EFFECTIVE DIVISOR CLASSES AND BLOWINGS-UP OF $\mathbb{P}^2$

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Let $X_n \to \mathbb{P}^2$ be the monoidal transformation of the (complex) projective plane centered at distinct points $P_1, \ldots, P_n$ of $\mathbb{P}^2$. We recall that the Néron-Severi group of $X_n$ is freely generated by the divisor class $[L]$ of the proper transform $L$ of a line in $\mathbb{P}^2$ and by the classes $[E_i]$ of the "exceptional" fibers $E_i$ over $P_i$; the intersection pairing is given by

$[L]^2 = 1; \quad [L] \cdot [E_i] = 0; \quad [E_i] \cdot [E_j] = -\delta_{ij}.$

Let $\mathcal{M}(X_n)$ denote the monoid of elements $F$ in the Néron-Severi group with the property that $F$ contains an effective divisor. In this paper we

(1) construct a finite generating set for $\mathcal{M}(X_n)$ for $n \leq 8$, and give a particularly simple geometric description of the generators when $P_1 \cdots P_n$ are in "general position";

(2) show that, for $n \geq 9$, $\mathcal{M}(X_n)$ need not be finitely generated, despite the finite generation of the whole Néron-Severi group;

(3) prove the related result that if a nonsingular surface $X$ contains an infinite number of exceptional curves of the first kind, then $X$ is necessarily rational.

We will let $K_{X_n}$ denote the canonical class on $X_n$; it is given by $K_{X_n} = \pi^* K_{\mathbb{P}^2} + \sum [E_i] = -3[L] + \sum [E_i]$. We observe that, for $n \leq 8$, the anti-cannonical class $-K_{X_n}$ contains an effective divisor (which will also be denoted by $-K_{X_n}$ when no confusion is possible), since $H^0(X_n, \omega_{X_n})$ can be regarded as the (complex) vector space of homogeneous forms in 3 variables of degree 3 vanishing at the points $P_1 \cdots P_n$.

**Lemma 1.** Let $X$ be any nonsingular rational surface, and let $C$ be a curve on $X$ with $p_a(C) \geq 1$. Then $[C] + K_X$ is an effective class.

**Proof.** The short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

yields, using Serre-duality and the rationality of $X$, $\dim H^0(X, \mathcal{O}_X(C) \otimes \omega_X) = \dim H^0(X, \mathcal{O}_X(-C)) = \dim H^1(C, \mathcal{O}_C) = p_a(C)$.

Recall that, for $n \leq 8$, the points $P_1 \cdots P_n$ of $\mathbb{P}^2$ are in general
position if no three $P_i$ are collinear and if no six of them lie on a conic.

**Theorem 1.** Let $X_n \to P^2$ be the monoidal transformation of $P^2$ centered at $P_1 \cdots P_n$, with $n \leq 8$ and $P_1 \cdots P_n$ in general position. Then $\mathcal{M}(X_n)$ is finitely generated, the generators being the classes of divisors on the following list:

