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We say that the locally countable sum theorem holds for a property $\mathscr{P}$ if whenever $\{F_\alpha: \alpha \in \Lambda\}$ is a locally countable closed covering of X such that each F_α has $\mathscr{P}$, then X has $\mathscr{P}$. We prove some general theorems which establish the locally countable sum theorem for properties satisfying certain conditions. It is then shown that a number of sum theorems hold for those properties which are closed hereditary and for which the locally countable sum theorem holds. We apply our theorems to some particular cases to obtain many new results and also to improve upon some known results.

Recently, there has been a growing interest in proving sum theorems for various properties of topological spaces. A sum theorem for a property is a theorem of the following type: If $\{A_\alpha: \alpha \in \Lambda\}$ is a covering of a space X such that each A_α has $\mathscr{P}$, then X has $\mathscr{P}$. Perhaps the simplest known sum theorem is the locally finite sum theorem (LFST) which states that if $\{F_\alpha: \alpha \in \Lambda\}$ is a locally finite closed covering of X such that each F_α has a property $\mathscr{P}$, then X has $\mathscr{P}$. For an extensive study of the LFST and for its several interesting consequences, one may refer to [1, 21 to 24]. The purpose of the present paper is to concentrate on the locally countable sum theorem (LCST) by which we mean the LFST with 'locally finite' replaced by 'locally countable'. We call a family $\mathscr{A}$ locally countable if each point has a neighborhood which intersects at most countably many elements of $\mathscr{A}$. We shall prove some general theorems which will establish the LCST for properties satisfying certain conditions (with some restrictions on the space of course). It will then be shown that a number of sum theorems hold for those properties which are closed hereditary and for which LCST holds. We shall apply our theorems to some particular cases to obtain many new results and also to improve upon some known results. In particulra, some results of Stone [25] and Okuyama [17] are improved.

No separation axioms are assumed. In particular, normal and regular spaces are not T_1.

1. The locally countable sum theorem.

1.1. We say that the locally countable sum theorem (to be abbreviated as LCST) holds for a property $\mathscr{P}$ if whenever $\{F_\alpha: \alpha \in \Lambda\}$ is a locally countable closed covering of X such that each F_α has $\mathscr{P}$, then X has $\mathscr{P}$.

Obviously, if the LCST holds for $\mathscr{P}$, then the LFST also holds for it. However, there are several important properties for which the LFST holds but the LCST does not hold. Metrizability is one example, as can be seen by considering any countably infinite, T_1, nonmetrizable space.

DEFINITION 1.2 [Burke, 6]. A space X is said to be subparacompact if every open covering of X has a σ-discrete closed refinement.

Obviously, every regular paracompact space is subparacompact. However, there is a subparacompact space which is not paracompact. In fact the space of the well-known example of a normal Hausdorff space which is not collectionwise normal due to Bing [3], has a σ-locally finite net and is therefore, a subparacompact space. This space cannot be paracompact as it is a Hausdorff non-collectionwise normal space.

1.3. We say that the countable sum theorem (to be abbreviated as CST) holds for a property $\mathscr{P}$ if whenever $\{F_i: i \in N\}$ is a countable closed covering of X such that each F_i has $\mathscr{P}$, then X has $\mathscr{P}$. If a property $\mathscr{P}$ is preserved under disjoint topological sums, then we say that $\mathscr{P}$ is additive. Also if every closed subspace of a space with $\mathscr{P}$ has $\mathscr{P}$, then we say that the property is closed hereditary (to be abbreviated as CH).

