ANALYTIC FUNCTIONS IN TUBES WHICH ARE REPRESENTABLE BY FOURIER-LAPLACE INTEGRALS

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ANALYTIC FUNCTIONS IN TUBES WHICH ARE REPRESENTABLE BY FOURIER-LAPLACE INTEGRALS

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Spaces of analytic functions in tubes in $\mathbb{C}^n$ which generalize the Hardy $H^p$ spaces are defined and studied. In addition Cauchy and Poisson integrals of distributions in $\mathcal{D}'$, are analyzed.

1. Introduction. Bochner ([1] and [2]) has defined the Hardy $H^2(T^C)$ spaces for tubes $T^C = \mathbb{R}^n + iC$ in $\mathbb{C}^n$ where $C \subset \mathbb{R}^n$ is an open convex cone. Stein and Weiss [11] have studied the $H^p(T^B)$ spaces for arbitrary $p > 0$ and with respect to tubes $T^B$, $B$ being an open proper subset of $\mathbb{R}^n$ [11, pp. 90-91]. Vladimirov [12, §§25.3-25.4] has considered analytic functions in $T^C$, $C$ being an open connected cone, which satisfy the growth [12, p. 224, (64)]. Vladimirov has stated [12, p. 227, lines 4-5] that the growth which defines the $H^2$ functions of Bochner is more restrictive than [12, p. 224, (64)]. We show in this paper that the $H^2$ growth is not more restrictive than [12, p. 224, (64)] by showing that the functions of Vladimirov are exactly the $H^2$ functions. However, Vladimirov's growth has led us to define new spaces of analytic functions in tubes which have growth estimates that are more general than that of the $H^p(T^B)$ spaces, and we analyze these new spaces in this paper. Further, we study Cauchy and Poisson integrals of distributions in $\mathcal{D}'$.

The $n$-dimensional notation in this paper is described in [7, p. 386]. The definitions of a cone in $\mathbb{R}^n$, projection of a cone $\text{pr}(C)$, compact subcone, and dual cone $C^* = \{t \in \mathbb{R}^n : \langle t, y \rangle \leq 0, y \in C\}$ of a cone $C$ are given in [12, p. 218]. Terminology concerning distributions is that of Schwartz [10]. The support of a distribution or function $g$ is denoted $\text{supp}(g)$. Definitions, properties, and relevant topologies of the function spaces $\mathcal{S}$, $\mathcal{D}_{L^p}$, $\mathcal{B} = \mathcal{D}_{L^\infty}$, and $\mathcal{B}$ and of the distribution spaces $\mathcal{S}'$ and $\mathcal{D}'_{L^p}$ are in [10]. The $L^1$ and $\mathcal{S}'$ Fourier and inverse Fourier transforms are defined in [7, pp. 387-388] and [10, p. 250], respectively. The limit in the mean Fourier and inverse Fourier transforms of functions in $L^p$, $1 < p \leq 2$, and $L^q$, $(1/p) + (1/q) = 1$, are in [8] and [3]. $\mathcal{F}[\phi(t); x]$ ($\mathcal{F}^{-1}[\phi(x); t]$) denotes the Fourier (inverse Fourier) transform of a function in the relevant sense. If $V \in \mathcal{S}'$ we denote its Fourier (inverse Fourier) transform by $\mathcal{F}[V] = \hat{V}$ ($\mathcal{F}^{-1}[V]$). For $\phi \in L^p$, $1 < p \leq 2$, the Parseval inequality is
2. The Cauchy and Poisson kernel functions and technical results. Let $C$ be an open connected cone, $C^*$ be the dual cone of $C$, and $0(C)$ be the convex envelope (hull) of $C$. The Cauchy kernel function [6, p. 201] is

$$K(z - t) = \int_{C^*} \exp(2\pi i \langle z - t, \eta \rangle) d\eta, z \in T^0(C) = R^n + i0(C), t \in R^n.$$  