(Note: $g(n) =$ number of generators of $\mathcal{M}(X_n)$)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$g(n)$</th>
<th>Divisor</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>$E_1$</td>
<td>Exceptional curve</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$L - E_1$</td>
<td>Proper transform of a line through $P$</td>
</tr>
<tr>
<td>2, 3, 4</td>
<td>2, 6, 10</td>
<td>$E_i(1 \leq i \leq n)$</td>
<td>Exceptional curve</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$L - E_i - E_j(1 \leq i &lt; j \leq n)$</td>
<td>Proper transform of the line through $P_i$ and $P_j$</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
<td>$E_i$</td>
<td>Exceptional curve</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1 \leq i \leq 5)$</td>
<td>Proper transform of the line through $P_i$ and $P_j$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$L - E_i - E_j(1 \leq i &lt; j \leq 5)$</td>
<td>Proper transform of the conic through all ${P_i}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2L - \sum E_i$</td>
<td>Proper transform of a sextic with a triple point at $P_k$ and with double points at $P_i$, $\forall i \neq k$</td>
</tr>
<tr>
<td>6</td>
<td>27</td>
<td>$E_i$</td>
<td>Exceptional curve</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1 \leq i \leq 6)$</td>
<td>Proper transform of the line through $P_i$ and $P_j$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$L - E_i - E_j(1 \leq i &lt; j \leq 6)$</td>
<td>Proper transform of the conic through all ${P_i}$ except $P_k$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2L - \sum_{i \neq k} E_i(1 \leq k \leq 6)$</td>
<td>Proper transform of a quintic through all ${P_i}$ with double points at $P_j$, $P_k$ and $P_i$</td>
</tr>
<tr>
<td>7</td>
<td>56</td>
<td>$E_i$</td>
<td>Exceptional curve</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1 \leq i \leq 7)$</td>
<td>Proper transform of the line through $P_i$ and $P_j$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$L - E_i - E_j(1 \leq i &lt; j \leq 7)$</td>
<td>Proper transform of the conic through all points ${P_i}$ except $P_j$ and $P_i$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2L - \sum_{i \neq k, l} E_i(1 \leq l &lt; k \leq 7)$</td>
<td>Proper transform of a quartic through all ${P_i}$ with double points at $P_j$, $P_k$ and $P_i$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$3L - 2E_j - \sum_{i \neq j} E_i(1 \leq j \leq 7)$</td>
<td>Proper transform of a sextic with a triple point at $P_k$ and with double points at $P_i$, $\forall i \neq k$</td>
</tr>
<tr>
<td>8</td>
<td>241</td>
<td>$E_i$</td>
<td>Exceptional curve</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(i = 1 \cdots 8)$</td>
<td>Proper transform of the line through $P_i$ and $P_j$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$L - E_i - E_j(1 \leq i &lt; j \leq 8)$</td>
<td>Proper transform of the conic through all ${P_i}$ except $P_j$, $P_k$ and $P_i$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2L - \sum_{i \neq j, k, l} E_i(1 \leq j &lt; k &lt; l \leq 8)$</td>
<td>Proper transform of a cubic through all ${P_i}$ except $P_j$, and with a double point at $P_i$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$3L - 2E_k - \sum_{i \neq j, k} E_i(1 \leq j, k \leq 8), j \neq k$</td>
<td>Proper transform of a quartic through all ${P_i}$ with double points at $P_j$, $P_k$ and $P_i$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$4L - 2E_j - 2E_k - 2E_i - \sum_{i \neq j, k} E_i(1 \leq j &lt; k &lt; l \leq 8)$</td>
<td>Proper transform of a quintic through all ${P_i}$ and with double points at all but $P_j$ and $P_k$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$5L - E_i - E_j - \sum_{i \neq j, k} 2E_i(1 \leq j &lt; k \leq 8)$</td>
<td>Proper transform of a sextic with a triple point at $P_k$ and with double points at $P_i$, $\forall i \neq k$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$6L - 3E_k - 2 \sum_{i \neq k} E_i(1 \leq k \leq 8)$</td>
<td>Proper transform of a quintic through all ${P_i}$ and with double points at all but $P_j$ and $P_k$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$3L - \sum_{i=1}^8 E_i$</td>
<td>Anti-cannonical curve</td>
</tr>
</tbody>
</table>
REMARK. For \( n = 6 \), we see that the generators of the monoid for the cubic hypersurface in \( \mathbb{P}^3 \) are the classes of the classical twenty-seven lines on \( X_6 \). More generally, the classes of the divisors listed above are, for \( 2 \leq n \leq 7 \), precisely the classes of all rational curves on \( X_n \) with self-intersection \(-1\). [1, Th. 26.2].

Before proving the theorem, we will first prove

**Lemma 2.** Let \( X_n \) be as in the theorem. Suppose that \( C \) is any curve on \( X_n \) for \( 1 \leq n \leq 7 \), or that \( C \) is a curve on \( X_8 \) whose class is not represented above for \( n = 8 \). Then for any divisor \( L \) on the above list, \( \dim H^0(X_n, \mathcal{O}_{X_n}(C - L)) = 0 \).

**Proof.** [Case 1: \( n \leq 7 \)]. A look at the proposed generating set of \( \mathcal{M}(X_n) \) shows that, given \( L \) as above, there is an effective nontrivial divisor \( D \) such that \( -K_{X_n} = [L] + [D] \). Therefore \( 0 = \dim H^0(X_n, \omega_{X_n} \otimes \mathcal{O}_{X_n}(L)) = \dim H^0(X_n, \omega_{X_n} \otimes \mathcal{O}(L - C)) \), and the result follows by duality.

