A NOTE ON DISCONJUGACY FOR SECOND ORDER SYSTEMS

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It is well-known that the equation

\[ x'' + A(t)x = 0 \]

is disconjugate on \([a, b]\) if and only if there exists a solution which is positive on \([a, b]\), in the case that \(A(t)\) is scalar-valued. In this note we generalize this simple result to the case where \(A(t) = (a_{ij}(t))\) is an \(n \times n\) matrix-valued function which satisfies certain generalized sign conditions. These results apply, for instance, if the off diagonal elements are nonnegative. Simple necessary and sufficient conditions are given for disconjugacy if \(A(t) = A\) and these are used to construct examples showing the necessity of sign conditions on \(A(t)\) for the above mentioned results and other results of Sturm type for systems to be valid.

Introduction. Many authors have considered the problem of extending the well-known results on disconjugacy for the scalar equation (1) to systems. We mention the work of Morse [8] and Hartman and Wintner [5], where \(A(t)\) is assumed symmetric or conditions are placed on the symmetric part of \(A\). Recently, many new results have been obtained in the papers of Ahmad and Lazer ([1], [2], [3]) and Schmitt and the author, [9], where symmetry assumptions have generally been avoided.

Recall that (1) is said to be disconjugate on the interval \([a, b]\) if no nontrivial solution of (1) vanishes twice on \([a, b]\), otherwise (1) is conjugate on \([a, b]\). If \(x \in \mathbb{R}^n\), we write \(x \geq 0\) if \(x_i \geq 0, 1 \leq i \leq n\); \(x > 0\) if \(x \geq 0\) and \(x \neq 0\); and \(x > 0\) if \(x_i > 0, 1 \leq i \leq n\). If \(A\) is an \(n \times n\) matrix we denote by \(\sigma(A)\) the spectrum of \(A\).

Below we state two corollaries of our main results and some examples to indicate the necessity of the hypotheses involved. The main results are stated in § 2 and the proofs are given in § 3.

**COROLLARY 1.** Let \(A(t) = (a_{ij}(t))\) be a continuous, matrix-valued function satisfying \(a_{ij}(t) \geq 0, i \neq j\). If (1) is disconjugate on \([a, b]\) then there is a solution \(x(t)\) of (1) satisfying \(x(t) > 0\) on \([a, b]\).

**COROLLARY 2.** Let \(A(t)\) satisfy the conditions of Corollary 1. If there exists a solution \(y(t)\) of the differential inequality \(y'' + A(t)y \leq 0\) satisfying \(y(t) \geq 0, a \leq t \leq b\), then (1) is disconjugate on \([a, b]\).
REMARK. Corollary 2 cannot be weakened with respect to the assumption that \( y(t) \gg 0 \) without additional conditions on \( A(t) \) as seen by the following example: the equation
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'' + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0
\]
is easily seen to be disconjugate on every interval of length less \( \pi \). However, a solution is given by
\[
x(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} > 0
\]
but \( x(t) \gg 0 \).

Corollary 2 generalizes Theorem 3 in [3].

We illustrate Corollary 2 by showing \( x'' + \left( \begin{array}{cc} -3t & 1 \\ 2 & -4t^2 \end{array} \right) x = 0 \) is disconjugate on \([1, \infty)\). To see this, let \( y(t) = \text{col}(t, t) \) and observe that \( y(t) \gg 0 \) on \( 1 \leq t < \infty \) and \( y'' + A(t)y \leq 0 \).

In case \( A(t) = A = (a_{ij}) \) we have the following necessary and sufficient conditions of a particularly simple form for (1) to be disconjugate on \([a, b]\) which do not involve sign conditions on \( A \).

**Lemma 3.** Let \( A(t) \equiv A \). Then (1) is disconjugate on \([a, b]\) if either \( \sigma(A) \cap (0, \infty) = \emptyset \) or if \( b-a < \pi/\sqrt{\lambda} \) for all \( \lambda \in \sigma(A) \cap (0, \infty) \). (1) is conjugate on \([a, b]\) if \( b-a \geq \pi/\sqrt{\lambda} \) for some \( \lambda \in \sigma(A) \cap (0, \infty) \).

