B}$ denote an infinite dimensional Banach space over the reals $\boldsymbol{R}$. Let $G = \boldsymbol{Z}_2 = \{-1, 1\}$ act on $\mathscr{B}$ by scalar multiplication and let Σ denote the closed invariant subsets of $E = \mathscr{B} - 0$. Define the real genus of $A \in \Sigma$ by

$$\operatorname{genus}_R A = \boldsymbol{Z}_2\text{-genus } A .$$

Then,

$$\operatorname{genus}_R A = \operatorname{cat}_B A^*$$

where $B = E/\boldsymbol{Z}_2$, $A^* = A/\boldsymbol{Z}_2$. As we have already observed, $\operatorname{genus}_R A = k < \infty$ is equivalent to saying that there is an equivalent (odd) map $f: A \to \boldsymbol{R}^k - 0$ and k is minimal with this property, so that genus_R is ordinary genus in the sense of Krasnoselskii [5].

3.3. Let $\mathscr{B}$ denote an infinite dimensional Banach space over the complex numbers $\boldsymbol{C}$. Let $G = S^1$, the complex numbers of norm 1. Then G acts freely on $E = \mathscr{B} - 0$, again by scalar multiplications. Let Σ denote the closed invariant subsets of E and define the complex genus of $A \in \Sigma$ by

$$\text{genus}_C A = S^1\text{-genus}\, A$$

then,

$$\text{genus}_C A = \text{cat}_B A^*$$

where $B = E/S^1$, $A^* = A/S^1$. We also mention here that $\text{genus}_C A = k < \infty$ is equivalent to saying that there is an equivariant map $f: A \to C^k - 0$ and k is minimal with this property.

Another consequence of Theorem 3.1 is the following result which asserts the independence of L-S category on the ambient Banach space.

COROLLARY 3.4. *If* $\mathscr{B}_i$, $i = 1, 2$ *are real (complex) Banach spaces (not necessarily infinite dimensional) and* $A_i \subset \mathscr{B}_i - 0$ *are closed invariant subsets admitting an equivariant homeomorphism* $\varphi: A_1 \to A_2$, *then*

$$\text{cat}_{B_1} A_1^* = \text{cat}_{B_2} A_1^*$$

where $B_i = \mathscr{B}_i - 0)/Z_2(S^1)$.

Proof. If both Banach spaces are infinite then

$$\text{cat}_{B_1} A_1^* = G\text{-genus}\, A_1 = G\text{-genus}\, A_2 = \text{cat}_{B_2} A_2^* .$$

To complete the proof it suffices to prove the following lemma.

LEMMA 3.5. *Let* $\mathscr{B}$ *denote an infinite dimensional Banach space over* $\boldsymbol{R}$ *or* $\boldsymbol{C}$ *and let* L *denote a finite dimensional subspace. Let* A *denote a closed invariant set in* $L - 0$. *If* $C = (L - 0)/G$, $B = (\mathscr{B} - 0)/G$, $A^* = A/G$, *where* $G = Z_2$ *or* S^1, *then*

$$\text{cat}_C A^* = \text{cat}_B A^* .$$

Proof. We consider only the real case. Wemay identify L with $\boldsymbol{R}^n$ and if Z_2-genus $A = k$, then $k \leqq n$ and we have a diagram of bundle maps

$$\begin{array}{ccccccc}
A & \xrightarrow{\varphi} & S^{k-1} & \xrightarrow{i} & S^{n-1} & \xrightarrow{j} & S^n \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p_{k-1}} & & \downarrow{\scriptstyle p_{n-1}} & & \downarrow{\scriptstyle p_n} \\
A^* & \xrightarrow{\bar{\varphi}} & \boldsymbol{R}P^{k-1} & \xrightarrow{\bar{i}} & \boldsymbol{R}P^{n-1} & \xrightarrow{\bar{j}} & \boldsymbol{R}P^n
\end{array}$$

where φ is the equivariant map obtained from the fact that Z_2-genus $A = k$ and i is inclusion. $\boldsymbol{R}P^{k-1}$ is the union of k contractible closed sets, $K_1^*, \cdots, K_k^*$ and hence if we set $A_l^* = \bar{\varphi}^{-1}(K_l^*)$, we have that each map

$$\bar{i} \circ (\bar{\varphi} | A_l^*) \sim \text{constant (in } \boldsymbol{RP}^{n-1}) .$$

We may assume without loss that $A_l = q^{-1}(A_l^*) \subset S^{n-1}$ and is a finite subcomplex of dimension $\leqq n - 1$. Since $(S^n, P_n, \boldsymbol{RP}^n)$ is n-universal [10]

$$j^* \circ \bar{i} \circ \bar{\varphi} | A_l^* \sim j_l \colon A_l^* \subset \boldsymbol{RP}^n .$$

Thus, A_l^* is contractible in $\boldsymbol{RP}^n$. This forces A_l^* to be a proper subset of $\boldsymbol{RP}^{n-1}$ and hence A_l^* is deformable in $\boldsymbol{RP}^{n-1}$ to $\boldsymbol{RP}^{n-2}$. Repeating the above argument then forces A_l^* to be contractible in $\boldsymbol{RP}^{n-1}$ and so

$$\text{cat}_C A^* \leqq k = \boldsymbol{Z}_2\text{-genus } A = \text{cat}_B A^* .$$

Since the inequality $\text{cat}_B A^* \leqq \text{cat}_C A^*$ is obvious the lemma follows and the proof of Corollary 3.4 is complete.

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