[Case 2: \( n = 8 \)]. Again, we will use duality and show that \( \dim H^0(X_8, \omega_{X_8} \otimes \mathcal{O}_{X_8}(L - C)) = 0 \). Suppose the contrary. Then \( K_{X_8} + [L] \) must be an effective class for some \( L \), and we may clearly assume that \( [L] \neq -K_{X_8} \). Then either

\[
[L] = \begin{cases} 
[4L - 2E_i - 2E_j - 2E_k - \sum_{i \neq j, k} E_i] \text{ some } i, j, k, \text{ or} \\
[5L - E_i - E_j - 2 \sum_{i \neq j} E_i] \text{ some } i, j, \text{ or} \\
[6L - 3E_k - 2 \sum_{i \neq k} E_i] \text{ some } k.
\end{cases}
\]

But by the general position of \( P_1 \cdots P_s \), the first two choices for \( L \) do not yield effective classes \([L] + K_{X_8}\); hence \( K_{X_8} + [L] \) is of the form \([3L - 2E_k - \sum_{i \neq k} E_i]\).

Now, since \( C \) is unequal to any \( E_i, C \cdot E_i \geq 0 \) and we may write \([C] = m[L] - \sum_{i=1}^s b_i[E_i]\), with \( m \geq 1 \) and \( b_i \geq 0 \). If \( K_{X_8} + [L - C] \) is to be effective, we must have \( m = 1, 2 \) or \( 3 \). If \( m = 1 \), the general position of the \( \{P_i\} \) forces all but two of the \( b_i \) to be 0 and the nonzero \( b_i \) to be 1, making \([K_{X_8} + L - C] = [2L - \sum c_i E_i]\) with \( \sum c_i \geq 6 \). This class is not effective since no six of the \( \{P_i\} \) lie on a conic. An analogous proof works for \( m = 2 \). If \( m = 3 \) we have, since \([C] \cdot [L - E_i - E_j] \geq 0 \) for all \( i, j \), three possibilities:

- (a) some \( b_i = 3 \), all others 0, or
- (b) all \( b_i \) are 0 or 1, or
- (c) some \( b_i = 2 \), all others are 0 or 1.

Neither (a) nor (b) can occur, as in these cases \( K_{X_8} + [L - C] = \sum c_i[E_i] \) with some \( c_i < 0 \), violating the effectiveness of \( K_{X_8} + [L - C] \).
Similarly, (c) can be dismissed unless \([C]\) is of the form \([3L - 2E_i - \sum_{k \neq i,j} E_k]\), some \(i, j\), which violates the hypothesis that \([C]\) not be represented on the list of divisors in the theorem.

**Proof of Theorem 1.** Fix a projective embedding of \(X_n\) into \(P^N\), some \(N \geq 3\). Then we may speak of the “degree” of a divisor on \(X_n\) with respect to this embedding. It suffices to show that, for \(C\) an effective divisor on \(X_n\), \([C - \mathscr{L}]\) is an effective class for some divisor \(\mathscr{L}\) listed in the theorem; the result will then follow by induction on “degree”. Furthermore, for \(n = 1, \cdots, 7\) we note that \(-K_{X_n}\) is a sum of classes of divisors listed, while for \(n = 8\) the anti-cannonical class is included on the list of proposed generators. Hence, by Lemma 1, we may assume that \(C\) is a curve with \(p_a(C) = 0\). Finally, we may assume that \(C\) is an irreducible curve whose class is not represented on the list in the theorem.

By Riemann-Roch, together with Lemma 2 and the rationality of \(X_n\), we have, for \(\mathscr{L}\) any divisor on the above list except \(-K_{X_n}\),

\[
\dim H^0(X_n, \mathcal{O}_{X_n}(C - \mathscr{L})) - \dim H^1(X_n, \mathcal{O}_{X_n}(C - \mathscr{L})) = 1/2(C^2 - 2\mathscr{L} \cdot C - K_{X_n} \cdot C).
\]

