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By a *submersion* we shall understand a C^∞ surjection f: $X \to Y$ between paracompact C^∞ manifolds with $\dim X \geq \dim Y$, subject to the condition that the differential of f have maximal rank at all points. This implies that the fiber f_y over any point $y \in Y$ will be a smooth regularly imbedded submanifold of X. Differentiable fiber bundles constitute a special class of submersions, characterized by the existence of local product structures, and in this particular case all fibers f_y are homeomorphic to a standerd fiber F. The central result in the homology theory of fiber bundles asserts the existence of a convergent spectral sequence whose E^∞ term is the bigraded group associated to some filtration of $H_*(X; G)$[1], and for which

$$E^2_{s,t} \approx H_s(Y; H_t(F; G))$$

in case the bundle is orientable over G. In the present paper this result is generalized to arbitrary submersions. The E^2 terms now come to be identified with certain groups $H_{s,t}(f; G)$ representing a homology functor from the category of submersions to the category of bigraded groups, which reduce of course to $H_s(Y; H_t(F; G))$ in the classical case.

The functor in question has been previously studied by Sekino [4], who has shown that in its relativised form it satisfies axioms of the Eilenberg-Steenrod type, for which a categoricity theorem can be established. In §1 we will give a direct constructive definition of the homology groups $H_{s,t}(f; G)$. The construction of the spectral sequence and identification of its E^2 terms take place in §§2 and 3, followed by a final section setting forth a few applications of the theory. We also note that §4 should be entirely comprehensible following §1.

A few remarks on the subject of applications may be in order. The usefulness of a spectral sequence clearly depends on the ease with which the homology (or cohomology) groups representing its E^2 terms can be evaluated or estimated in various geometric situations. For the spectral sequence under consideration we have found that these calculations can generally be effected with particular ease and directness, and so far as the study of submersions is concerned, this approach may well offer advantages over the sheaf-theoretic cohomology approach represented by the spectral sequence of Leray. Our

[1] Throughout this paper G will denote an arbitrary coefficient group and H_* the singular homology functor.

construction capitalizes on two fundamental facts: firstly, that every compact subset of a fiber can be enclosed in tubular a neighborhood, a property which generalizes the local product structures characteristic of fiber bundles, and which itself may be regarded as characterizing the class of submersions.[2] Secondly, we make stong use of the fact that homology commutes with direct limits. These premises, together with the circumstance that our base spaces are triangulable, enable one to construct a homology theory and corresponding spectral sequence constituting a particulary direct geometrical approach to the problem at hand, which is to study the relations between the topological structures of the total space, base space and fibers. To give at least one example of such a relation at this point, we will cite the following result established in § 4.

THEOREM. *Let $f: X \to Y$ be a submersion between orientable manifolds of dimensions n and m, respectively, having connected fibers; and let U denote the subspace of Y corresponding to compact fibers. The Betti number of X and U are then subject to the relation $R_{n-1}(X; G) \geqq R_{m-1}(U; G)$.*

In particular, when Y is the real line this theorem asserts that $R_{n-1}(X; G)$ is bounded below by the number of components in U, which might be a hitherto unnoticed result concerning real-valued functions with nowhere vanishing gradient. It should also be pointed out that our theorem pertains strictly to the theory of submersions, inasmuch as it obviously fails for more general maps (the map $f: R^2 \to R$ defined by the formula $f(x, y) = x^2 + y^2$ would provide an easy counter-example).

1. Homology of submersions. Our construction hinges on the concept of a *simplicial bundle* over a simplicial complex K, which we define to be a function that assigns to each simplex $\sigma \in K$ a commutative triangle

$$\begin{array}{ccc} |\sigma| \times F_\sigma & \xrightarrow{\Phi_\sigma} & E_\sigma \\ & {\scriptstyle \pi_\sigma} \searrow \quad \swarrow {\scriptstyle p_\sigma} & \\ & |\sigma| & \end{array}$$

where E_σ, F_σ are topological spaces, $|\sigma|$ is the closed space of σ, Φ_σ a homeomorphism and π_σ the natural projection, this data being subject to the following *descending face condition*: for every face $\tau < \sigma$ and $x \in |\tau|$,

[2] From this point of view one could dispense with the assumpation of differentiability.

$$p_\sigma^{-1}(x) \subset p_\tau^{-1}(x) .$$

This insures that the projections p_σ induce a projection $p: E \to |K|$, where E denotes the topological sum of the spaces E_σ. A simplicial bundle is said to *approximate* a submersion $f: X \to Y$ with respect to a compact subspace $C \subset X$ provided $C \subset E \subset X$, $f(C) \subset |K| \subset Y$ and $p = f|E$ (the restriction of f to E). The essential connection between simplicial bundles and submersions may now be expressed in terms of the following basic approximation theorem.

THEOREM 1.1. *Let $f: X \to Y$ be a submersion and C a compact subspace of X. Let M be a simplicial complex such that*[3] *$Y = |M|$. Then there exists a simplicial bundle over some barycentric subdivision K of a subcomplex of M which approximates f with respect to C, and whose total space E is compact.*

The proof is substantially identical[4] with the proof of Lemma 2 in Smith [6] and proceeds by an elementary inductive argument involving tubular neighborhood constructions. Essentially the approximation theorem permits us to view a submersion as the direct limit of an approximating system of simplicial bundles, an idea that has been formulated in category-theoretic terms by Sekino [4] and constitues the basis of his theory. For our present purpose it will suffice to associate with any given submersion a sequence of simplicial bundles satisfying the conditions of the following theorem, conditions which insure that the given sequence is cofinal in the full approximating system and consequently adequate for the computation of direct limits.

THEOREM 1.2. *Given a submersion $f: X \to Y$, there exists a sequence $\{{}_\alpha p: {}_\alpha E \to |{}_\alpha K|\}$ of simplicial bundles such that*

(1) *for every compact subspace $C \subset X$ there exists a positive integer α such that ${}_\alpha p: {}_\alpha E \to |{}_\alpha K|$ approximates f with respect to C;*

(2) *for $\alpha < \beta$, ${}_\alpha E \subset {}_\beta E$ and there exists a nonnegative integer n such that ${}^n_\alpha K \subset {}_\beta K$ (where ${}^n K$ denotes the nth barycentric subdivision of K).*

[3] Our differentiable manifolds are triangulable, and we will suppress the triangulating homeomorphism by assuming the corresponding identifications.

[4] The only difference being that in [**6**] we were concerned with simplicial bundles "dual" to the bundled here envisaged, i.e., bundles for which our *descending* face condition was replaced by a corresponding *ascending* face condition. While simplicial bundles of the descending type are adapted to homology theory, bundles of the ascending types are adapted to cohomology and obstruction theories. The only change required to adapt the proof in [**6**] to the present case is to reverse the direction of the induction procedure so as to go from high to low dimensions. See also Endicott [**2**], pp. 39-43.

To prove this on the basis of our first approximation theorem, one need only choose a covering of X in the form of a nested sequence $\{C_\alpha\}$ of a compact subsets and proceed by induction. Thus we suppose that for all $\alpha \leqq \gamma$ we have simplicial bundles $p_\alpha\colon {}_\alpha E \to |{}_\alpha K|$ satisfying condition (1) and (2), with the additional stipulation that each ${}_\alpha K$ is a barycentric subdivion of a subcomplex belonging to a given triangulation L of Y. Let m be a nonnegative integer such that ${}^m K_\gamma \subset L$. Applying 1.1 with $C = C_{\gamma+1} \cup {}_\gamma E$ and $M = {}^m L$ yields a simplicial bundle on the $(\gamma + 1)$-level which preserves our inductive hypothesis, and this completes the proof.

