Pacific Journal of Mathematics

SQUARE-FREE AND CUBE-FREE COLORINGS OF THE ORDINALS

JEAN ANN LARSON, RICHARD JOSEPH LAVER
AND GEORGE FRANK MCNULTY

Vol. 89, No. 1 May 1980

PACIFIC JOURNAL OF MATHEMATICS
Vol. 89, No. 1, 1980

SQUARE-FREE AND CUBE-FREE COLORINGS OF THE ORDINALS

JEAN A. LARSON, RICHARD LAVER AND GEORGE F. MCNULTY

We prove: Theorem 1. The class of all ordinals has a square-free 3-coloring and a cube-free 2-coloring. Theorem 2. Every kth **power-free** n**-coloring of** α **can be extended to a maximal** kth **power-free** n**-coloring of** β**, for some** $\beta \times \alpha \cdot \omega$**, where** $k, n \in \omega$**.**

Every ordinal is conceived as the set of all smaller ordinals; ω is the least infinite ordinal. By an *interval of ordinals* we mean any set $\{\delta: \beta \leqq \delta < \gamma\}$ where β and γ are ordinals; $[\beta, \gamma)$ abbreviates $\{\delta: \beta \leqq \delta < \gamma\}$. If S and T are intervals then there can be at most one order isomorphism from S onto T.

Let S be an interval of ordinals and κ be a cardinal. A *κ-coloring* of S is just a function with domain S and range included in κ. Suppose S and T are intervals of ordinals and that f is a coloring of S while g is a coloring of T. Then the coloring f of S is *similar* to the coloring g of T provided S and T are order isomorphic and $f(\alpha) = g(h(\alpha))$ for all $\alpha \in S$ where h is the unique order isomorphism from S onto T; if f and g are clear from the context we say that S is similar to T. A coloring f of the ordinal α is *square-free* if no two adjacent nonempty intervals of α are similar; it is *cube-free* if no three consecutive nonempty intervals are all similar to each other. All these notions extend naturally to the class of all ordinals.

In Bean, Ehrenfeucht, and McNulty [1] it was shown that α has a square-free 3-coloring and a cube-free 2-coloring whenever $\alpha < (2^{\aleph_0})^+$ and the question of extending this result to all ordinals was left open. This question is resolved here.

THEOREM 1. *The class of all ordinals has a square-free* 3*-coloring and a cube-free* 2*-coloring.*

If I is a class of ordinals and α_β is an ordinal for each $\beta \in I$, then $\sum_{\beta \in I} \alpha_\beta$ denotes the *ordinal sum* of the α_β's with respect to I. (See Sierpinski [2] for details.) Finite ordinal sums are written like $\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$. For each $\beta \in I$, let $\mathrm{Int}(\beta) = [\mu, \mu + \alpha_\beta)$ where $\mu = \sum_{\gamma \in J} \alpha_\gamma$ and $J = I \cap \beta$. For each $\beta \in I$, $\mathrm{Int}(\beta)$ is order isomorphic with α_β. In fact, $\sum_{\beta \in I} \alpha_\beta$ can be construed as the disjoint union of the $\mathrm{Int}(\beta)$'s as $\beta \in I$ where the intervals are given the order type of I. This means that if f_β is a κ-coloring of α_β, for each $\beta \in I$, then there is a κ-coloring f of $\sum_{\beta \in I} \alpha_\beta$ such that $f \restriction \mathrm{Int}(\beta)$ is similar to f_β.

An ordinal α is *(additively) indecomposable* provided $\alpha \neq \beta + \gamma$ whenever $\beta < \alpha$ and $\gamma < \alpha$. It is known (cf. Sierpinski [2]) that every ordinal is the ordinal sum of finitely many indecomposable ordinals and that the infinite indecomposable ordinals are exactly the ordinal powers of ω.

LEMMA 0. *If α is the class of all ordinals or α is an indecomposable ordinal with $\alpha > \omega$, then α is the sum of a strictly increasing sequence of smaller limit ordinals.*

Proof. There are three cases. First, suppose $\alpha = \omega^\beta$ where β is a limit ordinal. So $\alpha = \omega^\beta = \sum_{\gamma < \beta} \omega^\gamma$. Second, suppose $\alpha = \omega^{\beta+1}$. Then $\alpha = \omega^{\beta+1} = \omega^\beta \cdot \omega = \sum_{n \in \omega}(\omega^\beta \cdot n)$. Third, the class of all ordinals is $\sum_{\kappa \in I} \kappa$, where I is the class of cardinals. In each case the lemma holds.