Lemma 3 may be employed to construct some interesting examples. For instance, let
\[
A(\varepsilon) = \begin{pmatrix} 6 & 16 + \varepsilon^2 \\ -1 & -2 \end{pmatrix}.
\]
Then \( \sigma(A(\varepsilon)) = \{2 + \varepsilon i, 2 - \varepsilon i\} \). According to Lemma 3,
\[
x'' + A(1)x = 0
\]
is disconjugate on \([0, 4]\) while
\[
x'' + A(0)x = 0
\]
is conjugate on \([0, 4]\) since \( 4 \geq \pi/\sqrt{2} \). Thus the Sturm comparison test does not hold, in general, for systems since \( A(1) \geq A(0) \) (in the usual sense). In [9] it was shown that the Sturm test does hold if, for instance, both matrices are nonnegative (they need not be constant; see [9] for a more precise result). It is easy to construct examples showing that the sign conditions on \( A(t) \) in Corollary 1 are not superfluous.
2. Main results. Let $K$ be a cone in $\mathbb{R}^n$ with nonempty interior. We write $x \geq 0$ if $x \in K$, $x > 0$ if $x \in K - \{0\}$, and $x \gg 0$ if $x \in \text{int } K$ where $\text{int } K$ denotes the interior of $K$. Let $A(t)$ be a continuous matrix-valued function defined on $[a, b]$ satisfying:

(H) There exists $\lambda \geq 0$ such that $(A(t) + \lambda I)(K) \subseteq K$ for all $t \in [a, b]$ where $I$ denotes the identity matrix.

Where required, we assume $A(t)$ is defined on all of $\mathbb{R}$ satisfying condition (H). Simply let $A(t) = A(b)$ for $t > b$ and similarly for $t < a$.

**Theorem 1.** Assume that (H) holds and that (1) is disconjugate on $[a, b]$. Then there is a solution $y(t)$ of (1) satisfying $y(t) > 0$, $a \leq t \leq b$.

**Theorem 2.** If (H) holds and if $y(t)$ is twice differentiable, satisfies the differential inequality

$$y'' + A(t)y \leq 0,$$

and if $y(t) \geq 0$ on $a \leq t \leq b$, then (1) is disconjugate on $[a, b]$.

Finally, we point out that Vandergraft [10] has given sufficient conditions for a matrix $A$ to leave a cone with nonempty interior invariant involving only the spectral properties of $A$. In particular, every strictly triangular matrix has an invariant cone and if $A$ is symmetric then either $A$ or $-A$ leaves some cone invariant.

3. Proofs. First, we show that it suffices to prove Theorems 1 and 2 with the condition (H) replaced by the following: (H'): For each $t$, $A(t)(K) \subseteq (K)$, i.e., $A(t)$ is a positive operator.

To see this make the change in dependent variable by letting $t(s) = a + 1/2k \log (1/(1 - s))$ and change the independent variable by letting $v(s) = e^{-kt(s)}x(t(s))$. Then (1) is equivalent to

$$(2) \quad v''(s) + (t'(s))^2[k^2I + A(t(s))]v(s) = 0.$$ 

It is assumed that $k^2 = \lambda$ where $\lambda$ is as in assumption (H). Clearly (1) is disconjugate on $[a, b]$ if and only if (2) is disconjugate on the appropriate interval. Thus, if Theorem 1 holds under assumption (H'), then the assumption that (1) is disconjugate on $[a, b]$ implies the existence of a solution $v(s) > 0$ of (2) on the interval $t^{-1}([a, b])$ and hence a solution $x(t)$ of (1) on $[a, b]$ with $x(t) > 0$ on $[a, b]$. Similar reasoning shows that it suffices to prove Theorem 2 under
the assumption (H'). In all that follows we assume (H') holds.

At this point we require some notation. Let $X = BC(R, \mathbb{R}^n)$, the Banach space of bounded continuous functions of $R$ into $\mathbb{R}^n$ with supremum norm. Let $\mathcal{K} = \{x \in X: x(t) \in K \text{ for all } t \in R\}$. Then $\mathcal{K}$ is a cone in $X$ which is total, i.e., $\overline{K - K} = X$. If $a, b \in R, a < b$, define the compact linear operator $A_{a, b}: X \to X$ by

$$
(A_{a, b}x)(t) = \begin{cases}
0 & t > b \\
\int_a^b G(a, b; t, s)A(s)x(s)ds & t < a
\end{cases}
$$

where $G(a, b; t, s)$ is the nonnegative Green's function for $-d^2x/dt^2 = f(t), \ x(a) = x(b) = 0$. Notice, see [9], that (we assume (H') holds) $A_{a, b}$ is a positive operator, i.e., $A_{a, b}\mathcal{K} \subseteq \mathcal{K}$. If $a < b$ define $r(a, b) = \rho(A_{a, b})$, the spectral radius of $A_{a, b}$. We require the following lemma which is a trivial modification of lemmas 3.1 and 3.4 and the proof of Theorem 3.5 in [9].