To avoid the triviality of $K(z - t) = 0$ we assume in this section that $\overline{O(C)}$ does not contain an entire straight line [12, p. 222, Lemma 1]. In [6, Theorem 1] one of us proved $K(z - t) \in D_{L^q}$ for all $q$, $(1/p) + (1/q) = 1, 1 < p \leq 2$, as a function of $t \in R^n$ for fixed $z \in T^0(C)$. But $D_{L^q} \subset \mathcal{B} \subset D_{L^\infty}$ for every $q, 1 \leq q < \infty$, by [10, pp. 199-200]. We thus have

**Lemma 2.1.** Let $z \in T^0(C)$. As a function of $t \in R^n$,

$$K(z - t) \in \mathcal{B} \cap D_{L^q} \quad \text{for all } q, (1/p) + (1/q) = 1, 1 \leq p \leq 2.$$

For an open connected cone $C$ the Poisson kernel function [6, p. 204] is

$$Q(z; t) = \frac{K(z - t)K(z - t)}{K(2iy)}, \quad z = x + iy \in T^0(C), \quad t \in R^n.$$

**Lemma 2.2.** $Q(z; t) \in \mathcal{B} \cap D_{L^q}$ for all $q, 1 \leq q \leq \infty$, as a function of $t \in R^n$ for arbitrary $z \in T^0(C)$.

**Proof.** Let $\alpha$ be any $n$-tuple of nonnegative integers. By the Leibnitz rule

$$D_t^\alpha(Q(z; t)) = \frac{1}{K(2iy)} \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} D_t^\beta(K(z - t)) D_t^\gamma(\overline{K(z - t)})$$

$$z = x + iy \in T^0(C).$$

By (2.2) $D_t^\beta(K(z - t))$ and $D_t^\gamma(\overline{K(z - t)})$ are in $L^1 \cap L^\infty$ as functions of $t \in R^n$. Thus $D_t^\alpha(Q(z; t)) \in L^1 \cap L^\infty \subseteq L^q, 1 \leq q \leq \infty$. Hence $Q(z; t) \in \mathcal{B}$, $1 \leq q \leq \infty$; and $Q(z; t) \in \mathcal{B}$ also since $D_{L^q} \subset \mathcal{B}$, $1 \leq q < \infty$.

As a function of $x = \text{Re}(z) \in R^n$ for $y \in O(C)$ arbitrary we also have

$$Q(x; y) = \frac{K(x + iy)K(x + iy)}{K(2iy)} \in \mathcal{B} \cap D_{L^q} \quad \text{for all } q, 1 \leq q \leq \infty.$$
We conclude this section with two important and useful theorems.

**Theorem 2.1.** Let $B$ be an open connected subset of $\mathbb{R}^n$. Let $1 \leq p < \infty$ and $A \geq 0$. Let $g(t)$ be a measurable function on $\mathbb{R}^n$ which satisfies

\[(2.6) \quad \int_{\mathbb{R}^n} |g(t)|^p e^{-2\pi p(y, t)} dt \leq M_{A, g}^p e^{2\pi p A |y|}, \quad y \in B,\]

where the constant $M_{A, g}$ depends only on $A$ and $g(t)$ and not on $y \in B$. Then

\[(2.7) \quad F(z) = \int_{\mathbb{R}^n} g(t) e^{2\pi i (z, t)} dt, \quad z \in T^B,\]

is an analytic function of $z \in T^B$ and has an analytic extension to $T^{O(B)}$.

**Proof.** For arbitrary $y_0 \in B$ there is an open neighborhood of $y_0$, $N(y_0) \subset B$, and a $\delta > 0$ such that $\{y: |y - y_0| = \delta\} \subset N(y_0)$. There are $k$ cones $\Gamma_j$, $j = 1, \ldots, k$, having the properties as in [11, p. 92, lines 12-15] and such that whenever two points $v$ and $w$ are in a $\Gamma_j$ then $\langle v, w \rangle \geq (\sqrt{2}/2) |v| |w|$. For each $j = 1, \ldots, k$ choose $y_j$ such that $(y_0 - y_j) \in \Gamma_j$ and $|y_j - y_0| = \delta$. Then for each $p$, $1 \leq p < \infty$, and all $t \in \Gamma_j$, $j = 1, \ldots, k$, we have $(-2\pi p \langle y_j - y_0, t \rangle) \geq \varepsilon |t|$ where $\varepsilon = \sqrt{2} \pi p \delta > 0$. Using this fact, (2.6), and analysis as in [11, pp. 92-93] we have that the function

