FIXED POINT SETS OF 1-DIMENSIONAL PEANO CONTINUA

John Rowlay Martin and Edward D. Tymchatyn
FIXED POINT SETS OF 1-DIMENSIONAL PEANO CONTINUA

JOHN R. MARTIN AND E. D. TYMCATYN

It is shown that every nonempty closed subset of a 1-dimensional Peano continuum $X$ is the fixed point set of some continuous self-mapping of $X$.

1. Introduction. A topological space $X$ is said to have the complete invariance property (CIP) if every nonempty closed subset of $X$ is the fixed point set of some continuous self-mapping of $X$. The term CIP was suggested by L. E. Ward, Jr. in [5, p. 553] where it was asked if every Peano continuum had CIP. Examples have been given in [3], [4, 3.1] which show that $n$-dimensional Peano continua need not have CIP if $n > 1$. In [4, 3.4] it is asked if every 1-dimensional Peano continuum has CIP. The purpose of this note is to answer that question in the affirmative by showing that every 1-dimensional Peano continuum has CIP.

2. Preliminaries. Let $M$ be a metric space. A sequence of subsets of $M$ is called a null sequence provided that for any $\epsilon > 0$ at most a finite number of its elements has diameter greater than $\epsilon$. The space $M$ is said to have property $S$ provided that for each $\epsilon > 0$, $M$ is the union of a finite number of connected sets each of diameter less than $\epsilon$. A partitioning of $M$ is a finite collection $\mathcal{U}$ of pairwise disjoint connected open subsets of $M$ whose union is dense in $M$. If the mesh of $\mathcal{U}$ is less than $\epsilon$ (each element of $\mathcal{U}$ is of diameter less than $\epsilon$), $\mathcal{U}$ is called an $\epsilon$-partitioning. A sequence $\mathcal{U}_1$, $\mathcal{U}_2$, $\ldots$ of partitionings is called a decreasing sequence if, for each positive integer $i$, $\mathcal{U}_{i+1}$ is a refinement of $\mathcal{U}_i$ and the mesh of $\mathcal{U}_i$ approaches 0 as $i$ increases without limit. It is well-known [1, p. 545] that every Peano continuum has a decreasing sequence of partitionings.

A dendron is a connected, simply connected, finite graph. The closure of a subset $A$ of a topological space shall be denoted by $\text{Cl}(A)$.

3. The result.

**Theorem** Every 1-dimensional Peano continuum has the complete invariance property.

**Proof.** Let $X$ be a 1-dimensional Peano continuum and let $A$ be
a closed subset of $X$. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be a decreasing sequence of $(1/i)$-partitionings of $X$. Then each $\mathcal{U}_i$ is a finite collection of open connected pairwise disjoint sets of diameter less than $1/i$ such that for each $i \bigcup \{U|U \in \mathcal{U}_i\}$ is dense in $X$ and $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$. We suppose

$$\mathcal{U}_1 = \{U_{1,1}, \ldots, U_{1,m_0,1}\}$$

$$\mathcal{U}_2 = \{U_{2,t,j}|U_{2,t,j} \subset U_{1,t} \in \mathcal{U}_1 \text{ and } j \in \{1, \ldots, m_1,t}\}$$

and for $i > 1$

$$\mathcal{U}_i = \{U_{i,j_1,\ldots,j_i}|U_{i,j_1,\ldots,j_i} \subset U_{i-1,j_1,\ldots,j_{i-1}} \in \mathcal{U}_{i-1} \text{ and } j_i \in \{1, \ldots, m_{i-1,j_1,\ldots,j_{i-1}}\}\}.$$ 

For each $i = 1, 2, \ldots$ let

$$\mathcal{U}_i' = \{U \in \mathcal{U}_i|C1(U) \cap A \neq \emptyset\}.$$ 

Without loss of generality,

$$\mathcal{U}_1' = \{U_{1,1}, \ldots, U_{1,n_{0,1}}\} \text{ and for } i > 1$$

$$\mathcal{U}_i' = \{U_{i,j_1,\ldots,j_i}|U_{i,j_1,\ldots,j_i} \subset U_{i-1,j_1,\ldots,j_{i-1}} \in \mathcal{U}_i' \text{ and } j_i \in \{1, \ldots, n_{i-1,j_1,\ldots,j_{i-1}}\}\}.$$ 

Notice that $A \subset C1(\bigcup \mathcal{U}_i')$ for each $i$.

