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**FIXED POINT SETS OF 1-DIMENSIONAL PEANO CONTINUA**

JOHN ROWLAY MARTIN AND EDWARD D. TYMCHATYN

## FIXED POINT SETS OF 1-DIMENSIONAL PEANO CONTINUA

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**It is shown that every nonempty closed subset of a 1-dimensional Peano continuum  $X$  is the fixed point set of some continuous self-mapping of  $X$ .**

1. **Introduction.** A topological space  $X$  is said to have the *complete invariance property* (CIP) if every nonempty closed subset of  $X$  is the fixed point set of some continuous self-mapping of  $X$ . The term CIP was suggested by L. E. Ward, Jr. in [5, p. 553] where it was asked if every Peano continuum had CIP. Examples have been given in [3], [4, 3.1] which show that  $n$ -dimensional Peano continua need not have CIP if  $n > 1$ . In [4, 3.4] it is asked if every 1-dimensional Peano continuum has CIP. The purpose of this note is to answer that question in the affirmative by showing that every 1-dimensional Peano continuum has CIP.

2. **Preliminaries.** Let  $M$  be a metric space. A sequence of subsets of  $M$  is called a *null sequence* provided that for any  $\varepsilon > 0$  at most a finite number of its elements has diameter greater than  $\varepsilon$ . The space  $M$  is said to have *property S* provided that for each  $\varepsilon > 0$ ,  $M$  is the union of a finite number of connected sets each of diameter less than  $\varepsilon$ . A *partitioning* of  $M$  is a finite collection  $\mathcal{U}$  of pairwise disjoint connected open subsets of  $M$  whose union is dense in  $M$ . If the mesh of  $\mathcal{U}$  is less than  $\varepsilon$  (each element of  $\mathcal{U}$  is of diameter less than  $\varepsilon$ ),  $\mathcal{U}$  is called an  $\varepsilon$ -*partitioning*. A sequence  $\mathcal{U}_1, \mathcal{U}_2, \dots$  of partitionings is called a *decreasing sequence* if, for each positive integer  $i$ ,  $\mathcal{U}_{i+1}$  is a refinement of  $\mathcal{U}_i$  and the mesh of  $\mathcal{U}_i$  approaches 0 as  $i$  increases without limit. It is well-known [1, p. 545] that every Peano continuum has a decreasing sequence of partitionings.

A *dendron* is a connected, simply connected, finite graph. The closure of a subset  $A$  of a topological space shall be denoted by  $C1(A)$ .

### 3. The result.

**THEOREM** *Every 1-dimensional Peano continuum has the complete invariance property.*

*Proof.* Let  $X$  be a 1-dimensional Peano continuum and let  $A$  be

a closed subset of  $X$ . Let  $\mathcal{U}_1, \mathcal{U}_2, \dots$  be a decreasing sequence of  $(1/i)$ -partitionings of  $X$ . Then each  $\mathcal{U}_i$  is a finite collection of open connected pairwise disjoint sets of diameter less than  $1/i$  such that for each  $i$   $\bigcup\{U \mid U \in \mathcal{U}_i\}$  is dense in  $X$  and  $\mathcal{U}_{i+1}$  refines  $\mathcal{U}_i$ . We suppose

$$\begin{aligned}\mathcal{U}_1 &= \{U_{1,1}, \dots, U_{1,m_{0,1}}\} \\ \mathcal{U}_2 &= \{U_{2,i,j} \mid U_{2,i,j} \subset U_{1,i} \in \mathcal{U}_1 \text{ and } j \in \{1, \dots, m_{1,i}\}\}\end{aligned}$$

and for  $i > 1$

$$\begin{aligned}\mathcal{U}_i &= \{U_{i,j_1, \dots, j_i} \mid U_{i,j_1, \dots, j_i} \subset U_{i-1, j_1, \dots, j_{i-1}} \in \mathcal{U}_{i-1} \\ &\quad \text{and } j_i \in \{1, \dots, m_{i-1, j_1, \dots, j_{i-1}}\}\}.\end{aligned}$$

For each  $i = 1, 2, \dots$  let

$$\mathcal{U}'_i = \{U \in \mathcal{U}_i \mid C1(U) \cap A \neq \emptyset\}.$$

Without loss of generality,

$$\begin{aligned}\mathcal{U}'_1 &= \{U_{1,1, \dots, 1}, U_{1,n_{0,1}}\} \quad \text{and for } i > 1 \\ \mathcal{U}'_i &= \{U_{i,j_1, \dots, j_i} \mid U_{i,j_1, \dots, j_i} \subset U_{i-1, j_1, \dots, j_{i-1}} \in \mathcal{U}'_{i-1} \\ &\quad \text{and } j_i \in \{1, \dots, n_{i-1, j_1, \dots, j_{i-1}}\}\}.\end{aligned}$$

Notice that  $A \subset C1(\bigcup \mathcal{U}'_i)$  for each  $i$ .