\[G(t) = g(t) \exp(\varepsilon |t|/2p) \exp(-2\pi \langle y_0, t \rangle), \quad t \in \mathbb{R}^n, \quad 1 \leq p < \infty,\]

is an $L^1$ function. If $y = \text{Im}(z)$ is restricted so that $|y - y_0| < (\varepsilon/4\pi p)$ then

\[|g(t) e^{2\pi i (z, t)}| \leq |G(t)|, \quad t \in \mathbb{R}^n, \quad x = \text{Re}(z) \in \mathbb{R}^n.\]

Since $y_0 \in B$ was arbitrary it follows that $F(z)$ is analytic in $T^B$ and has an analytic extension to $T^{O(B)}$ by [4, p. 92, Theorem 9].

Note the indicatrix function $u_c(t)$ of a cone $C$ defined in [12, p. 219]. $O(C)$ may or may not contain an entire straight line in the next theorem.

**Theorem 2.2.** Let $C$ be any open connected cone and $A \geq 0$. Let $g(t) \in L^p$, $1 \leq p < \infty$, such that

\[(2.8) \quad \int_{\mathbb{R}^n} |g(t)|^p e^{-2\pi p(y, t)} dt \leq M_{A, g}^p \exp(2\pi p (A + \varepsilon) |y|), \quad y \in C,\]

for all $\varepsilon > 0$ where the constant $M_{A, g}$ depends on $A$, $\varepsilon$, and $g(t)$.
and not on \( y \in C \). Then \( \text{supp}(g) \subseteq S_A = \{ t : u_c(t) \leq A \} \) almost everywhere (a.e.).

**Proof.** Assume \( g(t) \neq 0 \) on a set of positive measure in \( S^A = R^n \setminus S_A = \{ t : u_c(t) > A \} \), an open set. Then there exists \( t_0 \in S^A \) such that \( g(t) \neq 0 \) on a set of positive measure in any open neighborhood of \( t_0 \). Using \( t_0 \in S^A \) and the continuity of the inner product, there is a point \( y_0 \in \text{pr}(C) \subseteq C \), a fixed number \( \sigma > 0 \), and a fixed open neighborhood \( N_\delta(t_0) \) of \( t_0 \) such that \( (\langle y_0, t \rangle) > (A + \sigma) > 0 \) for all \( t \in N_\delta(t_0) \). Then

\[
(2.9) \quad -\langle \lambda y_0, t \rangle = -\lambda \langle y_0, t \rangle > \lambda A + \lambda \sigma > 0, \quad t \in N_\delta(t_0), \quad \lambda > 0.
\]

Since \( y_0 \in \text{pr}(C) \subseteq C \) and \( C \) is a cone then \( \lambda y_0 \in C \) for all \( \lambda > 0 \) and \( |y_0| = 1 \). Using (2.9) and then (2.8) with \( y = \lambda y_0 \) we have for all \( \lambda > 0 \) that

\[
(2.10) \quad \exp(2\pi \rho(\lambda A + \lambda \sigma)) \int_{N_\delta(t_0)} |g(t)|^p dt \leq M_{\lambda, \epsilon, \sigma}^p \exp(2\pi \rho(\lambda A + \epsilon))
\]

and hence

\[
(2.11) \quad \exp(2\pi \rho(\sigma - \epsilon)) \int_{N_\delta(t_0)} |g(t)|^p dt \leq M_{\lambda, \epsilon, \sigma}^p
\]

for all \( \epsilon > 0 \). By fixing \( \epsilon > 0 \) such that \( \sigma > \epsilon > 0 \) and letting \( \lambda \to \infty \) in (2.11) we obtain a contradiction. The conclusion follows by noting that \( S_A \) is a closed set.

3. The analytic functions. The base \( B \) of the tube \( T^B = R^n + iB \) is an open proper subset of \( R^n \) in this section.