At this point the fundamental idea underlying our construction of the homology groups $H_{s,t}(f; G)$ can be explained. Given a simplicial bundle $p\colon E \to |K|$, coefficient group G and integer t, we will define a corresponding chain complex $C_*(K; H_t(F; G))$ in which the homology groups $\{H_t(F_\sigma; G), \sigma \in K\}$ function as a local coefficient system. Given a submersion $f\colon X \to Y$, we will choose an approximating sequence of simplicial bundles in accordance with 1.2, and this gives rise to a corresponding sequence of chain complexes $C_*({}_\alpha K; H_t({}_\alpha F; G))$, together with chain projections ϕ_α^β defined for $\alpha < \beta$, constituting a direct system of chain complexes. The homology groups in question may now be defined by setting

$$(H_{s,t}(f; G)) = H_s(\varinjlim C_*({}_\alpha K; H_t({}_\alpha F; G)))\ ;$$

or equivalently, as the direct limit of the direct system

$$\{H_s(C_*({}_\alpha K; H_t({}_\alpha F; G))), \phi_{\alpha^*}^{\beta}\}\ .$$

It follows from the previously noted cofinality of our approximating sequence $\{{}_\alpha p\colon {}_\alpha E \to |{}_\alpha K|\}$ that the groups $H_{s,t}(f; G)$ can be identified with the functorial homology groups investigated by Sekino, a fact which we need not presuppose, however, and which will play no role in the applications presented in § 4.

We proceed now to define the chain complex associated with a simplicial bundle $p\colon E \to |K|$. For $s \geqq 0$ we let

$$C_s(K; H_t(F; G)) = \bigoplus_{\sigma \in K^{(s)}} H_s(|\sigma|, |\dot{\sigma}|) \otimes H_t(F_\sigma; G) \tag{1.3}$$

where $K^{(s)}$ denotes the set of s-simplexes in K and $|\dot{\sigma}|$ the boundary of $|\sigma|$. To obtain boundary operators, we note that the inclusion maps[5]

$$j_\sigma\colon (|\sigma|, |\dot{\sigma}|) \subset (|K^s|, |K^{s-1}|)$$

induce a direct sum representation

[5] K^s denotes the s-skeleton of K.

$$\{j_{\sigma^*}\}: \bigoplus_{\sigma \in K^{(s)}} H_s(|\sigma|, |\dot{\sigma}|) \approx H_s(K^s, K^{s-1}) ,$$

and we let

$$q_\sigma: H_s(K^s, K^{s-1}) \longrightarrow H_s(|\sigma|, |\dot{\sigma}|)$$

denote the corresponding projections. For every $\tau \in K^{(s-1)}$ we can define a homomorphism

$$\varepsilon_\sigma^\tau: H_s(|\sigma|, |\dot{\sigma}|) \longrightarrow H_{s-1}(|\tau|, |\dot{\tau}|)$$

by setting $\varepsilon_\sigma^\tau = q_\tau \circ \partial_* \circ j_{\sigma^*}$, where

$$\partial_*: H_s(K^s, K^{s-1}) \longrightarrow H_{s-1}(K^{s-1}, K^{s-2})$$

denotes the connecting homomorphism for the triple (K^s, K^{s-1}, K^{s-2}). A simple direct calculation gives

LEMMA 1.4. *Let τ and ρ denote simplexes in K of dimensions $(s-1)$ and $(s+1)$, repectively. Then*

$$\sum_{\sigma \in K^{(s)}} \varepsilon_\sigma^\tau \circ \varepsilon_\rho^\sigma = 0 .$$

For $\tau < \sigma$ and $x \in |\tau|$ one can define (by virtue of the descending face condition) an injection $i_{\sigma,x}^\tau: F_\sigma \to F_\tau$ through the formula

$$i_{\sigma,x}^\tau(y) = \pi'_\tau \circ \Phi_\tau^{-1} \circ \Phi_\sigma(x, y) \tag{1.5}$$

where $\pi'_\tau: |\tau| \times F_\tau \to F_\tau$ denotes the natural projection. Since the homotopy class of $i_{\sigma,x}^\tau$ is clearly independent of x, one obtains canonical homomorphisms

$$i_\sigma^\tau: H_*(F_\sigma; G) \longrightarrow H_*(F_\tau; G)$$

henceforth referred to as *fiber projections*, for which the following transitivity condition is easily verified [4, p. 39]:

$$i_\sigma^\tau \circ i_\rho^\sigma = i_\rho^\tau \quad \text{for} \quad \tau < \sigma < \rho . \tag{1.6}$$

The desired boundary operators

$$\partial: C_s(K; H_t(F; G)) \longrightarrow C_{s-1}(K; H_t(F; G))$$

may now be defined for $s \geqq 0$ through the formula[6]

$$\partial(a \otimes c) = \sum_{\tau \in K^{(s-1)}} \varepsilon_\sigma^\tau(a) \otimes i_\sigma^\tau(c) \tag{1.7}$$

where

[6] Since $\varepsilon_\sigma^\tau = 0$ unless $\tau < \sigma$, the formula makes sense.

$$a \otimes c \in H_s(|\sigma|, |\dot{\sigma}|) \otimes H_t(F_\sigma; G) , \tag{1.8}$$

and it follows by (1.4) and (1.6) that $\partial \circ \partial = 0$.

This completes our construction of the homology groups $H_s(K; H_t(F; G))$ associated with a simplicial bundle. It should be pointed out, however, that for a simplicial bundle approximating a submersion $f: X \to Y$, the fiber projections i_σ^τ admit a simple geometric interpretation. The homeomorphisms $\Phi_\sigma: |\sigma| \times F_\sigma \to E_\sigma$ now constitute tubular neighborhoods in X, and without loss of generality one may assume that these tubular neighborhoods are associated with a given "horizontal distribution" on X, i.e., a distribution of m-planes which is transverse to the fibers. If for each $\sigma \in K$ we identify F_σ with $p_\sigma^{-1}(b_\sigma)$, where b_σ is (say) the barycenter of σ, then it turns out that the fiber projection i_σ^τ is induced by an injection $i: F_\sigma \to F_\tau$ induced by the horizontal liftings of some path in $|\sigma|$ from b_σ to b_τ.