Since \(p_a(C) = 0\), the adjunction formula applied to \(C\) yields \(C^2 = -K_{X_n} \cdot C - 2\), so we have, for all divisors \(\mathscr{L}\) on the list in the theorem except for \(-K_{X_n}\),

\[
\dim H^0(X_n, \mathcal{O}_{X_n}(C - \mathscr{L})) - \dim H^1(X_n, \mathcal{O}_{X_n}(C - \mathscr{L})) = (-K_{X_n} \cdot C) - 1 - (\mathscr{L} \cdot C).
\]

Thus, it suffices to show that for some divisor \(\mathscr{L}\) in the above list except for \(-K_{X_n}\),

\[
(*) -K_{X_n} \cdot C > \mathscr{L} \cdot C + 1.
\]

The proof of the validity of (*) is, for \(n = 1, \cdots, 5\), a simplified version of the cases \(n = 6, 7, 8\); hence we include only the later cases.

Let \([C] = m[L] - \sum_{i=1}^n b_i [E_i]\). Since \([C]\) is not represented on the above list, we intersect \(C\) with each element on the list to get

\[
n = 6: \begin{align*}
(1) \quad m &\geq 1 \\
(2) \quad b_i &\geq 0 \forall i \\
(3) \quad m - b_i - b_j &\geq 0 \forall i \neq j \\
(4) \quad 2m - \sum_{i \neq k} b_i &\geq 0 \forall k.
\end{align*}
\]

Since \(-K_{X_6} \cdot C = 3m - \sum_{i=1}^6 b_i\), our condition (*) to be fulfilled becomes

\[
(**) < \begin{cases}
3m > \sum_{i=1}^6 b_i + b_k + 1 \text{ for some } k, \text{ or} \\
2m > \sum_{k \neq i,j} b_k + 1 \text{ for some } i, j \text{ or} \\
m > b_k + 1 \text{ for some } k.
\end{cases}
\]
If \( m > 1 \), and if the third inequality of (***) fails, then, by conditions (2) and (3) above we have \( m = 2 \) and \( b_k = 1 \forall k \), violating (4) above. If \( m = 1 \), then by (2) and (3) at most one \( b_i \) can be nonzero, and the first two inequalities of (***) hold.

\( n = 7 \) we have

\[
(1) \quad m \geq 1 \quad (4) \quad 2m - \sum_{i \neq j, k} b_i \geq 0 \forall j \neq k \\
(2) \quad b_i \geq 0 \forall i \quad (5) \quad 3m - \sum_{j \neq i} b_j - 2b_i \geq 0 \forall i , \\
(3) \quad m - b_i - b_j \geq 0 \forall i \neq j
\]

and condition (*) becomes

\[
(**) < \begin{cases} 
3m > \sum_{i=1} b_i + b_k + 1 & \text{for some } k, \text{ or} \\
2m > \sum_{i \neq j, k} b_i + 1 & \text{for some } j, k, \text{ or} \\
m > b_j + b_k + 1 & \text{for some } j, k, \text{ or} \\
b_i > 1 & \text{for some } i . 
\end{cases}
\]

Assume that the fourth inequality of (***) fails. If all \( b_i \) are 1, and if the third inequality of (***) fails, then \( m \leq 3 \). By condition (4) we have \( m \geq 3 \), so \( m = 3 \) and \([C] = -K_{X_7}\), which we have already seen is a sum of proposed generators of \( \mathcal{M}(X_7) \). If some \( b_i \) is 0, then conditions (1) \( \cdots \) (4) and the first three conditions of (***) become the same as in the case \( n = 6 \).

\( n = 8 \) writing condition (*) in terms of \( m \) and the \( b_i (i=1, \cdots, 8) \) and assuming that (*) does not hold, we have:

\( (\alpha) \quad |3m - b_k - \sum_{i=1} b_i| \leq 1 \text{ for all } k \)

\( (\beta) \quad |2m - \sum_{i \neq j, k} b_i| \leq 1 \text{ for all } j, k \)

\( (\gamma) \quad |m - b_i - b_j - b_k| \leq 1 \text{ for all } i, j, k \)

\( (\delta) \quad |b_i - b_j| \leq 1 \text{ for all } i, j . \)

Let \( b = \min \{ b_i \} \), and \( B = \max \{ b_i \} \). Note that by (\( \delta \)), \( 0 \leq B - b \leq 1 \).