Let \( p > 0 \) and \( A \geq 0 \). \( V_A^p = V_A^p(T^p) \) is the space of all functions \( f(z) \) which are analytic in \( z \in T^B \) and which satisfy

\[
(3.1) \quad ||f(x + iy)||_{L^p} = \left( \int_{R^n} |f(x + iy)|^p dx \right)^{1/p} \leq M_{A, f} e^{2\pi A |y|}, \quad y \in B,
\]

where the constant \( M_{A, f} \) depends on \( A \geq 0 \) and \( f \) and does not depend on \( y \in B \).

\( V^p = V^p(T^B), p > 0 \), is the space of all functions \( f(z) \) which are analytic in \( T^B \) and which satisfy

\[
(3.2) \quad ||f(x + iy)||_{L^p} = \left( \int_{R^n} |f(x + iy)|^p dx \right)^{1/p} \leq M_{t, f} e^{2\pi |y|}, y \in B,
\]

for every \( \epsilon > 0 \) where the constant \( M_{t, f} \) depends on the arbitrary \( \epsilon > 0 \) and on \( f \) and does not depend on \( y \in B \).

The spaces defined above have been motivated by the growth \[12, p. 224, (64)\] of Vladimirov; we have denoted them as \( V_A^p \) and
V_p accordingly. Notice that \( V_p = \bigcap_{\epsilon > 0} V_{\epsilon}^p, \ p > 0; \) hence \( V_p \subseteq V_2^p, \ A > 0, p > 0. \) The Hardy spaces \( H^p(T) = V_0^p(T), \ p > 0; \) satisfy \( H^p \subseteq V_p, \ p > 0; \) hence \( H^p \subseteq V_2^p, \ p > 0, \ A \geq 0. \) There are tubes \( T^p \) and values of \( p \) such that \( H^p, V^p, \) and \( V_2^p \) contain nonzero functions and such that \( V_2^p \) contains functions which are not in \( H^p \) or \( V_p. \)

4. Representations of the analytic functions. Analysis as in [11, p. 99, Lemma 2.12], the \( L^p \) Fourier transform theory, \( 1 < p \leq 2, \) and a proof similar to that in [11, pp. 100-101] yield

**Lemma 4.1.** Let \( B \) be an open connected subset of \( \mathbb{R}^n \) and \( B' \subset B \) such that \( \inf\{|y_1 - y_2|: y_1 \in B', y_2 \in B| \geq \delta \) for some \( \delta > 0. \) Let \( f(z) \in V_2^p(T^p), \ p > 0, \ A \geq 0. \) There exists a constant \( K \) which does not depend on \( z \in T^p \) such that

\[
|f(z)| \leq Ke^{2\pi |y|}, \ z = x + iy \in T^p.
\]

If \( 1 < p \leq 2, \) then

\[
e^{2\pi(y',t)}h_y(t) = e^{2\pi(y',t)h_y(t)}
\]

for all \( y \) and \( y' \) in \( B \) and for almost every \( t \in \mathbb{R}^n \) where

\[
h_y(t) = \mathcal{F}^{-1}[f(x + iy); t], \ y \in B,
\]

is the \( L^q, (1/p) + (1/q) = 1, \) inverse Fourier transform of \( f(x + iy), \ y \in B. \)

We now represent some \( V_2^p(T^p) \) spaces using Fourier-Laplace integrals.

**Theorem 4.1.** Let \( B \) be an open connected subset of \( \mathbb{R}^n. \) Let \( f(z) \in V_2^p(T^p), 1 < p \leq 2, A \geq 0. \) There exists a measurable function \( g(t), t \in \mathbb{R}^n, \) such that

\[
(e^{-2\pi(y',t)}g(t)) \in L^q, \ (1/p) + (1/q) = 1,
\]

for all \( y \in B,

\[
\int_{\mathbb{R}^n} |g(t)|^q e^{-2\pi q(y',t)} dt \leq M_{A,f} e^{2\pi A|y|}, \ y \in B,
\]

where the constant \( M_{A,f} \) depends on \( A \) and on \( f \) but not on \( z \in T^p, \) and

\[
f(z) = \int_{\mathbb{R}^n} g(t) e^{2\pi(x,t)} dt, \ z \in T^p.
\]