THEOREM 1.4. *Let X be a subparacompact space. Suppose $\mathscr{P}$ is a property such that* (i) *$\mathscr{P}$ is additive* (ii) CST *holds for $\mathscr{P}$* (iii) *$\mathscr{P}$ is* CH. *Then the* LCST *holds for $\mathscr{P}$ in X.*

Proof. Let $\{F_\alpha: \alpha \in \Lambda\}$ be a locally countable closed covering of X such that each F_α has $\mathscr{P}$. Then, for each $x \in X$, there is an open set U_x containing x such that U_x intersects at most countably many F_α's. Thus, $\{U_x: x \in X\}$ is an open covering of X. Since X is subparacompact, there is a σ-discrete closed refinement $\mathscr{H} = \bigcup_{i=1}^\infty \mathscr{H}_i$, each $\mathscr{H}_i$ being a discrete family of closed sets. For each i, let $H_i = \bigcup\{H: H \in \mathscr{H}_i\}$. Now, each $H \in \mathscr{H}$ is contained in some U_x and each U_x can intersect at most countably many F_α's. This means that for each $H \in \mathscr{H}$, there exists a countable subfamily $\{F_{\alpha_i}: i \in N\}$ of $\{F_\alpha: \alpha \in \Lambda\}$ such that $H = \bigcup_{i=1}^\infty (H \cap F_{\alpha_i})$. Since $\mathscr{P}$ is CH, each $H \cap F_{\alpha_i}$ has $\mathscr{P}$. It follows that H has $\mathscr{P}$ in view of (ii). Now, $\{H: H \in \mathscr{H}_i\}$ is a discrete closed covering of H_i each member of which has $\mathscr{P}$. Then H_i has $\mathscr{P}$ in view of (i). Also, $X = \cup\{H_i: i \in N\}$ and hence X has $\mathscr{P}$ in view of (ii).

COROLLARY 1.5. *Suppose X is a fully normal space and $\mathscr{P}$ is a property such that* (i) *$\mathscr{P}$ is additive* (ii) CST *holds for $\mathscr{P}$* (iii) $\mathscr{P}$

is CH. *Then the* LCST *holds for* $\mathscr{P}$ *in* X.

Proof. Every fully normal space is subparacompact.

If in Theorem 1.4, 'subparacompact' is replaced by 'paracompact' and '$\mathscr{P}$ is additive' is replaced by 'LFST holds for $\mathscr{P}$', the following result is obtained:

THEOREM 1.6. *Let* X *be a paracompact space. If* $\mathscr{P}$ *is a property such that* (i) LFST *holds for* $\mathscr{P}$ (ii) CST *holds for* $\mathscr{P}$ (iii) $\mathscr{P}$ *is* CH, *then the* LCST *holds for* $\mathscr{P}$ *in* X.

Proof. Let $\{F_\alpha: \alpha \in \Lambda\}$ be a locally countable closed covering of X such that each F_α has $\mathscr{P}$. Since $\{F_\alpha: \alpha \in \Lambda\}$ is locally countable, therefore, for each $x \in X$, there is an open set U_x containing x which intersects at most countably many members of $\{F_\alpha: \alpha \in \Lambda\}$. Consider the open covering $\mathscr{U} = \{U_x: x \in X\}$. Since X is paracompact, there is a locally finite open refinement $\mathscr{V}$ of $\mathscr{U}$. For each $x \in X$, there is an open set N_x which contains x and which intersects at most finitely many members of $\mathscr{V}$. Since $\mathscr{V}$ is a refinement of $\mathscr{U}$, it follows that there is a finite subfamily $\{U_{x_i}: i = 1, 2, \cdots, n\}$ of $\mathscr{U}$ such that $\bar{N}_x \subseteq \bigcup_{i=1}^n U_{x_i}$. Since each U_{x_i} intersects at most countably many F_{α_i}'s, there are countably many F_{α_i}, $i \in N$, such that $\bar{N}_x = \bigcup_{i=1}^\infty (\bar{N}_x \cap F_{\alpha_i})$. Since each F_{α_i} has $\mathscr{P}$, each $\bar{N}_x \cap F_{\alpha_i}$ has $\mathscr{P}$ in view of (iii) and then each $\bar{N}_x$ has $\mathscr{P}$ in view of (ii). Now if $\mathscr{W}$ is a locally finite open refinement of $\{N_x: x \in X\}$, then $\{\bar{W}: W \in \mathscr{W}\}$ is a locally finite closed covering of X each member of which has $\mathscr{P}$. Hence X has $\mathscr{P}$ in view of (i).