Let $A_{1,i}$ be an arc in $X$ which meets $U_{1,1}$ and $U_{1,2}$. If $A_{1,1} \cap U_{1,2} \neq \emptyset$ let $A_{1,2} = \emptyset$. If $A_{1,1} \cap U_{1,3} = \emptyset$ let $A_{1,2}$ be an arc such that $A_{1,2}$ meets $U_{1,5}$ and $A_{1,1} \cap A_{1,2}$ is an endpoint of $A_{1,2}$. Suppose $A_{1,1} \cup \cdots \cup A_{1,i}$ is a finite dendron such that $A_{1,1} \cup \cdots \cup A_{1,i}$ meets $U_{1,j}$ for each $j \in \{1, \ldots, i + 1\}$. If $i + 2 \leq n_{0,1}$ let $A_{i+1,t+1} = \emptyset$ if $(A_{1,1} \cup \cdots \cup A_{1,i}) \cap U_{1,t+1} \neq \emptyset$, otherwise, let $A_{i+1,t+1}$ be an arc which meets $U_{1,t+1}$ and such that $(A_{i,1} \cup \cdots \cup A_{i,t}) \cap A_{i+1,t+1}$ is an endpoint of $A_{i+1,t+1}$. By induction $A_{i,t}$ is defined for each $i \in \{1, \ldots, n_{0,1} - 1\}$. Let

$$B_1 = A_{1,1} \cup \cdots \cup A_{1,n_{0,1} - 1}.$$ 

Suppose $B_1, \ldots, B_k$ are finite dendrons such that $B_1 \subset B_2 \subset \cdots \subset B_k$, $B_k$ meets $U$ for each $U \in \mathcal{U}_k'$ and

$$B_k - B_{k-1} \subset \bigcup \{U'|U \in \mathcal{U}_k'\}.$$ 

For each $U_{k,j_1,\ldots,j_k} \in \mathcal{U}_k'$ let $A_{k+1,j_1,\ldots,j_k} = \emptyset$ if $B_k$ meets $U_{k+1,j_1,\ldots,j_k,1}$, otherwise, let $A_{k+1,j_1,\ldots,j_k,1}$ be an arc in $U_{k,j_1,\ldots,j_k}$ which meets $U_{k+1,j_1,\ldots,j_k,1}$ and such that $B_k \cap A_{k+1,j_1,\ldots,j_k,1}$ is an endpoint of $A_{k+1,j_1,\ldots,j_k,1}$. Let $U_{k,j_1,\ldots,j_k} \in \mathcal{U}_k'$ and suppose $A_{k+1,j_1,\ldots,j_k,i}$ is defined for $i \in \{1, \ldots, m\}$ where $m < n_{k,j_1,\ldots,j_k}$. If $B_k \cup \bigcup_{i=1}^m A_{k+1,j_1,\ldots,j_k,i}$ meets $U_{k+1,j_1,\ldots,j_k,m+1}$ let $A_{k+1,j_1,\ldots,j_k,m+1} = \emptyset$, otherwise, let $A_{k+1,j_1,\ldots,j_k,m+1}$ be an arc in $U_{k,j_1,\ldots,j_k}$ which meets $U_{k+1,j_1,\ldots,j_k,m+1}$ and such that $(B_k \cup \bigcup_{i=1}^m A_{k+1,j_1,\ldots,j_k,i}) \cap A_{k+1,j_1,\ldots,j_k,m+1}$ is an endpoint of $A_{k+1,j_1,\ldots,j_k,m+1}$. Let
\[ B_{k+1} = B_k \cup \bigcup \{ A_{k+1, i_1, \ldots, i_k, j_{k+1}} \mid U_{k+1, i_1, \ldots, i_k, j_{k+1}} \in \mathcal{U}_{k+1} \} \].

By induction, \( B_k \) is defined for each \( k = 1, 2, \ldots \).

Let \( B = A \cup B_1 \cup B_2 \cup \cdots \). Then \( B \) is connected since \( B_1 \subset B_2 \subset \cdots \), each \( B_i \) is connected and \( \bigcup B_i \) is dense in \( B \). The set \( B \) is compact since \( B - U \) is contained in a finite dendron for each open neighborhood \( U \) of \( A \) and \( A \) is compact. It is easy to show that \( B \) has property \( S \). To see this, let \( \varepsilon > 0 \) and let \( n \) be a positive integer such that \( 3/n < \varepsilon \). Since \( B_\infty \) has property \( S \), there is a positive integer \( m \) and continua \( K_1, \ldots, K_m \) such that \( B_\infty = K_1 \cup \cdots \cup K_m \) and each \( K_i \) has diameter \( < 1/n \). Let \( U \in \mathcal{U}_m \). Let \( K_{t_1}, \ldots, K_{t_r} \) be the members of \( \{ K_1, \ldots, K_m \} \) which meet \( U \). Then \( (K_{t_1} \cup \cdots \cup K_{t_r} \cup U) \cap B \) has at most \( i_r \) components, and each of these has diameter \( < 3/n < \varepsilon \).

