

Pacific Journal of Mathematics

CYCLIC VECTORS FOR $L^p(G)$

VIKTOR LOSERT AND HARALD RINDLER

CYCLIC VECTORS FOR $L^p(G)$

VIKTOR LOSERT AND HARALD RINDLER

If G is a first countable locally compact group, then $L^p(G)$ has a cyclic vector with compact support, where $1 \leq p < \infty$.

In [3] Greenleaf and Moskowitz proved the existence of cyclic vectors for the left and right regular representation of $L^p(G)$, where G is a first countable, locally compact group, see also [4] and [5]. We generalize this result to $L^p(G)$ ($1 \leq p < \infty$) and certain other $L^1(G)$ -modules.

THEOREM. *Let G be a locally compact group.*

(i) *If G is first countable, then there exists a continuous function u on G with compact support such that the left invariant hull of u is dense in $L^p(G)$ for $1 \leq p < \infty$. The right hull of u (for the corresponding right action of G on $L^p(G)$) is also dense in $L^p(G)$.*

(ii) *Conversely, if $1 \leq p < \infty$ and $L^p(G)$ has a cyclic vector, then G is first countable.*

For the proof of the theorem we need two lemmas:

LEMMA 1. *Assume that H is a closed subgroup of G which is isomorphic to \mathbf{R} . If the nonzero measure μ is concentrated on a compact subset of H , then $\{f*\mu: f \in \mathcal{K}(G)\}$ is dense in $L^p(G)$ for $1 < p < \infty$.*

Proof of Lemma 1. Define q by $1/q + 1/p = 1$. If the space defined above is not dense in $L^p(G)$, there exists a nonzero continuous function $g \in L^q(G)$ such that $\langle f*\mu, g \rangle = 0$ for all $f \in \mathcal{K}(G)$, the space of continuous functions with compact support (if g is not continuous, replace g by $h*g \neq 0$, $h \in \mathcal{K}(G)$). Put $g^\vee(x) = g(x^{-1})$ ($x \in G$), then $\mu*g^\vee = 0$ on G . Put $\mu_1 = \Delta_G(\cdot)^{-1/q} \cdot \mu$ and for $y \in G$, $x \in H$, set $g_y(x) = g(y^{-1}x)\Delta_G(x)^{+1/q}$ (Δ_G denotes the modular function on G). By Weil's formula ([7], pp. 42-45) $g_y \in L^q(H)$ holds for a.e. $y \in G$. A short calculation shows that

$$\mu_1 * g_y^\vee(x) = \mu * g^\vee(xy) \Delta_G(x)^{-1/q} \quad \text{for } x \in H.$$

Since g is continuous we conclude that $\mu_1 * g_y^\vee = 0$ on H . μ_1 is concentrated on a compact subset of $H = \mathbf{R}$ and nonzero. The Fourier transform $\hat{\mu}_1$ is an analytic function. It follows that it has at most countably many zeros. By [1] the set $\{f*\mu_1: f \in \mathcal{K}(H)\}$ is dense in

$L^p(H)$ for $1 < p < \infty$. If $g_y \in L^q(H)$, it follows from this that $g_y = 0$. Again by Weil's formula we conclude that $g = 0$.

In a similar way we obtain:

LEMMA 2. *Let H be a closed subgroup of G , μ a bounded measure on H . If μ generates a dense left (right) ideal in $L^1(H)$ then it generates a dense left (right) ideal in $L^1(G)$. If H is compact, the same holds for $L^p(G)$ ($1 < p < \infty$).*

Proof of the theorem. (i) We use Yamabe's theorem to find an open subgroup G_1 of G , and a compact subgroup N of G_1 , normal in G_1 , such that G_1/N is a connected Lie group. Now we use the description of the Haar measure given in [2]. There exist closed subgroups H_1, \dots, H_n of G_1 , each of them being isomorphic to \mathbf{R} , and a compact subgroup $K \supseteq N$ such that $G_1 = H_1 \cdots H_n K$ and this is a topological decomposition of G_1 . The Haar measure on G_1 is simply the product of the Haar measures on H_1, \dots, H_n and K . Now let f be a continuous function on \mathbf{R} with compact support and nowhere vanishing Fourier transform. Let μ_i be the measure on $H_i = \mathbf{R}$ defined by f ($i = 1, \dots, n$). Since G is metrizable, the same holds for K and it follows that the dual of K is countable. Let g be a continuous function on K such that $U(g)$ is invertible for any continuous, irreducible representation of K . Let μ_{n+1} be the measure defined by g . It follows from Lemmas 1 and 2 that $\{h * \mu_1 * \cdots * \mu_{n+1} : h \in K(G)\}$ is dense in $L^p(G)$ for $1 \leq p < \infty$. The measure $\mu_1 * \cdots * \mu_{n+1}$ is absolutely continuous on G_1 , its derivative with respect to Haar measure is $u(x_1 \cdots x_{n+1}) = f(x_1) \cdots f(x_n) g(x_{n+1})$ ($x_i \in H_i$ $i = 1, \dots, n$, $x_{n+1} \in K$). It follows that u has the properties stated in the theorem. The proof for the right invariant hull is similar.