Let f be a coloring of the interval S of ordinals and let g be a coloring of the interval T. S and T are *mismatched* provided that U and V fail to be similar whenever U is an infinite subinterval of S and V is an infinite subinterval of T. Theorems 1.8 and 1.16 from Bean, Ehrenfeucht, and McNulty [1] are collected in the next lemma.

LEMMA 1. (a) *There is a collection $\mathscr{F}$ of square-free 3-colorings of ω such that $|\mathscr{F}| = 2^{\aleph_0}$ and C and D are mismatched whenever $C, D \in \mathscr{F}$ with $C \neq D$.*

(b) *There is a collection $\mathscr{S}$ of cube-free 2-colorings of ω such that $|\mathscr{S}| = 2^{\aleph_0}$ and C and D are mismatched whenever $C, D \in \mathscr{S}$ with $C \neq D$.*

Proof of Theorem 1. We will provide a proof that the class of all ordinals has a square-free 3-coloring. This proof can be easily modified to establish that the class of all ordinals has a cube-free 2-coloring. The property of having a square-free 3-coloring is hereditary in the sense that if α has a square-free 3-coloring and $\beta < \alpha$, then β has a square-free 3-coloring. Below we are concerned with providing each limit ordinal with a square-free 3-coloring and we proceed by induction.

Induction hypothesis. If α is an infinite limit ordinal or the class of all ordinals, and $f_0, f_1, \cdots$ are countably many square-free 3-colorings of ω such that f_i and f_j are mismatched whenever $i, j \in \omega$ with $i \neq j$, then there is a 3-coloring g of α such that

(i) g is square-free.

(ii) g and f_i are mismatched for each $i \in \omega$.

(iii) Any two similar infinite intervals of α are separated by an infinite interval.

Suppose the induction hypothesis holds for all infinite limit ordinals less than α and that $f_0, f_1, f_2, \cdots$ are countably many pairwise mismatched square-free 3-colorings of ω. There are two cases.

Case 1. $\alpha = \rho_0 + \rho_1 + \cdots + \rho_n$ where $\rho_0, \cdots, \rho_n$ are indecomposable and $0 < n \in \omega$.

According to Lemma 1 there must be $h_0, \cdots, h_n$, all square-free 3-colorings of ω, such that $h_0, h_1, \cdots, h_n, f_0, f_1, \cdots$ are all pairwise mismatched. By the induction hypothesis there are 3-colorings $d_0, \cdots, d_n$ of $\rho_0, \cdots, \rho_n$ respectively such that for each $i \leqq n$

(i)′ d_i is square-free.

(ii)′ $d_i, h_0, h_1, \cdots, h_n, f_0, f_1 \cdots$ are all pairwise mismatched.

(iii)′ Any two similar infinite intervals of ρ_i are separated by an infinite interval.

For each $i \leqq n$ and each $\gamma \in \rho_i$, let

$$d_i^*(\gamma) = \begin{cases} h_i(\gamma) & \text{if } \gamma \in \omega \\ d_i(\gamma) & \text{otherwise}, \end{cases}$$

and let g be the coloring of α induced by $d_0^*, \cdots, d_n^*$.

Condition (ii) of the induction hypothesis holds by (ii)′. To check condition (iii) suppose S and T are distinct similar infinite intervals of α. Since h_i and d_j are mismatched whenever $i, j \leqq n$ and since $h_0, h_1, \cdots, h_n$ are pairwise mismatched, for each $i \leqq n$ there is exactly one interval U of α (of order type ω) such that $f|U$ is similar to h_i. Since S and T are distinct but similar neither can have a subinterval similar to any of $h_0, h_1, \cdots, h_n$ or *any of their final segments*. Consequently there are $i, j \leqq n$, with finite initial segments δ of ρ_{i+1} and ε of ρ_{j+1} such that S is a subinterval of $\rho_i + \delta$ missing the initial segment of ρ_i of order-type ω, while T is a subinterval of $\rho_j + \varepsilon$ missing the initial segment of ρ_j of order-type ω. If $i \neq j$ then (iii) follows immediately, so suppose $i = j$. There must be cofinite initial segments S' of S and T' of T such that S' and T' are distinct yet similar and both S' and T' are subintervals of ρ_i missing the initial segment of ρ_i of order type ω. So S' and T' are colored by d_i and by (iii)′ they are separated by an infinite interval and therefore S and T are separated by an infinite interval as well.