1.7. Hypotheses (i), (ii) and (iii) of Theorems 1.4 and 1.6 are satisfied by each of the following properties:

Semi-stratifiability [9, 12], σ-space [18], space with a σ-closure-preserving net [18], space with a σ-discrete net [18], Σ-space [15], normality + P-space [14], α-space [2]. Hence the LCST holds for all these properties in a subparacompact (or paracompact) space.

Sometimes it is found that the conditions of Theorems 1.4 and 1.6 are satisfied with some additional conditions on the space. For instance, it is well-known that CST does not hold for metrizability. However, Stone [25] proved that the CST does hold for metrizability if the space is collectionwise normal, Hausdorff, locally countably compact. Let us first prove a metrization theorem which could be of some independent interest. The theorem will then be used to conclude a result which will improve upon the above-mentioned result of Stone and a subsequent result of Okuyama [1]. An application of Theorem

1.4 will then give results which will improve upon some other results of Stone [25] and Okuyama [17].

THEOREM 1.8. *For any space X, the following are equivalent*:
(a) *X is metrizable*;
(b) *X is a collectionwise normal, Hausdorff, locally ωM, σ-space*;
(c) *X is a paracompact, Hausdorff, locally ωM, σ-space.*

Proof. (a) $\Rightarrow$ (b) is obvious. Since every collectionwise normal, Hausdorff, σ-space is paracompact [17], therefore (b) $\Rightarrow$ (c). (c) $\Rightarrow$ (a). If (c) holds then every point has a neighborhood whose closure is a ωM-space which is also paracompact and Hausdorff σ-space. Hence, every point has a neighborhood whose closure is metrizable in view of Theorem 2.4 in [11]. Thus X is locally metrizable. Since X is paracompact, Hausdorff, it is then metrizable.

1.9. Since the CST holds for a σ-space, it follows that in view of Theorem 1.8 above, we can have the following result:

'Let X be a collectionwise normal, Hausdorff, locally ωM-space. If $\{F_i : i \in N\}$ is a countable closed covering of X such that each F_i is a σ-space, then X is metrizable.'

The above result improves Theorem 1 of Stone [25] and Theorem 3.8 of Okuyama [1]. Stone proved it for locally countably compact spaces and Okuyama proved it for locally M-spaces in place of locally ωM-spaces and also for metrizable in place of a σ-space.

1.10. Since we proved that the LCST holds for a σ-space in a subparacompact (or paracompact) space, we can have the following result again in view of Theorem 1.8:

'Let X be a paracompact, Hausdorff, locally ωM-space. If $\{F_\alpha : \alpha \in \Lambda\}$ is a locally countable closed covering of X such that each F_α is a σ-space, then X is metrizable.'

The above result improves Corollary 2 of Stone [25] and Corollary 3.10 of Okuyama [17]. In fact, Stone proved the result for locally countably compact in place of locally ωM and Okuyama proved it for locally M-spaces. Also, they assumed metrizability in place of a σ-space.

1.11. It was proved by Shiraki [20] that a Hausdorff ωM-space is metrizable if it is an α-space. Therefore, Theorem 1.4 applied to α-spaces gives the following result:

'Let X be a subparacompact Hausdorff ωM-space. If $\{F_\beta : \beta \in \Lambda\}$ is a locally countable closed covering of X such each that F_β is an α-space, then X is metrizable.'

1.12. As a further generalization of ωM-spaces, Borges [4] introduced $\omega\Delta$-spaces. Every regular T_1, $\omega\Delta$-space is developable if it is an α-space [7]. We, therefore, have the following result as in 1.11 above:

'Let X be a subparacompact, regular T_1, $\omega\Delta$-space. If $\{F_\beta: \beta \in \Lambda\}$ is a locally countable closed covering of X such that each F_β is an α-space, then X is developable.'