(ii) This part is entirely analogous to the case of $L^2(G)$ which was proved in [4] Theorem 2.1.

DEFINITION (see [6]). A symmetric Segal algebra $S(G)$ on G is a dense, left and right invariant linear subspace of $L^1(G)$, such that $S(G)$ is a Banach space with respect to a norm $\| \cdot \|_S$, $\|f\|_1 \leq \|f\|_S$, for $f \in S(G)$, $y \rightarrow L_y$ and $y \rightarrow R_y$ are strongly continuous representations of G by isometries on $S(G)$, [6], Ch. 6, §2.1, 2.2. (it follows in particular that $S(G)$ is a left and right $L^1(G)$ module and that the action of $L^1(G)$ is contractive).

COROLLARY. *If $S(G)$ is a symmetric Segal algebra on G and the function u of the theorem belongs to $S(G)$, the left and right invariant hulls of u are both dense in $S(G)$.*

Proof of the corollary. Take $g \in S(G)$. Since right translation is continuous on $S(G)$, there exists $h \in K(G)$ such that $\|g*h - g\|_S < \varepsilon$. By the theorem there exists $k \in K(G)$ such that $\|k*u - h\|_1 < \varepsilon$. It follows that

$$\|g*k*u - g\|_S \leq \|g*k*u - g*h\|_S + \|g*h - g\|_S < \varepsilon(\|g\|_S + 1).$$

The proof for the right invariant hull of u is similar.

REFERENCES

1. A. Beurling, *On a closure problem*, Ark. Mat., **1** (1950), 301-303.
2. H. F. Davis, *A note on Haar measure*, Proc. Amer. Math. Soc., **6** (1955), 318-321.
3. F. P. Greenleaf and M. Moskowitz, *Cyclic vectors for representations of locally compact groups*, Math. Ann., **190** (1971), 265-288.
4. ———, *Cyclic vectors for representations associated with positive definite measures: nonseparable groups*, Pacific. J. Math., **45** (1973), 165-186.
5. A. Hulanicki and T. Pytlik, *On commutative approximate identities and cyclic vectors of induced representations*, Studia Math., **48** (1973), 189-199.
6. H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Clarendon Press, Oxford, 1968.
7. A. Weil, *L'intégration dans les groupes topologiques et ses applications*, 2nd edn, Hermann, Paris, (1953).

Received March 21, 1979.

INSTITUT FÜR MATHEMATIK DER UNIVERSITÄT WIEN
 STRUDLHOFGASSE 4
 A-1090 WIEN, AUSTRIA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)

University of California
Los Angeles, California 90024

HUGO ROSSI

University of Utah
Salt Lake City, UT 84112

C. C. MOORE AND ANDREW OGG

University of California
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. FINN AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

David Bressoud, <i>A note on gap-frequency partitions</i>	1
John David Brillhart, <i>A double inversion formula</i>	7
Frank Richard Deutsch, Günther Nürnberger and Ivan Singer, <i>Weak Chebyshev subspaces and alternation</i>	9
Edward Richard Fadell, <i>The relationship between Ljusternik-Schnirelman category and the concept of genus</i>	33
Harriet Jane Fell, <i>On the zeros of convex combinations of polynomials</i>	43
John Albert Fridy, <i>An addendum to: "Tauberian theorems via block dominated matrices"</i>	51
Andrzej Granas, Ronald Bernard Guenther and John Walter Lee, <i>Applications of topological transversality to differential equations. I. Some nonlinear diffusion problems</i>	53
David E. Handelman and G. Renault, <i>Actions of finite groups on self-injective rings</i>	69
Michael Frank Hutchinson, <i>Local Λ sets for profinite groups</i>	81
Arnold Samuel Kas, <i>On the handlebody decomposition associated to a Lefschetz fibration</i>	89
Hans Keller, <i>On the lattice of all closed subspaces of a Hermitian space</i>	105
P. S. Kenderov, <i>Dense strong continuity of pointwise continuous mappings</i>	111
Robert Edward Kennedy, <i>Krull rings</i>	131
Jean Ann Larson, Richard Joseph Laver and George Frank McNulty, <i>Square-free and cube-free colorings of the ordinals</i>	137
Viktor Losert and Harald Rindler, <i>Cyclic vectors for $L^p(G)$</i>	143
John Rowlay Martin and Edward D. Tymchatyn, <i>Fixed point sets of 1-dimensional Peano continua</i>	147
Augusto Nobile, <i>On equisingular families of isolated singularities</i>	151
Kenneth Joseph Prevot, <i>Imbedding smooth involutions in trivial bundles</i>	163
Thomas Munro Price, <i>Spanning surfaces for projective planes in four space</i>	169
Dave Riffelmacher, <i>Sweedler's two-cocycles and Hochschild cohomology</i>	181
Niels Schwartz, <i>Archimedean lattice-ordered fields that are algebraic over their o-subfields</i>	189
Chao-Liang Shen, <i>A note on the automorphism groups of simple dimension groups</i>	199
Kenneth Barry Stolarsky, <i>Mapping properties, growth, and uniqueness of Vieta (infinite cosine) products</i>	209
Warren James Wong, <i>Maps on simple algebras preserving zero products. I. The associative case</i>	229