To see that g is a square-free coloring of α, observe that (iii) forces any two similar adjacent intervals to be finite. But g was devised so that *all* intervals of α of order type ω are colored in a square-free manner. Hence g is square-free and Case I of the induction is complete.

Case II. α is indecomposable with $\alpha > \omega$.

By Lemma 0 $\alpha = \sum_{\gamma \in \beta} \rho_\gamma$ for some β where $\rho_\gamma < \rho_\delta < \alpha$ and ρ_γ is an infinite limit ordinal, if $\gamma < \delta < \beta$. According to Lemma 1 there must be h_0 and h_1, both square-free 3-colorings of ω such that $h_0, h_1, f_0, f_1, f_2, \cdots$ are pairwise mismatched. By the induction hypothesis for each $\gamma \in \beta$ there is a 3-coloring d_γ of ρ_γ such that

(i)'' d_γ is square-free.

(ii)'' $d_\gamma, h_0, h_1, f_0, f_1, f_2, \cdots$ are pairwise mismatched.

(iii)'' Any two similar infinite intervals of ρ_γ are separated by an infinite interval

$$d^*_\gamma(\delta) = \begin{cases} h_0(\delta) & \text{if } \delta \in \omega \text{ and } \gamma \text{ is even} \\ h_1(\delta) & \text{if } \delta \in \omega \text{ and } \gamma \text{ is odd} \\ d_\gamma(\delta) & \text{otherwise}, \end{cases}$$

and let g be the coloring of α induced by $\langle d^*_\gamma : \gamma \in \beta \rangle$.

Conditions (i) and (ii) of the induction hypothesis can be established as in Case I. We argue that (iii) holds. Suppose S and T are similar infinite intervals in α. If S contains an interval of type ω colored the way h_0(or h_1) colors some final segment of ω then the same is true for T. According to the construction of g these kinds of colorings occur only on the initial segments of each ρ_γ of type ω. Since the ρ_γ's form an increasing sequence, no interval between an interval colored with h_0 and the next colored with h_1 occurs twice. So if S contains an interval of type ω colored the way h_0 (or h_1) colors some final segment of ω, then S does *not* contain an interval of type ω colored the way h_1 (alternatively h_0) colors some final segment of ω. The same is true for T. Consequently, if S and T were separated by a finite interval, then both S and T would lie entirely in $\rho_\gamma + \delta$ for some γ where δ is a finite initial segment of $\rho_{\gamma+1}$. From this point the argument proceeds as in Case I.

Since Lemma 1 guarantees the theorem when $\alpha = \omega$, the induction is complete and the theorem established.

For any $k \in \omega$, kth power-free colorings have definitions analogous to those of square-free and cube-free colorings. Every square-free coloring is kth power-free for all $k \geqq 2$. A kth power-free κ-coloring f of α is *maximal* provided f cannot be extended to a kth power-free κ-coloring of $\alpha + 1$. In Bean, Ehrenfeucht, and McNulty [1] it is shown that every kth power-free n-coloring f of m can be extended to a maximal kth power-free n-coloring of some natural number, whenever $k, n, m \in \omega$. We remark that the following theorem holds. The proof differs in no important way from the proof of Theorem 2.0 in [1].

THEOREM 2. *For any natural numbers n and k and any ordinal α, every kth power-free n-coloring of α can be extended to a maximal kth power-free n-coloring of β for some $\beta \in \alpha \cdot \omega$.*

References

1. D. Bean, A. Ehrenfeucht, and G. McNulty, *Avoidable patterns in strings of symbols*, Pacific J. Math., **85** (1979), 261-294.
2. V. Sierpinski, *Cardinal and Ordinal Numbers*, Warsawa 1958.

Received February 7, 1978.

UNIVERSITY OF FLORIDA
GAINESVILLE, FL 32611
UNIVERSITY OF COLORADO
BOULDER, CO 80302
AND
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA SC 29208

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)
University of Galifornia
Los Angeles, California 90024

HUGO ROSSI
University of Utah
Salt Lake City, UT 84112

C. C. MOORE AND ANDREW OGG
University of California
Berkeley, CA 94720

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. FINN AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH   B. H. NEUMANN   F. WOLF   K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFONIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 89, No. 1 May, 1980