Let  $A_{1,1}$  be an arc in  $X$  which meets  $U_{1,1}$  and  $U_{1,2}$ . If  $A_{1,1} \cap U_{1,3} \neq \emptyset$  let  $A_{1,2} = \emptyset$ . If  $A_{1,1} \cap U_{1,3} = \emptyset$  let  $A_{1,2}$  be an arc such that  $A_{1,2}$  meets  $U_{1,3}$  and  $A_{1,1} \cap A_{1,2}$  is an endpoint of  $A_{1,2}$ . Suppose  $A_{1,1} \cup \dots \cup A_{1,i}$  is a finite dendron such that  $A_{1,1} \cup \dots \cup A_{1,i}$  meets  $U_{1,j}$  for each  $j \in \{1, \dots, i+1\}$ . If  $i+2 \leq n_{0,1}$  let  $A_{1,i+1} = \emptyset$  if  $(A_{1,1} \cup \dots \cup A_{1,i}) \cap U_{1,i+2} \neq \emptyset$ , otherwise, let  $A_{1,i+1}$  be an arc which meets  $U_{1,i+2}$  and such that  $(A_{1,1} \cup \dots \cup A_{1,i}) \cap A_{1,i+1}$  is an endpoint of  $A_{1,i+1}$ . By induction  $A_{1,i}$  is defined for each  $i \in \{1, \dots, n_{0,1} - 1\}$ . Let

$$B_1 = A_{1,1} \cup \dots \cup A_{1,n_{0,1}-1}.$$

Suppose  $B_1, \dots, B_k$  are finite dendrons such that  $B_1 \subset B_2 \subset \dots \subset B_k$ ,  $B_k$  meets  $U$  for each  $U \in \mathcal{U}'_k$  and

$$B_k - B_{k-1} \subset \bigcup\{U \mid U \in \mathcal{U}'_k\}.$$

For each  $U_{k,j_1, \dots, j_k} \in \mathcal{U}'_k$  let  $A_{k+1, j_1, \dots, j_{k,1}} = \emptyset$  if  $B_k$  meets  $U_{k+1, j_1, \dots, j_{k,1}}$ , otherwise, let  $A_{k+1, j_1, \dots, j_{k,1}}$  be an arc in  $U_{k, j_1, \dots, j_k}$  which meets  $U_{k+1, j_1, \dots, j_{k,1}}$  and such that  $B_k \cap A_{k+1, j_1, \dots, j_{k,1}}$  is an endpoint of  $A_{k+1, j_1, \dots, j_{k,1}}$ . Let  $U_{k, j_1, \dots, j_k} \in \mathcal{U}'_k$  and suppose  $A_{k+1, j_1, \dots, j_{k,i}}$  is defined for  $i \in \{1, \dots, m\}$  where  $m < n_{k, j_1, \dots, j_k}$ . If  $B_k \cup \bigcup_{i=1}^m A_{k+1, j_1, \dots, j_{k,i}}$  meets  $U_{k+1, j_1, \dots, j_{k, m+1}}$  let  $A_{k+1, j_1, \dots, j_{k, m+1}} = \emptyset$ , otherwise, let  $A_{k+1, j_1, \dots, j_{k, m+1}}$  be an arc in  $U_{k, j_1, \dots, j_k}$  which meets  $U_{k+1, j_1, \dots, j_{k, m+1}}$  and such that  $(B_k \cup \bigcup_{i=1}^m A_{k+1, j_1, \dots, j_{k,i}}) \cap A_{k+1, j_1, \dots, j_{k, m+1}}$  is an endpoint of  $A_{k+1, j_1, \dots, j_{k, m+1}}$ . Let

$$B_{k+1} = B_k \cup \bigcup \{A_{k+1, j_1, \dots, j_k, j_{k+1}} \mid U_{k+1, j_1, \dots, j_k, j_{k+1}} \in \mathcal{U}'_{k+1}\}.$$

By induction  $B_k$  is defined for each  $k = 1, 2, \dots$ .