Let \( r \) of the \( b_i \)'s have value \( b \), and \( 8 - r \) of the \( b_i \)'s have value \( B \).

We will obtain our contradiction on a case-by-case basis:

\( r = 0 \). Then by (\( \alpha \)) \( m - 3B = 0 \) and \([C] = B(-K_{X_8})\), \( B \in \mathbb{Z} \); since \( p_s(C) = 0 \) the adjunction formula yields \( 2^2 - B + 2 = 0 \).

\( r = 8 \). Again by (\( \alpha \)), \([C] = b(-K_{X_8})\).

\( r = 1 \). By (\( \beta \)), \( m - 3B = 0 \), and by (\( \alpha \)) \( |3m - 7B - 2b| \leq 1 \), contradicting \( B - b = 1 \).

\( r = 7 \). Then \( m - 3b = 0 \) by (\( \beta \)), which is again impossible by (\( \alpha \)) and the fact that \( B - b = 1 \) for \( r \neq 0, 8 \).

\( r = 2 \). Since \( B - b = 1 \), (\( \beta \)) implies that \( 2m - 5B - b = 0 \), and (\( \gamma \)) implies that \( m - 2B - b = 0 \). Thus \( B - b = 0 \), a contradiction.

\( r = 6 \). Again, (\( \gamma \)) and (\( \beta \)) imply that \( B - b = 0 \).
We now examine the case in which the points \( P, \cdots, P_n \), with \( n \leq 8 \), of \( \mathbb{P}^2 \) are not in general position; in this case the classes of the divisors listed in Theorem 1 may contain reducible curves. For each \( n \leq 8 \), let \( F_1, \cdots, F_m \) be the classes of the formal sums of \( L \) and the \( \{ E_i \} \) listed in Theorem 1, and let \( D_i \in F_i \) be an effective divisor with the property that the number of distinct components of \( D_i \) is maximal for effective divisors in \( F_i \). (Such a divisor \( D_i \) exists since, for any effective divisor \( D \in F_i \), \# components of \( D \leq \deg D = \deg E \) for any \( E \in F_i \).) Write \( D_i = \sum n_{i,j} E_{i,j} \) with \( n_{i,j} > 0 \).

**Lemma 3.** Let \( P_1, \cdots, P_n \) be distinct points of \( \mathbb{P}^2 \) in arbitrary position, and let \( X \rightarrow \mathbb{P}^2 \) be the monoidal transformation centered at the \( \{ P_i \} \). Let \( D_i \in F_i \) be as above, for \( n = 8 \). Then there are only a finite number of divisor classes \( F \) on \( X \) with the property that \( F \) contains curve \( C \) with \( p_a(C) = 0 \) and with the property that \( \dim H^2(X_8, C_x(C - D_i)) \geq 1 \) for some \( i \).

**Proof.** If \( \dim H^2(X_8, C_x(C - D_i)) \geq 1 \), then, by duality, \( K_{X_8} + [D_i] - [C] \) must contain an effective divisor, and so must \( K_{X_8} + F_i \). Thus, as in the proof of Theorem 1, \( K_{X_8} + F_i \) must be of the form

\[
[L] - [E_i] - [E_j] - [E_k], \text{ some } i, j, k, \text{ or}
\]

\[
2[L] - \sum_{i=1}^3 [E_i], \text{ some } i, j, \text{ or}
\]

\[
3[L] - 2[E_k] - \sum_{i=1}^3 [E_i], \text{ some } k.
\]

Hence, if \( [C] = m[L] - \sum b_i [E_i] \), we must have \( 0 \leq m \leq 3 \), and since \( p_a(C) = 0 \), the adjunction formula yields \( (m^2 - 3m) - \sum b_i (b_i - b) = -2 \). Clearly with \( 0 \leq m \leq 3 \) there are only a finite number of solutions to this diaphantine equation.

Let \( R_1, \cdots, R_k \) be the divisor classes on \( X \) referred to in Lemma 3, and let \( S_i \in R_i \) be an effective divisor with maximal number of distinct components. Write \( S_i = \sum m_{i,j} Q_{i,j} \), with \( m_{i,j} > 0 \).