**Lemma 1.** The function $r(a, b)$ defined for $a < b$ is continuous in $a$ for fixed $b$ and continuous in $b$ for fixed $a$. Moreover, $r(a, b)$ is nondecreasing in $b$ (for fixed $a$) and nonincreasing in $a$ (for fixed $b$), and $r(a, b) \to 0 +$ as $b - a \to 0 +$. In addition, (1) is disconjugate on $[a, b]$ if and only if $r(a, b) < 1$.

**Proof of Theorem 1.** If (1) is disconjugate on $[a, b]$ then $r(a, b) < 1$ by Lemma 1. Also by Lemma 1, we can choose $a_i < a$ and $b_i > b$ such that $r(a_i, b_i) < 1$. Now either (i) $r(a_i, b_i) < 1$ for all $b_i \geq b$, or (ii) there exists $b_* > b_i$ such that $r(a_i, b_*) = 1$. In case (ii) we may conclude (by the Krein-Rutman theorem as applied in [9]) the existence of a solution $y(t)$ of (1) satisfying $y(a_i) = y(b_*) = 0$ and $y(t) > 0, a_i < t < b_*$. Thus Theorem 1 is proved in this case. In case (i), (1) is disconjugate on $[a_i, \infty)$ and Theorem 3.11 of [9] completes the proof of this case.

**Proof of Theorem 2.** For this argument let $X = C([a, b]; \mathbb{R}^n)$ and $\mathcal{K}$ the corresponding cone. If $y(t) \geq 0$ on $a \leq t \leq b$ is a solution of the differential inequality $y'' + A(t)y \leq 0$, then we observe that $y \in \text{int } \mathcal{K}(y \geq 0)$. Let $z = A_{a, b}y$ so $z(t)$ satisfies

$$
z'' + A(t)y = 0, \ z(a) = z(b) = 0, \ z(t) \geq 0 \quad a \leq t \leq b.
$$

Then $y(t) - z(t)$ satisfies

$$
(y - z)'' \leq 0 \quad \text{and} \quad (y - z)(a) \geq 0, \ (y - z)(b) \geq 0.
$$
Hence, if \( \varphi \) is a positive linear functional with respect to \( K \subseteq \mathbb{R}^n \) and \( v(t) = \varphi(y(t) - z(t)) \) then \( v'' \leq 0 \) and \( v(a) > 0, v(b) > 0 \). Thus \( v(t) > 0 \) on \( a \leq t \leq b \). Since \( \varphi \) was an arbitrary positive linear functional we conclude that \( y(t) - z(t) \geq 0 \) on \( a \leq t \leq b \), i.e., \( y \geq z \) in \( \mathcal{H} \).

If (1) were not disconjugate on \([a, b]\), then \( r(a, b) \geq 1 \) and thus there exists \( b' \leq b \) with \( r(a, b') = 1 \) and hence (Theorem 3.5 in [9]) a solution \( u(t) \) of (1) satisfying \( u(a) = u(b') = 0 \) and \( u(t) > 0 \) on \([a, b']\). Define \( u(t) = 0 \) on \((b', b]\) so \( u \in \mathcal{H} \). Since \( y \in \text{int} \mathcal{H}^\ast \) we may choose \( \alpha > 0 \) maximal such that \( \alpha u \leq y \) (i.e., if \( \beta u \leq y \) then \( \beta \leq \alpha \)). Then we have

\[
\alpha u = \alpha A_{a, b'}(u) \leq \alpha A_{a, b}(u) \leq A_{a, b}y = z \leq y.
\]

But \( \alpha u \leq y \) implies we may choose \( \eta > \alpha \) such that \( \eta u \leq y \), a contradiction to the maximality of \( \alpha \). This contradiction proves the theorem. Notice that we used the easily established fact that if \( a \leq a' < b' \leq b \) then \( A_{a', b'}x \leq A_{a, b}x \) for all \( x \in \mathcal{H} \).