It remains now to construct the chain projections ϕ_α^β associated with an approximating sequence (1.2), for which purpose we need to consider *canonical* (or "barycentric") *subdivisions* of simplicial bundles [4, p. 42]. Let $p: E \to |K|$ be a simplicial bundle. For each s-simplex ω belonging to the nth barycentric subdivision ${}^nK$ of K there exists a unique $\sigma \in K^{(s)}$ such that $|\omega| \subset |\sigma|$, and setting ${}^nF_\omega = F_\sigma$, ${}^nE_\omega = p_\sigma^{-1}(|\omega|)$, ${}^n\Phi_\omega = \Phi_\sigma \,|\, |\omega| \times F_\sigma$ and ${}^np_\omega = p_\sigma \,|\, {}^nE_\omega$ one obtains a new simplicial bundle ${}^np: E \to |{}^nK|$, which is the nth canonical subdivision. On the chain level one obtains a corresponding chain homomorphism

$$Sd^n: C_s(K; H_t(F; G)) \longrightarrow C_s({}^nK; H_t({}^nF; G))$$

through the formula

$$Sd^n(a \otimes c) = Sd_\sigma^n(a) \otimes c \tag{1.9}$$

where again $a \otimes c$ is given as in (1.8), and

$$Sd_\sigma^n: H_s(|\sigma|, |\dot{\sigma}|) \longrightarrow \bigoplus H_s(|\omega|, |\dot{\omega}|)$$

is essentially the classical subdivision operator, which may be defined as follows. Let $A = |{}^nK^{s-1}| \cap |\sigma|$, and let $i: (|\sigma|, |\dot{\sigma}|) \subset (|\sigma|, A)$ and $i_\sigma^\omega: (|\omega|, |\dot{\omega}|) \subset (|\sigma|, A)$ denote inclusions. Then

$$\{i_{\sigma^*}^\omega\}: \bigoplus H_s(|\omega|, |\dot{\omega}|) \approx H_s(|\sigma|, A)$$

constitutes a direct-sum representation [8, p. 474], and one has

$$Sd_\sigma^n = \{i_{\sigma^*}^\omega\}^{-1} \circ i_* \equiv \sum_\omega B_{\sigma,\omega}^n . \tag{1.10}$$

Now let $\{{}_\alpha p: {}_\alpha E \to |{}_\alpha K|\}$ denote an approximating sequence as given by 1.2, and let $\alpha < \beta$. This determines an integer n such that ${}_\alpha^nK \subset {}_\beta K$. Moreover, for every $\sigma \in {}_\alpha K^{(s)}$ and $\omega \in {}_\beta K^{(s)}$ with $|\omega| \subset |\sigma|$,

we may identify ${}_\alpha F_\sigma$ and ${}_\beta F_\omega$ with ${}_\alpha p_\sigma^{-1}(x)$ and ${}_\beta p_\omega^{-1}(x)$, respectively, for some $x \in |\omega|$. Since ${}_\alpha E \subset E_\beta$, one thus obtains an inclusion induced injection $j_\sigma^\omega : {}_\alpha F_\sigma \to {}_\beta F_\omega$, whose homotopy class is independent of x. The desired chain projections

$$\phi_\alpha^\beta : C_s({}_\alpha K; H_t({}_\alpha F; G)) \longrightarrow C_s({}_\beta K; H_t({}_\beta F; G))$$

are defined on the generators $a \otimes c$ by setting

$$\phi_\alpha^\beta(a \otimes c) = \sum_\omega B_{\sigma,\omega}^n(a) \otimes j_{\sigma *}^\omega(c) , \tag{1.11}$$

and it is obvious that the transitivity conditions for a direct system are satisfied.

2. **The spectral sequence of a simplicial bundle.** As previously affirmed, we shall define a spectral sequence corresponding to a given submersion $f : X \to Y$ whose $E_{s,t}^2$ term may be identified with the homology group $H_{s,t}(f; G)$ constructed in the course of §1. Again our procedure will involve two basic steps: firstly, we will define a spectral sequence associated with a simplicial bundle $p : E \to |K|$, whose $E_{s,t}^2$ term may be identified with the corresponding homology group $H_s(K; H_t(F; G))$. In § 3 we shall once more consider an approximating sequence of simplicial bundles, which gives rise to a direct system of spectral sequences, from which we obtain the desired spectral sequence in the direct limit.

Our first step follows an approach introduced by Chern and Spanier [1], and subsequently extended by Spanier [8], which considers the filtration of the total space X induced from the skeletal filtration resulting from a triangulation of the base space. This filtration of X induces a filtration[7] of the singular chain complex $S(X; G)$, which in turn gives rise to a spectral sequence [8; §§ 9.1 & 9.2]. Applying this idea to the case of a simplicial bundle $p : E \to |K|$, we let E_s denote the empty set for $s < 0$ and set

$$E_s = \bigcup_{\sigma \in K^s} E_\sigma \quad \text{for} \quad s \geqq 0 ;$$

to obtain the following result, which is a special case of Theorem 2 [8; p. 469].

THEOREM 2.1. *There exists a convergent E^1 spectral sequence with*

$$E_{s\ t}^1 \approx H_{s+t}(E_s, E_{s-1}; G) ,$$

d^1 corresponding to the connecting homomorphism ∂_* of the triple

[7] This filtration is convergent and bounded below.

(E_s, E_{s-1}, E_{s-2}), and E^∞ isomorphic to the bigraded group associated to the filtration of $H_*(E; G)$ defined by

$$F_sH_*(E; G) = \mathrm{Im}\,[H_*(E_s; G) \longrightarrow H_*(E; G)]\ .$$

It will be shown that there exist isomorphisms

$$\psi_s\colon C_s(K; H_t(F; G)) \approx H_{s+t}(E_s, E_{s-1}; G)$$

under which the boundary operators (1.7) correspond precisely to ∂_*. This fact, together with 2.1, will give our first main result,

THEOREM 2.2. *Given a simplicial bundle $p\colon E \to |K|$ and coefficient group G, there exists a convergent E^2 spectral sequence with*

$$E^2_{s,t} \approx H_s(K; H_t(F; G))\ ,$$

and E^∞ isomorphic to the bigraded group associated with the filtration of $H_*(E; G)$ described in 2.1.

We begin our construction[8] of ψ_s with

LEMMA 2.3. *For every $\sigma \in K^{(s)}$ the homology cross product defines an isomorphism*

$$\mu_\sigma\colon H_s(|\sigma|, |\dot\sigma|) \otimes H_t(F_\sigma; G) \approx H_{s+t}((|\sigma|, |\dot\sigma|) \times F_\sigma; G)\ .$$

This follows directly from the Künneth formula [8; 10, p. 235]. Next, the homeomorphism Φ_σ of the simplicial bundle furnishes as isomorphism

$$\Phi_{\sigma*}\colon H_*((|\sigma|, |\dot\sigma|) \times F_\sigma; G) \approx H_*(E_\sigma, E_{\dot\sigma}; G)$$

where $E_{\dot\sigma} = p_\sigma^{-1}(|\dot\sigma|)$. The third step is given by

LEMMA 2.4. *The inclusion*

$$i'_\sigma\colon (E_\sigma, E_{\dot\sigma}) \subset p^{-1}(|\sigma|, |\dot\sigma|)$$

induces an isomorphism between corresponding homolgy groups.

To prove this, let e denote a closed s-cell in the interior of $|\sigma|$, let $\mathring{e}$ denote the interior of e, and consider the commutative diagram[9]

$$\begin{array}{ccc} H_*(E_\sigma, E_{\dot\sigma}) & \xrightarrow{h_1} & H_*(E_\sigma, p_\sigma^{-1}(|\sigma| - \mathring{e})) \\ {\scriptstyle i'_{\sigma*}}\downarrow & & \downarrow{\scriptstyle h_2} \\ H_*(p^{-1}(|\sigma|, |\dot\sigma|)) & \xrightarrow[h_3]{} & H_*(p^{-1}(|\sigma|, |\dot\sigma| - \mathring{e})) \end{array}$$

[8] Here one should recall (1.3).