**Theorem 2.** Let \( X_n \rightarrow \mathbb{P}^2 \) be the monoidal transformation centered at points \( P_1, \cdots, P_n \) of \( \mathbb{P}^2 \), with \( n \leq 8 \) and with the points \( \{ P_i \} \) in arbitrary positions. Then \( \mathcal{M}(X_n) \) is finitely generated, the generators being \( \{ E_{i,j} \} \) for \( n \leq 7 \), and \( \{ [E_{i,j}] \} \cup \{ [Q_{i,j}] \} \) if \( n = 8 \).

**Proof.** [Case 1: \( n \leq 7 \)]. We will show that, for \( C \) an irreducible
curve on $X_n$, $C - E_{i,j}$ is equivalent to an effective divisor, for some $i, j$. As in the proof of Theorem 1, we may assume that $p_a(C) = 0$. Moreover, the proof of Lemma 2 for $n \leq 7$ did not rely on the general position of the $\{P_i\}$; hence for any curve $C$ on $X_n$, $n \leq 7$, $\dim H^q(X_n, \mathcal{O}_{x_n}(C - D_i)) = 0$ for all $i$. Thus it suffices to show that

(a) if $p_a(C) = 0$, $C$ irreducible and $[C] \neq [E_{i,j}]$ for all $i, j$, then $\chi(\mathcal{O}_{x_n}(C - D_i)) \geq 1$ for some $i$, and

(b) $[E_{i,j}]$ cannot be written nontrivially as a sum of effective divisor classes.

Part (b) follows from the maximality of the number of components of $D_i$ for effective divisors in $F_i$. For part (a) we note that, since the intersection-theoretic properties of the $\{F_i\}$ are the same as in Theorem 1, it suffices to show that

\[ (*) \quad -K_{X_n} \cdot C > (D_i \cdot C) + 1 \quad \text{for some } i, \]

with $[C] \neq [E_{i,j}] \vee i, j$. Writing $[C] = m[L] - \sum_{i=1}^{r} b_i[E_i]$ and writing $(*)$ in terms of $m$ and the $\{b_i\}$, the condition $(*)$ becomes precisely the condition $(**)$ of Theorem 1.

Since $[C] \neq [E_{i,j}]$ for all $i, j$, we have $C \cdot D_i \geq 0 \forall i$, i.e., the constraints on $m$ and the $\{b_i\}$ are the same as in the proof of Theorem 1. Since the truth of $(**)$ depended only on these constraints, we are done.

[Case 2: $n = 8$]. As in the case $n \leq 7$, it suffices to show that for $C$ an irreducible curve on $X_8$ with $p_a(C) = 0$, either $C - E_{i,j}$ or $C - Q_{i,j}$ is equivalent to an effective divisor. Clearly, if $C \in R_i$, for some $i$, then $C - Q_{i,j}$ is equivalent to an effective divisor for some $i, j$. If $C \notin R_i$ for any $i$, it suffices to show that, with $C \neq E_{i,j}$ for all $i, j$,

\[ (*) \quad \chi(\mathcal{O}_{x_8}(C - D_i)) \geq 1 \quad \text{for some } i . \]

Since $C \cdot D_i \geq 0$ for all $i$, the verification of $(*)$ reduces to the case $n = 8$ of Theorem 1.

In contrast with the above, if $n \geq 9$, $\mathcal{M}(X_n)$ need not be finitely generated.

**Example.** Let $C_1$ be a cuspidal cubic curve in $P^2$, and let $C_2$ be any cubic curve intersecting $C_1$ in nine distinct points, none of which is a singular point of $C_1$. Let $Y$ be the surface obtained by blowing up $P^2$ at $C_1 \cap C_2$. **Claim:** $\mathcal{M}(Y)$ is not finitely generated.

Let $F_1(X_0, X_1, X_2)$ be the (cubic) defining polynomials of $C_i(i = 1, 2)$. Then the rational function $F_1/F_2$ on $P^2$ has its only inde-
terminate points on $C_1 \cap C_2$. Since $C_1$ and $C_2$ are transversal, the rational function $F_1/F_2$ pulls back to $Y$ to give a holomorphic map $\phi: Y \to \mathbf{P}^1$, with fibers the proper transforms under the blowing up $\pi: Y \to \mathbf{P}^2$ of the curves in the pencil generated by $C_1$ and $C_2$.