**Proof.** Define \( h_y(t) \) as in (4.3) and put
\( g(t) = e^{\pi i \langle y, t \rangle} h_y(t), \quad y \in B. \)

By (4.2) \( g(t) \) is independent of \( y \in B \). From (4.3) and (4.7) we have
\[
\begin{align*}
\langle 4.8 \rangle 
\end{align*}
\]

\[ e^{-2\pi \langle y, t \rangle} g(t) = \mathcal{F}^{-1}[f(x + iy); t], \quad y \in B; \]

hence (4.4) holds by the Fourier transform theory. Since \( f(x) \in V_p^\alpha(T^\alpha), 1 < p \leq 2, (1.1) \) holds for \( \mathcal{F}^{-1}[f(x + iy); t]; \) and by (4.8) and (1.1) we have
\[
\begin{align*}
\langle 4.9 \rangle 
\end{align*}
\]

\[ ||e^{-2\pi \langle y, t \rangle} g(t)||_{L^p} \leq ||f(x + iy)||_{L^p} \leq M_{A,f} e^{2\pi A |y|}, \quad y \in B, \]

from which (4.5) follows. The Fourier transform theory and (4.8) yield
\[
\begin{align*}
\langle 4.10 \rangle 
\end{align*}
\]

\[ f(z) = \mathcal{F}[e^{-2\pi \langle y, t \rangle} g(t); x], \quad z = x + iy \in T^B. \]

By Theorem 2.1 the integral on the right of (4.6) is analytic in \( T^B \) and is the \( L^1 \) Fourier transform of \( (\exp(-2\pi \langle y, t \rangle) g(t)) e^{L^1} ; y \in B. \)

(4.6) now follows by the Fourier transform theory and (4.10).

**Corollary 4.1.** Let \( C \) be an open connected cone. Let \( f(x) \in V_p^\alpha(T^\alpha), 1 < p \leq 2, A \geq 0. \) There exists a function \( g(t) \in L^q, (1/p) + (1/q) = 1, \) with \( \text{supp}(g) \subseteq \{ t : u \in C \} \) a.e. such that (4.4), (4.5), and (4.6) hold.

**Proof.** The existence of a measurable function \( g(t) \) such that (4.4), (4.5), and (4.6) hold corresponding to \( C \) follows from Theorem 4.1. Let \( k > 0 \) be arbitrary. For any \( y \in C \)
\[
\begin{align*}
\langle 4.11 \rangle 
\end{align*}
\]

\[ \int_{|t| \leq k} |g(t)|^q dt \leq \int_{|t| \leq k} |g(t)|^q e^{-2\pi q \langle y, t \rangle} e^{2\pi q |y|} dt \]
\[ \leq M_{A,f} \exp(2\pi q (A + k) |y|) \]

since \( g(t) \) satisfies (4.5). Choose \( y_k = (y_0)/(A + k), y_0 \in \text{pr}(C) \), the projection of \( C \). Then \( y_k \in C, k > 0, \) since \( C \) is a cone and \( A \geq 0. \) By (4.11) with \( y = y_k \)
\[
\begin{align*}
\langle 4.12 \rangle 
\end{align*}
\]

\[ \int_{|t| \leq k} |g(t)|^q dt \leq M_{A,f} \exp(2\pi q (A + k) |y_k|) = M_{A,f} e^{2\pi q} \]

since \( y_0 \in \text{pr}(C). \) From Theorem 4.1 \( g(t) \) is independent of \( y \in C, \) and the right side of (4.12) is independent of the arbitrary \( k > 0. \) Hence (4.12) proves \( g(t) \in L^q. \) Theorem 2.2 now yields \( \text{supp}(g) \subseteq \{ t : u \in C \} \) a.e.