[9] For brevity we will suppress G in some diagrams.

induced by inclusion maps. Since E_σ and $p^{-1}(|\sigma|)$ can differ only over $|\dot{\sigma}|$, and $|\dot{\sigma}|$ lies in the interior of $(|\sigma| - \mathring{e})$, one sees that h_2 is an excision induced isomorphism. Moreover, since $|\dot{\sigma}|$ is obviously a deformation retract of $(|\sigma| - \mathring{e})$, the isomorphism Φ_σ guarantees that $E_{\dot{\sigma}}$ is likewise a deformation retract of $p_\sigma^{-1}(|\sigma| - \mathring{e})$, which implies that h_1 is an isomorphism. But since any deformation retraction

$$r_\sigma\colon p_\sigma^{-1}(|\sigma| - \mathring{e}) \longrightarrow E_{\dot{\sigma}}$$

extends trivially to a deformation retraction

$$r'_\sigma\colon p^{-1}(|\sigma| - e) \longrightarrow p^{-1}(|\dot{\sigma}|)\ , \tag{2.5}$$

it also follows that h_3 is an isomophism, and this proves our lemma.

The final step in our construction of ψ_s is provided by the following result, which is an exact analogue of Lemma 2 [8; p. 474].

LEMMA 2.6. *The inclusions*

$$i''_\sigma\colon p^{-1}(|\sigma|, |\dot{\sigma}|) \subset (E_s, E_{s-1})$$

induce a direct-sum representation

$$\{i''_{\sigma *}\}\colon \bigoplus_{\sigma \in K^{(s)}} H_*(p^{-1}(|\sigma|, |\dot{\sigma}|); G) \approx H_*(E_s, E_{s-1}; G)\ .$$

Moreover, in view of the deformation retractions (2.5), Spanier's proof carries over verbatim to the present case.

Combining the isomorphisms established in the four preceding steps, we may now define ψ_s by the formula

$$\psi_s = \{i_{\sigma *}\} \circ [\bigoplus \Phi_{\sigma *}] \circ [\bigoplus \mu_\sigma] \tag{2.7}$$

where the direct sums extend over all $\sigma \in K^{(s)}$ and $i_\sigma = i''_\sigma \circ i'_\sigma$ denotes the inclusion $(E_\sigma, E_{\dot{\sigma}}) \subset (E_s, E_{s-1})$.

It remains to be shown that the isomorphisms ψ_s commute with the respective boundary operators. Before embarking upon this somewhat delicate calculation, we shall establish the following lemma, which turns out to be crucial.

LEMMA 2.8. *Let $\sigma \in K^{(s)}$ and τ an $(s-1)$-face of σ. Let $x_0 \in |\tau|$, and let*

$$\Phi_\sigma^{\tau}\colon (|\tau|, |\dot{\tau}|) \times F_\sigma \longrightarrow (E_\sigma, E_{\dot{\sigma}})$$

denote the map induced by Φ_σ. There exists a homotopy

$$H\colon (|\tau|, |\dot{\tau}|) \times F_\sigma \times I \longrightarrow (E_s, E_{s-1})$$

such that

$$H\colon i_\tau \circ \Phi_\tau \circ (1_\tau \times i^\tau_{\sigma,x_0}) \cong i_\sigma \circ \Phi^\tau_\sigma$$

where $1_\tau\colon (|\tau|, |\dot{\tau}|) \to (|\tau|, |\dot{\tau}|)$ *is the identity and* $i^\tau_{\sigma\, x_0}\colon F_\sigma \to F_\tau$ *the injection given by* (1.5).

To construct H, we first define a map

$$\rho\colon (|\tau|, |\dot{\tau}|) \times F_\sigma \longrightarrow (|\tau|, |\dot{\tau}|) \times F_\tau$$

by setting

$$\rho(x, y) = (x, i^\tau_{\sigma,x}(y)) .$$

One verifies by a simple direct calculation that

$$\Phi_\tau \circ \rho = j \circ \Phi^\tau_\sigma \tag{2.9}$$

where

$$j\colon p_\sigma^{-1}(|\tau|, |\dot{\tau}|) \subset (E_\tau, E_\tau)$$

denotes the inclusion. Composing both sides of (2.9) with i_τ, and observing that $i_\tau \circ j$ is a restriction of i_σ, one obtains

$$i_\tau \circ \Phi_\tau \circ \rho = i_\sigma \circ \Phi^\tau_\sigma .$$

It will therefore suffice to construct a homotopy

$$\bar{H}\colon (|\tau|, |\dot{\tau}|) \times F_\sigma \times I \longrightarrow (|\tau|, |\dot{\tau}|) \times F_\tau$$

such that

$$\bar{H}\colon \rho \cong 1_\tau \times i^\tau_{\sigma,x_0} \tag{2.10}$$

from which the desired homotopy H can be obtained by setting $H = i_\tau \circ \Phi_\tau \circ \bar{H}$. To construct $\bar{H}$, we observe that there exists for each $x \in |\tau|$ an obvious homotopy $H_x\colon F_\sigma \times I \to F_\tau$ such that

$$H_x\colon i^\tau_{\sigma,x} \cong i^\tau_{\sigma,x_0}$$

which results from (1.5) by identifying I with the line segment in $|\tau|$ from x to x_0 under the canonical affine map. One can then define $\bar{H}$ through the formula

$$\bar{H}(x, y, t) = (x, H_x(y, t))$$

which gives (2.10), and establishes our lemma.

The desired commutativity reduces to the formula

$$\psi_{s-1} \circ \partial(a \otimes c) = \partial_* \cdot \psi_s(a \otimes c) \tag{2.11}$$

which we will now establish by direct calculation. Since ∂ is given by (1.7), one obtains

$$\psi_{s-1}\cdot\partial(a\otimes c)=\psi_{s-1}\sum_{\tau}\varepsilon_{\sigma}^{\tau}(a)\otimes i_{\sigma}^{\tau}(c) \tag{2.12}$$

where the sum can be thought of as extending over all $\tau\in K^{(s-1)}$. Applying (2.7), and bearing in mind that μ_{τ} represents the homology cross product, one may rewrite (2.12) in the form

$$\psi_{s-1}\circ\partial(a\otimes c)=\sum_{\tau} i_{\tau *}\cdot\Phi_{\tau *}(\varepsilon_{\sigma}^{\tau}(a)\times i_{\sigma}^{\tau}(c))\ . \tag{2.13}$$

It follows from our definition of the fiber projections (given in §1) that

$$i_{\sigma}^{\tau}=(i_{\sigma,x_0}^{\tau})_{*}$$

for any $x_0\in|\tau|$, which together with a well-known property of the cross product [8; 11, p. 235] gives

$$\varepsilon_{\sigma}^{\tau}(a)\times i_{\sigma}^{\tau}(c)=(1_{\tau}\times i_{\sigma,x_0}^{\tau})_{*}(\varepsilon_{\sigma}^{\tau}(a)\times c)\ .$$

Substituting this into (2.13) and applying (2.8) gives

$$\psi_{s-1}\circ\partial(a\otimes c)=\sum_{\tau} i_{\sigma *}\circ\Phi_{\sigma *}^{\tau}(\varepsilon_{\sigma}^{\tau}(a)\times c)\ . \tag{2.14}$$