Let  $B = A \cup B_1 \cup B_2 \cup \dots$ . Then  $B$  is connected since  $B_1 \subset B_2 \subset \dots$ , each  $B_i$  is connected and  $\bigcup B_i$  is dense in  $B$ . The set  $B$  is compact since  $B - U$  is contained in a finite dendron for each open neighborhood  $U$  of  $A$  and  $A$  is compact. It is easy to show that  $B$  has property  $S$ . To see this, let  $\varepsilon > 0$  and let  $n$  be a positive integer such that  $3/n < \varepsilon$ . Since  $B_n$  has property  $S$ , there is a positive integer  $m$  and continua  $K_1, \dots, K_m$  such that  $B_n = K_1 \cup \dots \cup K_m$  and each  $K_i$  has diameter  $< 1/n$ . Let  $U \in \mathcal{U}'_n$ . Let  $K_{i_1}, \dots, K_{i_r}$  be the members of  $\{K_1, \dots, K_m\}$  which meet  $U$ . Then  $(K_{i_1} \cup \dots \cup K_{i_r} \cup U) \cap B$  has at most  $i_r$  components, and each of these has diameter  $< 3/n < \varepsilon$ . It follows that  $B$  has property  $S$  and hence is locally connected (see [6, p. 20]). By [2, p. 174]  $B$  is a retract of  $X$ .

It suffices to prove that there is a continuous mapping  $f: B \rightarrow B$  such that  $f(x) = x$  if and only if  $x \in A$ . Since  $B$  is locally connected, each component of  $B - A$  is open in  $B$ . Hence,  $B - A$  has at most countably many components  $C_1, C_2, \dots$ . Notice that every component of  $B - A$  is a simply connected local graph. It follows from the last sentence and from the construction of the sets  $B_k$  that every sequence of pairwise disjoint arcs in  $B - A$  is a null sequence. Hence, the sequence  $C_1, C_2, \dots$  is null. It suffices to prove, therefore, that for each  $i \geq 1$  there exists a continuous mapping  $g_i: C1(C_i) \rightarrow C1(C_i)$  such that  $g_i(x) = x$  if and only if  $x \in C1(C_i) - C_i$ . The existence of  $g_i$  follows easily from the fact that  $C_i$  is a simply connected local graph in which every sequence of pairwise disjoint arcs is null.

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David Bressoud, <i>A note on gap-frequency partitions</i> .....	1
John David Brillhart, <i>A double inversion formula</i> .....	7
Frank Richard Deutsch, Günther Nürnberger and Ivan Singer, <i>Weak Chebyshev subspaces and alternation</i> .....	9
Edward Richard Fadell, <i>The relationship between Ljusternik-Schnirelman category and the concept of genus</i> .....	33
Harriet Jane Fell, <i>On the zeros of convex combinations of polynomials</i> .....	43
John Albert Fridy, <i>An addendum to: "Tauberian theorems via block dominated matrices"</i> .....	51
Andrzej Granas, Ronald Bernard Guenther and John Walter Lee, <i>Applications of topological transversality to differential equations. I. Some nonlinear diffusion problems</i> .....	53
David E. Handelman and G. Renault, <i>Actions of finite groups on self-injective rings</i> .....	69
Michael Frank Hutchinson, <i>Local <math>\Lambda</math> sets for profinite groups</i> .....	81
Arnold Samuel Kas, <i>On the handlebody decomposition associated to a Lefschetz fibration</i> .....	89
Hans Keller, <i>On the lattice of all closed subspaces of a Hermitian space</i> .....	105
P. S. Kenderov, <i>Dense strong continuity of pointwise continuous mappings</i> .....	111
Robert Edward Kennedy, <i>Krull rings</i> .....	131
Jean Ann Larson, Richard Joseph Laver and George Frank McNulty, <i>Square-free and cube-free colorings of the ordinals</i> .....	137
Viktor Losert and Harald Rindler, <i>Cyclic vectors for <math>L^p(G)</math></i> .....	143
John Rowlay Martin and Edward D. Tymchatyn, <i>Fixed point sets of 1-dimensional Peano continua</i> .....	147
Augusto Nobile, <i>On equisingular families of isolated singularities</i> .....	151
Kenneth Joseph Prevot, <i>Imbedding smooth involutions in trivial bundles</i> .....	163
Thomas Munro Price, <i>Spanning surfaces for projective planes in four space</i> .....	169
Dave Riffelmacher, <i>Sweedler's two-cocycles and Hochschild cohomology</i> .....	181
Niels Schwartz, <i>Archimedean lattice-ordered fields that are algebraic over their <math>o</math>-subfields</i> .....	189
Chao-Liang Shen, <i>A note on the automorphism groups of simple dimension groups</i> .....	199
Kenneth Barry Stolarsky, <i>Mapping properties, growth, and uniqueness of Vieta (infinite cosine) products</i> .....	209
Warren James Wong, <i>Maps on simple algebras preserving zero products. I. The associative case</i> .....	229