1.13. Since every paracompact Σ-space with a point-countable base is metrizable [13], Theorem 1.6 as applied to Σ-spaces gives the following result:

'Let X be a paracompact space with a point-countable base. If $\{F_\alpha: \alpha \in \Lambda\}$ is a locally countable closed covering of X such that each F_α is a Σ-space, then X is metrizable.'

1.14. Nagami [16] proved that the LCST holds for the property of having $\dim \leqq n$ in a paracompact Hausdorff space. Now, Ostrand [19] showed that the LFST holds for the property of having $\dim \leqq n$ in any space. Also, Nagami proved that it is CH in normal spaces and Čech [8] had shown that the CST holds for it in normal spaces. In view of Theorem 1.4, we then have the following slightly improved version of Nagami's result:

'Let X be a subparacompact normal space. If $\{F_\alpha: \alpha \in \Lambda\}$ is a locally countable closed covering of X such that $\dim F_\alpha \leqq n$ for each $\alpha \in \Lambda$, then $\dim X \leqq n$.'

1.15. Nagami [16] proved that the LCST holds for the property of having $\operatorname{Ind} \leqq n$ in a hereditarily paracompact Hausdorff space. Also, Nagami [16] proved that the LFST holds for the property of having $\operatorname{Ind} \leqq n$ in a totally normal space. Dowker [10] proved that it is CH and the CST holds for it in a totally normal space. Thus, in view of Theorem 1.4, we obtain the following slightly improved version of Nagami's result:

'Let X be a totally normal, subparacompact space. If $\{F_\alpha: \alpha \in \Lambda\}$ is a locally countable closed covering of X such that $\operatorname{Ind} F_\alpha \leqq n$ for each $\alpha \in \Lambda$, then $\operatorname{Ind} X \leqq n$.'

DEFINITION 1.16 [Boyte, 5]. A space X is said to be point paracompact if for each open covering $\mathscr{U}$ of X, there is an open refinement $\mathscr{V}_p$ of $\mathscr{U}$ for each $p \in X$ such that $\mathscr{V}_p$ is locally finite at p.

Every paracompact space as also every regular space is point paracompact. Also, every Hausdorff point paracompact space is regular.

THEOREM 1.17. *Let X be a point paracompact space and let $\mathscr{P}$*

be a property such that (i) $\mathscr{P}$ *is* CH (ii) CST *holds for* $\mathscr{P}$ (iii) X *has* $\mathscr{P}$ *if* X *has* $\mathscr{P}$ *locally. Then the* LCST *holds for* $\mathscr{P}$ *in* X.

Proof. Let $\{F_\alpha: \alpha \in \Lambda\}$ be a locally countable closed covering of X such that each F_α has $\mathscr{P}$. For each $x \in X$, there is an open neighborhood U_x of x which intersects at most countably many members of $\{F_\alpha: \alpha \in \Lambda\}$. Consider the open covering $\{U_x: x \in X\}$ of X. Since X is point paracompact, for each $p \in X$, there is an open refinement $\mathscr{V}_p$ of $\{U_x: x \in X\}$ which is locally finite at p. It follows that there is a neighborhood N_p of p and a finite subfamily $\{U_{x_i}: i = 1, \cdots, n\}$ of $\{U_x: x \in X\}$ such that $\bar{N}_p \subseteqq \bigcup_{i=1}^n U_{x_i}$. Since each U_{x_i} intersects countably many F_α's, there is a countable subfamily $\{F_{\alpha_i}: i = 1, 2, \cdots\}$ of $\{F_\alpha: \alpha \in \Lambda\}$ such that $\bar{N}_p = \bigcup_{i=1}^n (\bar{N}_p \cap F_{\alpha_i})$. If each F_α has $\mathscr{P}$, then $\bar{N}_p$ has $\mathscr{P}$ in view of (i) and (ii). Hence X has $\mathscr{P}$ in view of (iii).

DEFINITION 1.18 [Worrell and Wicke, 27]. A space X is said to be θ-refinable if for each open covering $\mathscr{U}$ of X, there is a sequence $\{\mathscr{U}_n: n \in N\}$ of open refinements such that for each point $p \in X$, there is an $n \in N$ such that p belongs to atmost finitely many members of U_n.