Before going further, we must introduce some additional spaces and maps. We will let $|\ddot{\sigma}|$ denote the union of all closed $(s-2)$-faces in σ, and set $E_{\sigma}^{\cdot\cdot}=p_{\sigma}^{-1}(|\ddot{\sigma}|)$. This brings into play the new inclusions

$$i_{\sigma}^{\cdot}\colon(E_{\sigma}^{\cdot},E_{\sigma}^{\cdot\cdot})\subset(E_{s-1},E_{s-2})\ ,$$
$$i^{\tau}\colon(|\tau|,|\dot{\tau}|)\subset(|\dot{\sigma}|,|\ddot{\sigma}|)\ ,$$

together with the homeomorphism

$$\Phi_{\sigma}^{\cdot}\colon(|\dot{\sigma}|,|\ddot{\sigma}|)\times F_{\sigma}\approx(E_{\sigma}^{\cdot},E_{\sigma}^{\cdot\cdot})$$

constituting a restriction of Φ_{σ}. Since

$$i_{\sigma}\circ\Phi_{\sigma}^{\tau}=i_{\sigma}^{\cdot}\circ\Phi_{\sigma}^{\cdot}\circ(i^{\tau}\times 1_{F_{\sigma}})$$

one may rewrite (2.14) in the form

$$\psi_{s-1}\circ\partial(a\otimes c)=\sum_{\tau} i_{\sigma *}^{\cdot}\circ\Phi_{\sigma *}^{\cdot}\circ(i^{\tau}\times 1_{F_{\sigma}})_{*}(\varepsilon_{\sigma}^{\tau}(a)\times c)\ . \tag{2.15}$$

Using [8; 11, p. 235] once more, one has

$$(i^{\tau}\times 1_{F_{\sigma}})_{*}(\varepsilon_{\sigma}^{\tau}(a)\times c)=i_{*}^{\tau}\circ\varepsilon_{\sigma}^{\tau}(a)\times c\ . \tag{2.16}$$

By a tedious but elementary calculation, for which we refer to [2; pp. 36-37], one verifies that

$$\sum_{\tau} i_{*}^{\tau}\circ\varepsilon_{\sigma}^{\tau}(a)=\partial'_{*}(a) \tag{2.17}$$

where

$$\partial'_*\colon H_s(|\sigma|, |\dot{\sigma}|) \longrightarrow H_{s-1}(|\dot{\sigma}|, |\ddot{\sigma}|)$$

denotes the connecting homomorphism of the triple $(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|)$. Letting

$$\partial''_*\colon H_{s+t}(E_\sigma, E_{\dot{\sigma}}; G) \longrightarrow H_{s+t-1}(E_{\dot{\sigma}}, E_{\ddot{\sigma}}; G)$$

denote the connecting homomorphism of the triple $(E_\sigma, E_{\dot{\sigma}}, E_{\ddot{\sigma}})$, one obtains

$$\partial'_*(a) \times c = \partial''_*(a \times c) \tag{2.18}$$

which follows as a special case of [8; 15, p. 235], bearing in mind that the exact sequence of a triple constitutes a special case of the relative Mayer-Vietoris sequence [8; pp. 180–190]. Combining (2.16), (2.17), and (2.17), and (2.18) with (2.15) gives

$$\psi_{s-1} \circ \partial(a \otimes c) = i_{\dot{\sigma}^*} \circ \Phi_{\dot{\sigma}^*} \circ \partial''_*(a \times c)\ . \tag{2.19}$$

Now let ∂'''_* denote the connecting homomorphism of the triple $(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \times F_\sigma$. By naturality of the connecting homomorphism one obtains

$$\Phi_{\dot{\sigma}^*} \circ \partial''_* = \partial'''_* \circ \Phi_{\sigma^*} \tag{2.20}$$

and for the same reason

$$i_{\dot{\sigma}^*} \circ \partial'''_* = \partial_* \circ i_{\sigma_*}\ . \tag{2.21}$$

Substituting (2.20) and (2.21) into (2.19), one obtains

$$\psi_{s-1} \circ \partial(a \otimes c) = \partial_* \circ i_{\sigma^*} \circ \Phi_{\sigma_*}(a \times c)\ . \tag{2.22}$$

Recalling the definition (2.7) of ψ_s, together with the fact that μ_σ represents the cross product, one sees at last that the right side of (2.22) reduces to $\partial_* \circ \psi_s(a \otimes c)$, which confirms (2.11) and completes the proof of Theorem 2.2.

3. The spectral sequence of a submersion. We now turn to the main task, which is to construct a spectral sequence a spectral sequence associated with a given submersion $f\colon X \to Y$. Let $\{{}_\alpha p\colon {}_\alpha E \to |{}_\alpha K|\}$ be an approximating sequence of simplicial bundles supplied by (1.2), and for each α let $\{{}_\alpha E^r, {}_\alpha d^r\}$ denote the spectral sequence associated with the corresponding simplicial bundle and given coefficient group by the construction of § 2. Condition (2) in Theorem 1.2 implies that the inclusion ${}_\alpha E \subset {}_\beta E$ (defined for $\alpha < \beta$) is filtration preserving, i.e., that ${}_\alpha E_s \subset {}_\beta E_s$ for all s, and consequently induces a homomorphism ${}^\beta_\alpha\phi^r$ between the respective spectral sequences. This system of

homomorphisms obviously satisfies the requisite transitivity conditions, and one therefore obtains a direct system of spectral sequences. Taking the direct limits

$$E^r = \varinjlim {}_\alpha E^r$$
$$d^r = \varinjlim {}_\alpha d^r$$

gives, for each $r \geqq 1$, a bigraded group E^r, together with a differential d^r of bidegree $(-r, r-1)$. The remaining ingredient of a spectral sequence, i.e., a system of bigraded group isomorphisms $\theta^r: H(E^r) \approx E^{r+1}$, can likewise be obtained as a direct limit of the corresponding isomorphisms ${}_\alpha\theta^r: H({}_\alpha E^r) \approx {}_\alpha E^{r+1}$. More precisely, the fact that ${}^\beta_\alpha\phi^r$ constitutes a homomorphism of spectral sequences implies commutativity of the diagram

$$\begin{array}{ccc} H({}_\alpha E^r) & \xrightarrow{{}_\alpha\theta^r} & {}_\alpha E^{r+1} \\ \downarrow{\scriptstyle {}^\beta_\alpha\phi^r_*} & & \downarrow{\scriptstyle {}^\beta_\alpha\phi^{r+1}} \\ H({}_\beta E^r) & \xrightarrow{{}_\beta\theta^r} & {}_\beta E^{r+1} \end{array}$$

so that $\{{}_\alpha\theta^r\}$ constitutes an isomorphism between the given direct systems of bigraded groups. Since homology commutes with direct limits, one obtains the desired isomorphism $\theta^r: H(E^r) \approx E^{r+1}$ by setting

$$\theta^r = \varinjlim {}_\alpha\theta^r \ .$$

We have thus constructed a (first quadrant) E^1 spectral sequence $\{E^r, d^r\}$ associated with the given submersion, and it remains to verify our claims concerning its E^2 and E^∞ terms.