Let $Y^*$ denote the set $Y - \bigcap_{t \in \mathbf{P}^1} \text{sing } \phi^{-1}(t)$, and let $\phi^{-1}(t_0)$ be the proper transform of the cuspidal curve $C_1$. The fibers of an elliptic fibering have been classified by [2, Th. 6.2 and 9.1], along with the possible group structures of the set of nonsingular points; we see by the classification that $\phi^{-1}(t_0) \cap Y^*$ has the structure of a torsion-free abelian group, with any point serving as the identity element.

Let $\Gamma$ denote the set of sections of $\phi$ (which necessarily map into $Y^*$); then after choosing some element of $\Gamma$ (such as one of the nine exceptional curves lying over a point of $C_1 \cap C_2$) as an identity element, $\Gamma$ has the structure of an abelian group under pointwise addition (the addition being the group operations on the nonsingular sets of the fibers of $\phi$). We have, for each $t \in \mathbf{P}^1$, a natural evaluation homomorphism

$$\psi_t: \Gamma \longrightarrow \phi^{-1}(t) \cap Y^*, \text{ defined by } \sigma \longrightarrow \sigma(t).$$

Since $\Gamma$ contains at least nine disjoint sections (i.e., the nine exceptional curves lying over $C_1 \cap C_2$), the map $\psi_{t_0}$ maps $\Gamma$ nontrivially into a torsion-free group, so $\Gamma$ must be infinite.

By [2, Th. 9.2], each $\eta \in \Gamma$ induces a fiber-preserving automorphism

$$L_\eta: Y^* \longrightarrow Y^*, \text{ defined by } L_\eta(z) = z + \eta \circ \phi(z),$$

which actually extends to an automorphism of $Y$. Thus, any two elements of $\Gamma$ differ by an automorphism of $Y$.

Hence, the orbits of the exceptional curves lying over $C_1 \cap C_2$ under the action of Aut($Y$) yield an infinite number of exceptional curves of the first kind on $Y$. The following fact shows that $\mathcal{M}(Y)$ is not finitely generated, while of course N.S. ($Y$) $\approx$ PIC($Y$) $\cong \mathbf{Z} \oplus 10$.

**Fact.** Let $Y$ be any surface containing an infinite number of curves of negative self-intersection. Then $\mathcal{M}(Y)$ is not finitely generated.

**Proof.** Suppose to the contrary that $L_1', \ldots, L_n'$ is a (finite) generating set of $\mathcal{M}(Y)$. To obtain a contradiction it suffices to show that if $C_i$ is a fixed curve in the algebraic equivalence class $L_i$, and if $E$ is a curve on $Y$ with negative self-intersection, then
For the curves $C_i$ and $E$ as stated, write

$$[E] = \sum_{i=1}^{n} m_i [C_i] = \sum_{i=1}^{n} m_i [C_i], \text{ with } m_i \geq 0.$$ 

Therefore $E^2 = \sum_{i=1}^{n} m_i (C_i \cdot E)$. If $E$ is not a component of $C_i$ for any $i$, then the right-hand side of the above equation is nonnegative, which is a contradiction.

REMARK. The elliptic surface constructed above is only one of a large number of known examples of surfaces which contain an infinite number of rational curves with self-intersection $-1$ and which are obtained by blowing up the projective plane at nine points. For other examples, see [5, p. 164], or [1, p. 407].

REMARK. It is not hard to show, using the projection formula [1, p. 426 A. 4] that if $X \to Y$ is a monoidal transformation of surfaces, and if $\mathcal{M}(X)$ is finitely generated, then $\mathcal{M}(Y)$ is also finitely generated. Hence $\mathcal{M}(X_n)$ need not be finitely generated for $n \geq 9$.

In view of the fact used above, the question naturally arises as to which surfaces can contain an infinite number of curves with negative self-intersection. A partial answer is given by a conjecture of A. Kas, a proof of which is provided below:

**Theorem 3.** Let $X$ be nonsingular algebraic surface over $C$ which contains an infinite number of exceptional curves of the first kind. Then $X$ is rational.

**Proof.** Let $\phi_1, \ldots, \phi_n$ be a basis of holomorphic 1-forms on $X$, for $n \geq 0$. We will first reduce to the case $n = 0$.