Every subparacompact as also every metacompact space is θ-refinable.

1.19. We say that the point-finite open cover sum theorem (to be abbreviated as PF-OCST) holds for a property $\mathscr{P}$, if whenever $\{G_\alpha: \alpha \in \Lambda\}\}$ is a point-finite open cover of X such that each G_α has $\mathscr{P}$, then X has $\mathscr{P}$.

THEOREM 1.20. *Let X be a point paracompact, normal and θ-refinable space. Suppose $\mathscr{P}$ is a property such that* (i) $\mathscr{P}$ *is* CH (ii) CST *holds for* $\mathscr{P}$ (iii) PF-OCST *holds for* $\mathscr{P}$. *Then the* LCST *holds for* $\mathscr{P}$ *in* X.

Proof. We shall first show that X has $\mathscr{P}$ if X has $\mathscr{P}$ locally. The result will then follow from Theorem 1.17 above. If X has $\mathscr{P}$ locally, then for each $x \in X$, there is a neighborhood N_x of x such that N_x has $\mathscr{P}$. Consider the open covering $\{N_x^0: x \in X\}$ of X. Since X is θ-refinable, there is a countable family $\{F_i: i \in N\}$ of closed subsets of X covering X such that for each F_i there is an open refinement $\mathscr{V}_i = \{V_\alpha: \alpha \in \Lambda_i\}$ of $\mathscr{U}$ such that $\mathscr{V}_i$ is point-finite at each point of F_i [27]. Now $\{V_\alpha \cap F_i: \alpha \in \Lambda_i\}$ is a point-finite open (in F_i) covering of F_i and F_i is normal. Therefore, there is a closed refinement $\{A_\alpha: \alpha \in \Lambda_i\}$ of $\{V_\alpha \cap F_i: \alpha \in \Lambda_i\}$ such that $A_\alpha \subset V_\alpha \cap F_i$ for each $\alpha \in \Lambda_i$. Now F_i is normal, and hence there is an open F_σ subset H_α

of F_i such that $A_\alpha \subset H_\alpha \subset V_\alpha \cap F_i$ for each $\alpha \in \Lambda_i$. Since each N_x has $\mathscr{P}$ and each H_α is contained in some N_x, therefore H_α has $\mathscr{P}$ in view of (i) and (ii). Thus $\{H_\alpha: \alpha \in \Lambda_i\}$ is a point-finite open cover of F_i such that each H_α has $\mathscr{P}$. Hence X has $\mathscr{P}$ in view of (iii). It follows now that the LCST holds for $\mathscr{P}$ in view of Theorem 1.17.

COROLLARY 1.21. *Let X be point paracompact, normal and θ-refinable. Suppose $\mathscr{P}$ is a property such that* (i) *$\mathscr{P}$ is* CH (ii) CST *holds for $\mathscr{P}$* (iii) *$\mathscr{P}$ is additive* (iv) *$\mathscr{P}$ is preserved under finite-to-one open continuous maps. Then the* LCST *holds for $\mathscr{P}$ in X.*

Proof. It is enough to show that the PF-OCST holds for $\mathscr{P}$. Let $\{G_\alpha: \alpha \in \Lambda\}$ be a point-finite open covering of X such that G_α has $\mathscr{P}$. Let X^* be the disjoint topological sum of G_α's and let $P: X^* \to X$ be the natural projection map. Since $\{G_\alpha: \alpha \in \Lambda\}$ is a point-finite open cover, therefore, $\mathscr{P}$ is a finite-to-one, open continuous map. Now, X^* has $\mathscr{P}$ in view of (iii) and then X has $\mathscr{P}$ in view of (iv). Thus the LCST holds for $\mathscr{P}$ in view of Theorem 1.20 above.