To begin with the E^∞ term, we must first exhibit an increasing filtration $F_s H_*(X; G)$ of the graded group $H_*(X; G)$, and must subsequently identify E^∞ with the bigraded group associated to the given filtration. Consider the filtrations $\{F_s H_*({}_\alpha E; G)\}$ of $H_*({}_\alpha E; G)$ defined in 2.1. The inclusions ${}_\alpha E_s \subset {}_\beta E_s$ induce homomorphisms

$${}^\beta_\alpha i_s: F_s H_*({}_\alpha E; G) \longrightarrow F_s H_*({}_\beta E; G)$$

which give rise to a commutative diagram

$$\begin{array}{ccccccccc} \longrightarrow & F_s H_*({}_\alpha E) & \xrightarrow{{}_\alpha j_s} & F_{s+1} H_*({}_\alpha E) & \longrightarrow \cdots \longrightarrow & F_m H_*({}_\alpha E) & = & H_*({}_\alpha E) \\ & \downarrow{\scriptstyle {}^\beta_\alpha i_s} & & \downarrow{\scriptstyle {}^\beta_\alpha i_{s+1}} & & \downarrow & & \\ \longrightarrow & F_s H_*({}_\beta E) & \xrightarrow{{}_\beta j_s} & F_{s+1} H_*({}_\beta E) & \longrightarrow \cdots \longrightarrow & F_m H_*({}_\beta E) & = & H_*({}_\beta E) \end{array}$$

where the ${}_\alpha j_s$ are actual inclusions and $m = \dim Y$. This constitutes a direct system of filtrations, and we define

$$F_sH_*(X;G) = \varinjlim F_sH_*({}_\alpha E;G)$$
$$j_s = \varinjlim {}_\alpha j_s \ .$$

Since the direct limit of monomorphisms is a monomorphism, and since also

$$\varinjlim H_*({}_\alpha E;G) = H_*(X;G)$$

one concludes that $\{F_sH_*(X;G), j_s\}$ is a filtration of $H_*(X;G)$. To achieve the desired identification of E^∞, we note that for every α there is a short exact sequence

$$0 \longrightarrow F_{s-1}H_{s+t}({}_\alpha E;G) \xrightarrow{{}_\alpha j_{s-1}} F_sH_{s+t}({}_\alpha E;G) \xrightarrow{{}_\alpha \rho} {}_\alpha E^\infty_{s,t} \longrightarrow 0 \ .$$

By functoriality of this sequence [3; 3.1, p. 327] there is a commutative diargram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{s-1}H_{s+t}({}_\alpha E) & \xrightarrow{{}_\alpha j_{s-1}} & F_sH_{s+t}({}_\alpha E) & \xrightarrow{{}_\alpha \rho} & {}_\alpha E^\infty_{s,t} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle {}^\beta_\alpha i_{s-1}} & & \downarrow{\scriptstyle {}^\beta_\alpha i_s} & & \downarrow{\scriptstyle {}^\beta_\alpha \phi^\infty} & & \\
0 & \longrightarrow & F_{s-1}H_{s+t}({}_\beta E) & \xrightarrow{{}_\beta j_{s-1}} & F_sH_{s+t}({}_\beta E) & \xrightarrow{{}_\beta \rho} & {}_\beta E^\infty_{s,t} & \longrightarrow & 0
\end{array}$$

for $\alpha < \beta$, giving a direct system of short exact sequences. Moreover, since $\{E^r, d^r\}$ is a convergent spectral sequence, it is obvious that

$$E^\infty = \varinjlim {}_\alpha E^\infty \ .$$

One therefore obtains a short exact sequence

$$0 \longrightarrow F_{s-1}H_{s+t}(X;G) \xrightarrow{j_s} F_sH_{s+t}(X;G) \xrightarrow{\rho} E^\infty_{s,t} \longrightarrow 0$$

in the direct limit, as was to be shown.

It remains to establish an isomorphism

$$E^2_{s,t} \approx H_{s,t}(f;G)$$

taking Theorem 2.2 as our starting point. By naturality of the isomorphism [3; 3.1, p. 327]

$$E^1_{s,t} \approx H_{s+t}(E_s, E_{s-1};G)$$

and once again taking into account that homology commutes with direct limits, it will suffice to establish commutativity of the diagram

$$\begin{array}{ccc}
C_s({}_\alpha K; H_t({}_\alpha F;G)) & \xrightarrow{{}_\alpha \psi_s} & H_{s+t}({}_\alpha E_s, {}_\alpha E_{s-1};G) \\
\downarrow{\scriptstyle \phi^\beta_\alpha} & & \downarrow{\scriptstyle \pi^\beta_\alpha} \\
C_s({}_\beta K; H_t({}_\beta F;G)) & \xrightarrow{{}_\beta \psi_s} & H_{s+t}({}_\beta E_s, {}_\beta E_{s-1};G)
\end{array}$$

for $\alpha < \beta$, where ϕ_α^β is given by (1.11), ψ_s by (2.7) and π_α^β is inclusion induced. Now this diagram splits into three diagrams, corresponding to the three factors in terms of which ψ_s is defined. Taking into account (1.9), (1.10), and (1.11), the first of these subdiagrams becomes

$$\begin{array}{ccc}
H_s(|\sigma|, |\dot\sigma|) \otimes H_t({}_\alpha F_\sigma; G) & \xrightarrow{\mu_\sigma} & H_{s+t}((|\sigma|, |\dot\sigma|) \times {}_\alpha F_\sigma; G) \\
\downarrow{\scriptstyle i_* \otimes 1} & & \downarrow{\scriptstyle (i \times 1)_*} \\
H_s(|\sigma|, A) \otimes H_t(F; G) & \xrightarrow{\mu'_\sigma} & H_{s+t}(|\sigma|, A) \times ({}_\alpha F_\sigma; G) \\
\uparrow{\scriptstyle \{i_{\sigma *}^\omega\} \otimes 1} & & \uparrow{\scriptstyle \{(i_\sigma^\omega \times 1)_*\}} \\
\oplus H_s(|\omega|, |\dot\omega|) \otimes H_t({}_\alpha F_\sigma; G) & \xrightarrow{\oplus \mu_\omega} & \oplus H_{s+t}((|\omega|, |\dot\omega|) \times {}_\alpha F_\sigma; G) \\
\downarrow{\scriptstyle \oplus (1 \otimes j_{\sigma *}^\omega)} & & \downarrow{\scriptstyle \oplus (1 \times j_\sigma^\omega)_*} \\
\oplus H_s(|\omega|, |\dot\omega|) \otimes H_t({}_\beta F_\omega; G) & \xrightarrow{\oplus \mu'_\omega} & \oplus H_{s+t}((|\omega|, |\dot\omega|) \times {}_\beta F_\omega; G)
\end{array}$$

which commutes by naturality of the homology cross product. Next we have the subdiagram