The next result follows by the techniques used to prove Theorem 4.1 and Corollary 4.1 together with the facts that \( \{ t : u \in C \} \subseteq C^* \) and measure \( (C^*) = 0 \) if \( \overline{O}(C) \) contains an entire straight line [12, p. 222, Lemma 1].
Corollary 4.2. Let $C$ be an open connected cone. Let $f(z) \in V^p(T^C)$, $1 < p \leq 2$. There exists a function $g(t) \in L^q$, $(1/p) + (1/q) = 1$, with $\text{supp}(g) \subseteq C^*$ a.e. such that

$$(4.13) \quad \int_{\mathbb{R}^n} |g(t)|^q e^{-2\pi y \cdot t} dt \leq M_{\varepsilon,f} e^{2\pi q \varepsilon |y|}, \quad y \in C,$$

for every $\varepsilon > 0$ where the constant $M_{\varepsilon,f}$ depends at most on $\varepsilon$ and $f$; and (4.6) holds for $z \in T^C$. Further, if $\overline{O(C)}$ contains an entire straight line then $f(z) = 0$, $z \in T^C$.

If we assumed that $g(t) \in L^q$ in Corollary 4.2 satisfies $g(t) = \mathcal{F}^{-1}[h(\eta); t]$ for some $h \in L^2$ then we can prove

$$(4.14) \quad f(z) = \int_{\mathbb{R}^n} g(t) e^{2\pi i (z, t)} dt = \int_{\mathbb{R}^n} h(\eta) K(z - \eta) d\eta, \quad z \in T^C,$$

in Corollary 4.2. If $p = 2$ the assumption of such a function $h \in L^2$ is redundant [3].

Since $H^p(T^B) \subseteq V^p(T^B)$, $p > 0$, and $H^p(T^B) \subseteq V^q_T(T^B)$, $p > 0$, $A \geq 0$, Theorem 4.1 and Corollaries 4.1 and 4.2 hold for $f(z) \in H^p(T^B)$, $1 < p \leq 2$.

Corollary 4.3. Let $C$ be an open connected cone. We have $V^\varepsilon(T^C) = H^\varepsilon(T^C)$.

Proof. Given $f(z) \in V^\varepsilon(T^C)$, Corollary 4.2 yields $g(t) \in L^2$ with $\text{supp}(g) \subseteq C^*$ a.e. such that (4.13) and (4.6) hold. The Parseval equality (1.1) for $p = 2$ yields

$$\|f(x + iy)\|_{L^2} = \|g(t) e^{-2\pi(y, t)}\|_{L^2} \leq \|g\|_{L^2};$$

hence $f(z) \in H^\varepsilon(T^C)$. The proof is complete since $H^p(T^C) \subseteq V^p(T^C)$, $p > 0$.

The proof of the preceding corollary combined with the representation [12, p. 225, (67)] and the properties obtained for $g(t)$ there show that the analytic functions of Vladimirov in [12, §§25.3-25.4] are exactly the $H^\varepsilon(T^C) = V^\varepsilon(T^C)$ functions.

5. Converse and dual theorems. We now prove a dual result to Theorem 4.1.

Theorem 5.1. Let $B$ be an open connected subset of $\mathbb{R}^n$. Let $1 < p \leq 2$ and $A \geq 0$. Let $g(t)$ be a measurable function on $\mathbb{R}^n$ which satisfies (2.6). Then the function $F(z)$, $z \in T^B$, defined by (2.7) is an element of $V^\lambda_T(T^B)$, $(1/p) + (1/q) = 1$. 

Proof. $F(z)$ is analytic in $T^g$ by Theorem 2.1, which also implies $(\exp(-2\pi<y, t>)g(t)) \in L^r$, $y \in B$; and by (2.6) this function is in $L^p$ also, $y \in B$. Thus (1.1) and (2.6) yield

$$||F(x + iy)||_{L^q} \leq ||e^{-2\pi<y, t>}g(t)||_{L^p} \leq M_A\rho e^{2\pi A|y|}, \quad y \in B,$$

and $F(z) \in V_4^g(T^g)$ as desired.

**Corollary 5.1.** Let $C$ be an open connected cone. Let $1 < p \leq 2$ and $A \geq 0$. Let $g(t)$ be a measurable function on $R^n$ which satisfies (2.6) for every $y \in C$. Then $g(t) \in L^p$, $\text{supp}(g) \subseteq \{t: u_0(t) \leq A\}$ a.e., and the function $F(z), z \in T^c$, defined by (2.7) is an element of $V_4^g(T^c)$, $(1/p) + (1/q) = 1$.