THEOREM 1.22. *Let X be a point paracompact, θ-refinable space. If $\mathscr{P}$ is a property such that* (i) *$\mathscr{P}$ is* CH (ii) *$\mathscr{P}$ is* OH (*that is, every open subspace of a space with $\mathscr{P}$ has $\mathscr{P}$*) (iii) CST *holds for $\mathscr{P}$* (iv) PF-OCST *holds for $\mathscr{P}$. Then the* LCST *holds for $\mathscr{P}$ in X.*

Proof. It is enough to show that X has $\mathscr{P}$ if X has $\mathscr{P}$ locally. So, suppose each point $x \in X$ has a neighborhood N_x which has $\mathscr{P}$. Then, proceeding as in the proof of Theorem 1.20, we get $\{V_\alpha \cap F_i: \alpha \in F_i\}$ as a point-finite open covering of F_i such that each $V_\alpha \cap F_i$ has $\mathscr{P}$ in view of (i) and (ii). Then F_i has $\mathscr{P}$ in view of (iv) and therefore X has $\mathscr{P}$ in view of (iii). Then the LCST holds for $\mathscr{P}$ in view of Theorem 1.17.

COROLLARY 1.23. *Let X be a point paracompact θ-refinable space. If $\mathscr{P}$ is a property such that* (i) *$\mathscr{P}$ is* CH (ii) *$\mathscr{P}$ is* OH (iii) CST *holds for $\mathscr{P}$* (iv) *$\mathscr{P}$ is additive* (v) *$\mathscr{P}$ is preserved under finite-to-one, open continuous maps. Then the* LCST *holds for $\mathscr{P}$ in X.*

Proof. Similar to the proof of Corollary 1.21.

1.24. A θ-refinable space X has property $\mathscr{P}$ if it has that property locally where $\mathscr{P}$ is any of the following properties:

Semi-stratifiability [26], σ-space [26], α-space [2], P-space (if X is normal) [26]. It is clear that in view of Theorem 1.22, the LCST holds for all the above properties in a θ-refinable, point paracompact space, with the additional condition of normality on X in case of a P-space.

2. Some more applications. In this section, we shall obtain three general sum theorems which hold for all those properties which are CH and for which the LCST is known to hold.

DEFINITION 2.1. A family $\{A_\alpha: \alpha \in \Lambda\}$ of subsets of X is said to be order locally countable if there is a linear ordering '$<$' of the index set Λ such that for each $\alpha \in \Lambda$, the family $\{A_\beta: \beta < \alpha\}$ is locally countable at each point of A_α.

The concept of order locally countable families generalizes the concept of order locally finite families due to Katuta [21]. Also, every σ-locally countable family can be easily seen to be order locally countable.

Now let us assume that for all the following results in this section, $\mathscr{P}$ is a property such that

(a) $\mathscr{P}$ is CH.

(b) the LCST holds for $\mathscr{P}$.

THEOREM 2.2. *Let $\mathscr{V}$ be an order locally countable open covering of X such that $\bar{V}$ has $\mathscr{P}$ for each $V \in \mathscr{V}$. Then X has $\mathscr{P}$.*

Proof. Let $\mathscr{V} = \{V_\alpha: \alpha \in \Lambda\}$ where Λ has a linear ordering with respect to which $\mathscr{V}$ is order locally countable. For each $\alpha \in \Lambda$, let $F_\alpha = \bar{V}_\alpha \sim \cup\{V_\beta: \beta < \alpha\}$. Since $\bar{V}_\alpha$ has $\mathscr{P}$, therefore, F_α has $\mathscr{P}$ for each α. It is easy to verify that $\{F_\alpha: \alpha \in \Lambda\}$ is locally countable. Thus $\{F_\alpha: \alpha \in \Lambda\}$ is a locally countable closed covering of X such that each F_α has $\mathscr{P}$. Therefore, X has $\mathscr{P}$ because the LCST holds for $\mathscr{P}$.