$$\begin{array}{ccc}
H_{s+t}((|\sigma|, |\dot\sigma|) \times {}_\alpha F_\sigma; G) & \xrightarrow{{}_\alpha\Phi_{\sigma *}} & H_{s+t}({}_\alpha E_\sigma, {}_\alpha E_{\dot\sigma}; G) \\
\downarrow{\scriptstyle (i \times 1)_*} & & \downarrow \\
H_{s+t}((|\sigma|, A) \times {}_\alpha F_\sigma; G) & \xrightarrow{{}_\alpha\Phi'_{\sigma *}} & H_{s+t}({}_\alpha E_\sigma, {}_\alpha p_\sigma^{-1}(A); G) \\
\uparrow{\scriptstyle \{(i_\sigma^\omega \times 1)_*\}} & & \uparrow \\
\oplus H_{s+t}((|\omega|, |\dot\omega|) \times {}_\alpha F_\sigma; G) & \xrightarrow{\oplus {}_\alpha\Phi_{\omega *}} & \oplus H_{s+t}({}_\alpha E_\omega, {}_\alpha E_{\dot\omega}; G) \\
\downarrow{\scriptstyle \oplus (1 \times j_\sigma^\omega)_*} & & \downarrow{\scriptstyle \oplus h_{\sigma *}^\omega} \\
\oplus H_{s+t}((|\omega|, |\dot\omega|) \times {}_\beta F_\omega; G) & \xrightarrow{\oplus {}_\beta\Phi_{\omega *}} & \oplus H_{s+t}({}_\beta E_\omega, {}_\beta E_{\dot\omega}; G)
\end{array}$$

in which the vertical maps on the right are inclusion induced. The top two rectangles derive from commutative diagrams on the space level and consequently commute by functoriality. On the other hand, since the homeomorphisms ${}_\alpha\Phi_\omega$ and ${}_\beta\Phi_\omega$ derive from independent tubular heighborhood constructions, the space level diagram corresponding to the bottom rectangle need not commute. But once again we are saved by the fact that the space level diagram in question is homotopy commutative; i.e., we shall prove

LEMMA 3.1. *There exists a homotopy*

$$H\colon (|\omega|, |\dot\omega|) \times {}_\alpha F_\sigma \times I \longrightarrow ({}_\beta E_\omega, {}_\beta E_{\dot\omega})$$

such that

$$H\colon h_\sigma^\omega \circ {}_\alpha\Phi_\omega \cong {}_\beta\Phi_\omega \circ j_\sigma^\omega .$$

For convenience we will identify the fibers ${}_\alpha F_\sigma$ and ${}_\beta F_\omega$ with

${}_\alpha p_\sigma^{-1}(x_0)$ and ${}_\beta p_\omega^{-1}(x_0)$, respectively, where $x_0 \in |\omega|$, so that $j_\sigma^\omega : {}_\alpha F_\sigma \to {}_\beta F_\omega$ becomes an actual inclusion. Moreover, this implies that

$$ {}_\alpha \Phi_\sigma(x_0, y) = y $$

for all $y \in {}_\alpha F_\sigma$, and likewise for ${}_\beta \Phi_\omega$. Let

$$ \pi : |\omega| \times {}_\alpha F_\sigma \longrightarrow {}_\alpha F_\sigma $$

denote the natural projection, and for each $x \in |\omega|$, let $\phi_x : I \to |\omega|$ denote the affine line segment from x to x_0. The desired homotopy H may now be defined through the formula

$$ H(x, y, t) = {}_\beta \Phi_\omega(x, \pi \circ {}_\beta \Phi_\omega^{-1} \circ {}_\alpha \Phi_\sigma(\phi_x(t), y)) \ . $$

It follows immediately that

$$ H(x, y, 0) = {}_\alpha \Phi_\sigma(x, y) $$
$$ H(x, y, 1) = {}_\beta \Phi_\omega(x, y) $$

and this establishes our lemma.

So far as the third and last subdiagram is concerned (we will not exhibit it), one sees that it is composed entirely of inclusion maps and therefore commutes trivially. This concludes our proof of the following

THEOREM 3.2. *Let $f : X \to Y$ be a submersion and G a coefficient group. There exists a convergent E^2 spectral sequence with*

$$ E^2_{s,t} \approx H_{s,t}(f; G) $$

and E^∞ isomorphic to the bigraded group associated with a filtration of $H_*(X; G)$.

4. Some applications. The applications to be set forth arise by way of our main result 3.2 from the following fact regarding the homology groups $H_{s,t}(f; G)$.

THEOREM 4.1. *Let $f : X \to Y$ be a submersion and U an open subset of Y. If $H_t(f_y; G) = 0$ for all $y \notin U$, then*

$$ H_{s,t}(f; G) \approx H_{s,t}(f \mid U; G) \ , $$

where $f \mid U$ denotes the restriction of f to the preimage of U.

To establish this results, it will be convenient to introduce a few definitions relating to tubular neighborhoods. By a *compact tubular neighborhood* we shall understand a homeomorphism $\Phi : B \times F \to V$,

where B is a compact neighborhood in Y and V a compact subspace of X, such that $f \circ \Phi$ is the projection $B \times F \to B$. Given $y \in B$ we will set $F_y = V \cap f_y$, and will let

$$\Phi_y^{y'}: F_y \longrightarrow F_{y'},$$

denote the homeomorphism induced by Φ for $y, y' \in B$.

LEMMA 4.2. *Let $\Phi: B \times F \to V$ be a compact tubular neighborhood such that B does not meet U, and let $z \subset F_y$ be a singular t-cycle, where $y \in B$. Then there exists a compact subspace $C \subset X$ such that*

$$\Phi_{y\#}^{y'} z \sim 0 \quad \text{in} \quad C \cap f_{y'}, \tag{4.3}$$

for all $y' \in B$.

The proof runs as follows. Our acyclicity assumption implies that for every point $y' \in B$ there exists a compact subspace $C_{y'}$ such that

$$\Phi_{y\#}^{y'} z \sim 0 \quad \text{in} \quad C_{y'}.$$

By compactness of B one may conclude with the aid of an elementary tubular neighborhood theorem [7] that there exists a finite sequence of points $y_1, \cdots, y_s$ and corresponding compact tubular neighborhoods

$${}_i\Phi: B_i \times F_i \longrightarrow V_i$$

such that

(i) each B_i is a convex neighborhood of y_i;
(ii) $\{B_1, \cdots, B_s\}$ covers B;
(iii) $C_{y_i} \subset V_i$ for $i = 1, \cdots, s$;
(iv) $V \cap f^{-1}(B_i) \subset V_i$ for $i = 1, \cdots, s$.

We claim now that (4.3) holds with

$$C = V_1 \cup \cdots \cup V_s.$$

For let $y' \in B$. By (ii) there exists an index i such that $y' \in B_i$, and by (iii) one has

$${}_i\Phi_{y_i\#}^{y'}(\Phi_{y\#}^{y_i} z) \sim 0 \quad \text{in} \quad C \cap f_{y'}.$$

Since

$$\Phi_{y\#}^{y'} z = \Phi_{y_i\#}^{y'}(\Phi_{y\#}^{y_i} z),$$

it will suffice to show that

$$\Phi_{y_i}^{y'} \cong {}_i\Phi_{y_i}^{y'}.$$

But the desired homotopy may be constructed exactly as in Lemma 3.1, by virtue of conditions (i) and (iv).

We may now proceed to the proof of Theorem 4.1. Let $\{U_\alpha\}$ be a covering of U by an increasing nested sequence of compact sets, and let $\{{}_\alpha p\colon {}_\alpha E \to |{}_\alpha K|\}$ be an approximating sequence of simplicial bundles satisfying the conditions of Theorem 1.2, together with

(4.4) *for every* α *and* $\sigma \in {}_\alpha K$, *if* $|\sigma|$ *meets* U_α, *then* $|\sigma| \subset U$.