**Case 1.** $n \geq 2$ and $\phi_i \wedge \phi_j \neq 0$, some $i, j$.

We write the canonical map $\pi: X \to \text{Alb}(X)$, given by

$$z \mapsto \left[ \int_{\phi_1}, \ldots, \int_{\phi_n} \right]$$

modulo the lattice in $C^n$ generated by the $2n$ vectors

$$\begin{bmatrix} \int_{\phi_i}, & \cdots, & \int_{\phi_n} \\ \Gamma_i & \cdots & \Gamma_i \end{bmatrix}, \quad i = 1, \ldots, 2n,$$
where $P$ is a fixed point of $X$ and $\Gamma_1, \ldots, \Gamma_{2n}$ are 1-cycles whose homology classes generate the free subgroup of $H_1(X, \mathbb{Z})$.

The hypotheses imply that the Jacobian of the Albanese map $\pi$ has rank 2; hence $\pi$ is generically finite-to-one in the sense that there are only a finite number of points $p \in \text{Alb}(X)$ such that $\text{dim} \pi^{-1}(p) = 1$. Let $\{p_1, \ldots, p_k\}$ be this finite set, and let $\pi^{-1}(p_i)$ be the divisor $\sum n_{ij} D_j$, with $n_{ij} > 0$ and $D_i$ irreducible. If $C$ is a rational curve on $X$, then $\pi(C)$ is a single point; hence the number of rational curves on $X$ is bounded by $\sum n_{ij}$. (Actually it is not hard to see that a rational curve on $X$ must be a component of a fixed divisor in the canonical class of $X$.)

**Case 2.** $n = 1$, or $n \geq 2$ and $\phi_i \wedge \phi_j = 0 \forall i, j$.

If $n = 1$, then $\text{dim} \pi(X) = \text{dim} \text{Alb}(X) = 1$. If $n \geq 2$, the fact that $\phi_i \wedge \phi_j = 0 \forall i, j$ implies that the Jacobian matrix of $\pi$ has rank 1, and $\text{dim} \pi(X) = 1$ in this case as well.

Let $D$ be the curve $\pi(X) \subset \text{Alb}(X)$, and let $\{a_1, \ldots, a_r\} \subset D$ be the (finite) set of points such that $\forall t \in D$, $\pi^{-1}(t)$ is singular if and only if $t = a_i$, some $i$. Let $C$ be a rational curve on $X$ with nonzero self-intersection. Then $\pi(C)$ is a point of $D$, so $C$ is a component of $\pi^{-1}(t_0)$, some $t_0 \in D$. Since $(\pi^{-1}(t))^2 = 0 \forall t$, and since $C^2 \neq 0$, $t_0 \in \{a_1, \ldots, a_r\}$. Thus the number of rational curves on $X$ with nonzero square is bounded by $\sum_{i, j} n_{i, j}$, where $\pi^*(a_i)$ is the effective divisor $\sum n_{i, j} D_j$. Therefore, we have reduced to

**Case 3.** $X$ has no (global) holomorphic 1-forms. For $C$ an exceptional curve of the first kind on $X$, the adjunction formula yields $C \cdot K_X = -1$, and so $C \cdot mK_X < 0 \forall m > 0$.

**Case 3a.** $2K_x$ contains an effective divisor $D$. Then since $D \cdot C < 0$, $C$ must be a component of $D$, and the number of exceptional curves of the first kind on $X$ is bounded by $\sum n_i$, where $D = \sum n_i D_i$, with $D_i$ integral and $n_i > 0$.

**Case 3b.** $2K_x$ does not contain an effective divisor, i.e., $P_2(X) = 0$. Since $X$ has no global holomorphic 1-forms, $q(X) = \dim H^1(X, O_X^*) = 0$. Since $q(X) = P_2(X) = 0$, $X$ is rational by the classification theorem of Castelnuovo [3. Th. 49]).

**Remark.** Among the standard surface types, it is also known that certain K3 surfaces contain an infinite number of $-2$ curves. In addition, it seems to be a part of the folklore that, for each positive integer $n$, there is an elliptic surface containing an infinite
number of curves with self-intersection $-n$.

We end this paper with a conjecture, a discussion of which is to appear in the near future:

Conjecture. Let $X$ be a nonsingular algebraic surface of general type. Then $\mathcal{H}(X)$ is finitely generated.

REFERENCES


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