**Proof of Lemma 3.** The lemma follows immediately from the following assertion: Equation (1) has a nontrivial solution satisfying \( x(0) = x(T) = 0 \) if and only if there exists \( \lambda \in \sigma(A) \cap (0, \infty) \) such that \( \sqrt{\lambda} T = k\pi \) for some positive integer \( k \). To prove the assertion, first assume that \( 0 \in \sigma(A) \) so that there exists a complex matrix \( B \) satisfying \( B^2 = A \). A \( \mathbb{C}^n \)-valued function \( x(t) \) satisfies (1) and \( x(0) = 0 \) if and only if there exists \( x_0 \in \mathbb{C}^n \) such that \( x(t) = (\sin Bt)x_0 \). Thus (1) has a nontrivial solution satisfying \( x(0) = x(T) = 0 \) if and only if \( \det [\sin BT] = 0 \). Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the eigenvalues of \( A \). Then by the spectral mapping theorem and elementary properties of the determinant,

\[
\det [\sin BT] = \prod_{i=1}^{n} \sin \sqrt{\lambda_i} T.
\]

Thus \( \det [\sin BT] = 0 \) if and only if \( \sqrt{\lambda_j} T = k\pi \) for some \( j, 1 \leq j \leq n \) and some integer \( k \). This last holds only if \( \sqrt{\lambda_j} \) is real, in particular \( \lambda_j \) must be positive and \( k \) must be positive. Hence a necessary and sufficient condition for there to be a nontrivial \( \mathbb{C}^n \)-valued solution of (1) satisfying \( x(0) = x(T) = 0 \) is for \( \sqrt{\lambda} T = k\pi \) for some \( \lambda \in \sigma(A) \cap (0, \infty) \) and some positive integer \( k \). Such a solution will be of the form \( x(t) = (\sin Bt)x_0 \) where \( x_0 \neq 0 \) is in the null space of \( \sin BT \). The real and imaginary parts of \( x_0 \), at least one of which is nonzero, will also be solutions of (1) satisfying \( x(0) = x(T) = 0 \). This completes the proof of the assertion in case \( 0 \in \sigma(A) \). In case \( 0 \in \sigma(A) \) write \( \mathbb{R}^n = M \oplus N \) where \( M \) is the generalized nullspace of
A, \( M = \bigcup_{n=1}^{\infty} \text{Ker } A^n \), \( \text{Ker } A^p \), \( p \) some positive integer which we may assume is the smallest such) and \( N = \text{Range } A^p \). The complementary subspaces \( M \) and \( N \) reduce \( A \) and \( A/M \) is nilpotent on \( M \). Write \( A/M = B, A/N = C \). Then (1) becomes

\[
(2) \quad y'' + By = 0 \\
(3) \quad z'' + Cz = 0 \\
x = y + z
\]

The previous analysis applies to (3) since \( \sigma(C) = \sigma(A) - \{0\} \). Since \( B \) is nilpotent it is easy to see that the only solution of (2) satisfying \( y(0) = y(T) = 0 \) is the trivial solution (multiply (2) by \( B^{p-1} \) where \( B^p = 0 \)). This completes the proof in this case.

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Received July 15, 1977 and in revised form April 21, 1980.
Frank Hayne Beatrous, Jr. and R. Michael Range, *On holomorphic approximation in weakly pseudoconvex domains* .............................................. 249
Lawrence Victor Berman, *Quadratic forms and power series fields* .......... 257
John Bligh Conway and Waclaw Szymański, *Singly generated antisymmetric operator algebras* .................................................. 269
Patrick C. Endicott and J. Wolfgang Smith, *A homology spectral sequence for submersions* ...................................................... 279
Sushil Jajodia, *Homotopy classification of lens spaces for one-relator groups with torsion* .......................................................... 301
Herbert Meyer Kamowitz, *Compact endomorphisms of Banach algebras* .............................................................. 313
Keith Milo Kendig, *Moiré phenomena in algebraic geometry: polynomial alternations in \( \mathbb{R}^n \) ......................................................... 327
Cecelia Laurie, *Invariant subspace lattices and compact operators* ........ 351
Ronald Leslie Lipsman, *Restrictions of principal series to a real form* .... 367
Douglas C. McMahon and Louis Jack Nachman, *An intrinsic characterization for PI flows* .................................................. 391
Norman R. Reilly, *Modular sublattices of the lattice of varieties of inverse semigroups* .................................................... 405
Jeffrey Arthur Rosoff, *Effective divisor classes and blowings-up of \( \mathbb{P}^2 \) ...... 419
Zalman Rubinstein, *Solution of the middle coefficient problem for certain classes of \( C \)-polynomials* ........................................... 431
Alladi Sitaram, *An analogue of the Wiener-Tauberian theorem for spherical transforms on semisimple Lie groups* ...................... 439
Hal Leslie Smith, *A note on disconjugacy for second order systems* ...... 447
J. Wolfgang Smith, *Fiber homology and orientability of maps* ............... 453
Audrey Anne Terras, *Integral formulas and integral tests for series of positive matrices* ........................................... 471