Let ${}_\alpha\bar{K}$ denote the largest subcomplex of ${}_\alpha K$ such that $|{}_\alpha\bar{K}| \subset U$, and let ${}_\alpha\bar{p}\colon {}_\alpha\bar{E} \to |{}_\alpha\bar{K}|$ denote the corresponding simplicial bundle. We note that (4.4) implies $U_\alpha \subset |{}_\alpha K|$. One therefore obtains a direct subsystem

$$\bar{\mathscr{C}} = \{C_*({}_\alpha\bar{K}; H_t({}_\alpha\bar{F}; G)), \bar{\phi}_\alpha^\beta\}$$

of chain complexes such that

$$H_*(\varinjlim \bar{\mathscr{C}}) = H_{*,t}(f \mid U; G) .$$

It will therefore suffice to show that the inclusion

$$\bar{\mathscr{C}} \subset \mathscr{C} \equiv \{C_*({}_\alpha K; H_t({}_\alpha F; G)), \phi_\alpha^\beta\}$$

induces an isomorphism in the direct limit. Since the direct limit of monomorphisms is again a monomorphism, it only remains to check surjectivity. Given a generator

$$a \otimes c \in H_s(|\sigma|, |\dot{\sigma}|) \otimes H_t({}_\alpha F_\sigma; G)$$

for $\sigma \in {}_\alpha K$, we must show that there exists $\beta > \alpha$ such that

(4.5) $$\phi_\alpha^\beta(a \otimes c) \in C_s({}_\beta\bar{K}; H_t({}_\beta\bar{F}; G)) .$$

Let A denote the complement of U in $|\sigma|$, and assume without loss of generality that A is nonempty. We may identify ${}_\alpha F_\sigma$ with $p_\sigma^{-1}(y)$ for some $y \in A$ and let c be represented by a singular t-cycle $z \subset p_\sigma^{-1}(y)$. Since A is compact, we conclude by Lemma 4.2 that there exists a compact $C \subset X$ such that

(4.6) $$ {}_\sigma\Phi_{y\#}^{y'} z \sim 0 \quad \text{in} \quad C \cap f_y ,$$

for all $y' \in A$. Choose $\beta > \alpha$ such that $C \subset {}_\beta E$. Recalling (1.11), one sees that (4.5) may be established by showing that for every $\omega \in {}_\beta K^{(s)}$ with $|\omega| \subset |\sigma|$ and $\omega \notin {}_\beta\bar{K}$ one has $j_{\sigma *}^\omega(c) = 0$. But given such an ω, there must exist a point $y' \in |\omega| \cap A$, and one may identify ${}_\beta F_\omega$ with $p_\omega^{-1}(y')$. The map $j_\sigma^\omega\colon {}_\alpha F_\sigma \to {}_\beta F_\omega$ is then represented by ${}_\sigma\Phi_y^{y'}$ (followed by the inclusion $p_\sigma^{-1}(y') \subset p_\omega^{-1}(y')$), and consequently (4.6) implies the vanishing of $j_{\sigma *}^\omega(c)$.

Theorem 4.1 will be of interest in two special cases. In the first

place, if $f: X^n \to Y^m$ is any submersion having connected fibers, the conditions of (4.1) are seen to hold if one takes U to be the subspace of Y corresponding to compact fibers, and takes $t = n - m$. Moreover, the restriction $f \mid U$ will now be a fiber bundle, and if X, Y are both oriented, the homology groups $H_{n-m}(f_y; G)$ may be canonically identified with G for all $y \in U$. It is therefore easy to evaluate $H_{s,n-m}(f \mid U; G)$, and one obtains

$$H_{s,n-m}(f; G) \approx H_s(U; G) . \tag{4.7}$$

Secondly, the conditions of (4.1) may hold for an empty set U, in which case one has

$$H_{s,t}(f; G) = 0 \quad \text{for all} \quad s . \tag{4.8}$$

It should also be noted that (4.1) fails when U is not open.

Theorems 3.2 and 4.1 together give rise to an indefinite number of geometric propositions, which we shall illustrate by means of a few examples. The most immediate of these consequences is no doubt the Vietoris-Begle theorem for submersions, which we have previously established [7] by elementary methods. Turning to new results, one has

THEOREM 4.9. *Let $f: X \to Y$ be a submersion between orientable manifolds of dimensions n and m, respectively, such that every fiber f_y is either a homology $(n - m)$-sphere or else acyclic over a given coefficient group G. Then there exists an exact sequence*

$$\cdots \longrightarrow H_s(X; G) \longrightarrow H_s(Y; G) \longrightarrow H_{s-n+m-1}(U; G) \longrightarrow \cdots$$

where $U \subset Y$ denotes the subspace corresponding to compact fibers.

This clearly generalizes the Thom-Gysin sequence for sphere bundles, and we also note that for submersions of codimention 1 the result has been previouly established [5] by different methods. To prove our theorem, one concludeds by (3.2) and (4.8) that $E^2_{s,t}$ is trivial except when t equals 0 or $(n - m)$, which means that d^{n-m+2} is the only nontrivial differential in our spectral sequence. One therefore obtains exact sequences

$$0 \longrightarrow E^\infty_{s\,0} \longrightarrow E^2_{s\,0} \xrightarrow{d^{n-m+2}} E^2_{s-n+m-1,n-m} \longrightarrow E^\infty_{s-n+m-1,n-m} \longrightarrow 0$$

and

$$0 \longrightarrow E^\infty_{s-n+m,n-m} \longrightarrow H_s(X; G) \longrightarrow E^\infty_{s,0} \longrightarrow 0$$

which by the standard "splicing trick" give rise to a long exact sequence

$$\cdots \longrightarrow H_s(X; G) \longrightarrow E^2_{s,0} \longrightarrow E^2_{s-n+m-1,n-m} \longrightarrow \cdots .$$

Since

$$E^2_{s-n+m-1,n-m} \approx H_{s-n+m-1}(U; G)$$

by (3.2) and (4.7), and clearly

$$E^2_{s,0} \approx H_s(Y; G) ,$$

one obtains the result in question.

It is of interest to note that some of the information contained in the generalized Thom-Gysin sequence for large values of s carries over under much weaker assumptions; for example, one can obtain

THEOREM 4.10. *Let $f: X \to Y$ be a submersion between orientable manifolds of dimensions n and m, respectively, whose fibers are connected. Then there is an inequality*

$$R_{n-1}(X; G) \geqq R_{m-1}(U; G)$$

involving the Betti numbers of X and U, where $U \subset Y$ corresponds to the compact fibers.

To show this we note that $E^2_{s\,t}$ is trivial for $s > m$, and by (4.8) it is also trivial for $t > n - m$. Since all differentials touching $E^r_{m-1,n-m}$ are therefore trivial for $r \geqq 2$, one has

$$E^2_{m-1,n-m} \approx E^\infty_{m-1,n-m} .$$

But by (3.2) and (4.7), this implies that

$$E^\infty_{m-1,n-m} \approx H_{m-1}(U; G) ,$$

and the desired rank inequality follows now by the last assertion contained in (3.2).

We will close with another result of this nature, which may be established through a simlar consideration.

THEOREM 4.11. *Let $f: X \to Y$ be as in* 4.10, *with the additional stipulation that the fibers are G-acyclic in positive dimension $n - m - 1$. Then there is an additional rank inequality*

$$R_{n-2}(X; G) \geqq R_{m-2}(U; G) ,$$

where $U \subset Y$ corresponds to compact fibers.

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