It follows that \( B \) has property \( S \) and hence is locally connected (see [6, p. 20]). By [2, p. 174] \( B \) is a retract of \( X \).

It suffices to prove that there is a continuous mapping \( f: B \to B \) such that \( f(x) = x \) if and only if \( x \in A \). Since \( B \) is locally connected, each component of \( B - A \) is open in \( B \). Hence, \( B - A \) has at most countably many components \( C_1, C_2, \ldots \). Notice that every component of \( B - A \) is a simply connected local graph. It follows from the last sentence and from the construction of the sets \( B_k \) that every sequence of pairwise disjoint arcs in \( B - A \) is a null sequence. Hence, the sequence \( C_1, C_2, \ldots \) is null. It suffices to prove, therefore, that for each \( i \geq 1 \) there exists a continuous mapping \( g_i: C_1(C_i) \to C_1(C_i) \) such that \( g_i(x) = x \) if and only if \( x \in C_1(C_i) - C_i \). The existence of \( g_i \) follows easily from the fact that \( C_i \) is a simply connected local graph in which every sequence of pairwise disjoint arcs is null.

**References**


Received February 23, 1979 and in revised form July 5, 1979. The first author’s research was supported in part by the National Research Council of Canada (Grant A8205), and the second author’s research was supported in part by the National Research Council of Canada (Grant A5616).
Pacific Journal of Mathematics
Vol. 89, No. 1 May, 1980

David Bressoud, A note on gap-frequency partitions ........................................... 1
John David Brillhart, A double inversion formula ............................................... 7
Frank Richard Deutsch, Günther Nürnberger and Ivan Singer, Weak
Chebyshev subspaces and alternation ................................................................. 9
Edward Richard Fadell, The relationship between Ljusternik-Schnirelman
category and the concept of genus ................................................................. 33
Harriet Jane Fell, On the zeros of convex combinations of polynomials ....... 43
John Albert Fridy, An addendum to: “Tauberian theorems via block
dominated matrices” .......................................................................................... 51
Andrzej Granas, Ronald Bernard Guenther and John Walter Lee, Applications
of topological transversality to differential equations. I. Some nonlinear
diffusion problems ............................................................................................... 53
David E. Handelman and G. Renault, Actions of finite groups on self-injective
rings ....................................................................................................................... 69
Michael Frank Hutchinson, Local Δ sets for profinite groups ......................... 81
Arnold Samuel Kas, On the handlebody decomposition associated to a
Lefschetz fibration ................................................................................................. 89
Hans Keller, On the lattice of all closed subspaces of a Hermitian space ...... 105
P. S. Kenderov, Dense strong continuity of pointwise continuous
mappings ............................................................................................................... 111
Robert Edward Kennedy, Krull rings ................................................................. 131
Jean Ann Larson, Richard Joseph Laver and George Frank McNulty,
Square-free and cube-free colorings of the ordinals ......................................... 137
Viktor Losert and Harald Rindler, Cyclic vectors for $L^p(G)$ ......................... 143
John Rowlay Martin and Edward D. Tymchatyn, Fixed point sets of
1-dimensional Peano continua ........................................................................ 147
Augusto Nobile, On equisingular families of isolated singularities .............. 151
Kenneth Joseph Prevote, Imbedding smooth involutions in trivial bundles ... 163
Thomas Munro Price, Spanning surfaces for projective planes in four
space .................................................................................................................... 169
Dave Riffelmacher, Sweedler’s two-cocycles and Hochschild
cohomology ........................................................................................................ 181
Niels Schwartz, Archimedean lattice-ordered fields that are algebraic over
their o-subfields .................................................................................................. 189
Chao-Liang Shen, A note on the automorphism groups of simple dimension
groups ................................................................................................................ 199
Kenneth Barry Stolarsky, Mapping properties, growth, and uniqueness of
Vieta (infinite cosine) products ......................................................................... 209
Warren James Wong, Maps on simple algebras preserving zero products. I.
The associative case .......................................................................................... 229