COROLLARY 2.3. *If $\mathscr{V}$ is a σ-locally countable open covering of X such that $\bar{V}$ has $\mathscr{P}$ for each $V \in \mathscr{V}$, then X has $\mathscr{P}$.*

COROLLARY 2.4. *If $\{V_\alpha: \alpha \in \Lambda\}$ is an order locally countable family of closed subsets of X such that $\cup\{V_\alpha^\circ: \alpha \in \Lambda\} = X$ and if each V_α has $\mathscr{P}$, then X has $\mathscr{P}$.*

THEOREM 2.5. *Let $\mathscr{V}$ be an order locally countable open covering of X such that* BdV *is compact for each $V \in \mathscr{V}$. If X is regular and if each $V \in \mathscr{V}$ has $\mathscr{P}$, then X has $\mathscr{P}$.*

Proof. In view of Theorem 2.2, it is enough to show that $\bar{V}$ has $\mathscr{P}$ for each $V \in \mathscr{V}$. Let $A = \bar{V} \sim V$ where V is any member of

$\mathscr{V}$. Let $A = \bar{V} \sim V$ where V is any member of $\mathscr{V}$. Since A is compact and X is regular, there exist finitely many open sets $U_1, \cdots, U_n$ such that $A \subset \bigcup_{i=1}^{n} U_i$ and each U_i is contained in some member of $\mathscr{V}$. Since each member of $\mathscr{V}$ has $\mathscr{P}$, therefore each $\bar{U}_i$ has $\mathscr{P}$. If $F_i = \bar{U}_i \cap \bar{V}$ for each i, then F_i has $\mathscr{P}$ since $\bar{U}_i$ has $\mathscr{P}$. Also, if $F_0 = \bar{V} \sim \bigcup_{i=1}^{n} U_i$, then $F_0 \subset V$ and so F_0 has $\mathscr{P}$. Thus, $\{F_i: i = 0, 1, 2, \cdots, n\}$ is a finite and hence an order locally countable closed covering of $\bar{V}$ every member of which has $\mathscr{P}$. Hence each $\bar{V}$ has $\mathscr{P}$ and the result follows from Theorem 2.2.

COROLLARY 2.6. *If $\mathscr{V}$ is a σ-locally countable open covering of a regular space X such that* $\operatorname{Bd} V$ *is compact for each $V \in \mathscr{V}$, then X has $\mathscr{P}$ if each $V \in \mathscr{V}$ has $\mathscr{P}$.*

THEOREM 2.7. *Let $\mathscr{V}$ be a σ-locally countable open covering of X such that each $V \in \mathscr{V}$ has $\mathscr{P}$. If X is normal and if each $V \in \mathscr{V}$ is an F_σ, then X has $\mathscr{P}$.*

Proof. Let $\mathscr{V} = \bigcup_{i=1}^{\infty} \mathscr{V}_i$, where each $\mathscr{V}_i$ is locally countable. Let $\mathscr{V}_i = \{V_\alpha: \alpha \in \Lambda\}$. Since each V_α is an F_σ and X is normal, for each $\alpha \in \Lambda_i$, we can write $V_\alpha = \bigcup_{p=1}^{\infty} V_{\alpha,p}$ where each $V_{\alpha,p}$ is an open set whose closure is contained in V_α. Let $\mathscr{U}_{i,p} = \{V_{\alpha,p}: \alpha \in \Lambda_i\}$. Then $\mathscr{U}$ is a σ-locally countable open covering of X such that $\bar{U}$ has $\mathscr{P}$ for each $U \in \mathscr{U}$. Hence X has $\mathscr{P}$ in view of Theorem 2.2.

2.8. We have already established in §1, the LCST for various properties under various restrictions on the space. In view of these results, theorems of this section can be used to assert many more results. For instance, we have established the LCST for semi-stratifiability in a θ-refinable, point paracompact space. Thus, Theorem 2.2 as applied to semi-stratifiability would give us the following result:

'If $\{U_\alpha: \alpha \in \Lambda\}$ is an order locally countable open covering of a θ-refinable, point paracompact space X such that $\bar{U}_\alpha$ is semi-stratifiable for each $\alpha \in \Lambda$, then X is semi-stratifiable.'

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