**Proof.** Theorem 5.1, the proof of Corollary 4.1, and Theorem 2.2 yield the results.

If $p = 2$, Theorem 5.1 and Corollary 5.1 are converses of Theorem 4.1 and Corollary 4.1, respectively. Similarly the next corollary is a converse of Corollaries 4.2 and 4.3 together with (4.14) for $p = 2$.

**Corollary 5.2.** Let $C$ be an open connected cone. Let $1 < p \leq 2$. Let $g(t)$ be a measurable function on $R^n$ such that (4.13) holds with $q$ replaced by $p$ and $M_{i,r}$ replaced by $M_{i,q}$. Then $g(t) \in L^p$, $\text{supp}(g) \subseteq C^*$ a.e.; the function $F(z), z \in T^c$, defined by (2.7) is an element of $H^q(T^c)$, $(1/p) + (1/q) = 1$; and there exists a function $h \in L^q$ such that $F(x + iy) \to h(x)$ in $L^q$ as $y \to 0$, $y \in C$, with this boundary value being obtained independently of how $y \to 0$, $y \in C$. Further, if $p = 2$ then $F(z)$ has the representation (4.14); and if $\overline{O(C)}$ contains an entire straight line then $F(z) = 0, z \in T^c$.

**Proof.** Because of previous analysis the only new idea is the boundary value property. Since $g \in L^p$ there exists $h \in L^q$ such that $h(x) = \mathcal{F}[g(t); x]$ in $L^q$. Then $(F(x + iy) - h(x)) = \mathcal{F}[(\exp(-2\pi<y, t>)g(t)) - g(t); x]$ in $L^q, y \in C$. Using (1.1) and the Lebesgue dominated convergence theorem the proof is completed.

6. Generalized Cauchy and Poisson integrals. Throughout this section $C$ is an open connected cone such that $\overline{O(C)}$ does not contain an entire straight line.

Let $U \in \mathcal{D}'_{lp}, 1 \leq p \leq 2$. By Lemma 2.1, the generalized Cauchy integral of $U$

$$C(U; z) = \langle U, K(z - t)\rangle, \ z \in T^{0(C)},$$

is a well defined function of $z \in T^{0(C)}$. 

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Using similar proofs we see that [6, Lemma 4] holds for \( p = 1 \), and the convergence in [6, Lemma 5] holds in the topology of \( \mathcal{D}' \). The analysis used to prove [6, Theorems 2, 9, and 10] can be adapted where necessary to show that these results hold also for \( p = 1 \), and we have the following extension of these results.

**Theorem 6.1.** Let \( U \in \mathcal{D}'_{L^p}, 1 \leq p \leq 2 \), and let \( C \) be an open connected cone. \( C(U; z) \) is an analytic function of \( z \in T^{0}(C) \) which satisfies [6, p. 202, (8)] for \( z \in T_c', \) \( C' \) being any compact subcone of \( O(C) \). For any \( \phi \in \mathcal{S}' \) we have

\[
\lim_{y \to 0} \left< C(U; x + iy), \phi(x) \right> = \left< \mathcal{F}(I_C(\eta)) \mathcal{F}^{-1}[U], \phi(x) \right> 
\]

with the transforms being in the \( \mathcal{S}' \) sense. If \( U = \hat{V} \) where \( V \in \mathcal{S}' \) with \( \text{supp}(V) \subseteq C^* \), then \( V = \sum_{|a| \leq m} t^a h_a(t), h_a(t) \in L^2, (1/p) + (1/q) = 1, \) for some nonnegative integer \( m \); we have

\[
C(U; z) = \left< V, e^{2\pi i(x, t)} \right>, \quad z \in T^{0}(C),
\]

as elements of \( \mathcal{S}' \); and

\[
\lim_{y \to 0} \left< C(U; x + iy), \phi(x) \right> = \left< U, \phi \right>, \quad \phi \in \mathcal{S}.
\]

[6, Corollary 1, Theorems 11, 12, and 15] hold for \( p = 1 \) also. [6, Theorem 16] can now be extended to include \( p = 1 \) and to conclude the analyticity of \( C(U; z) \) in \( T^{0}(C) \), the growth [6, p. 202, (8)] for \( z \in T_c', \) \( C' \subset O(C) \), and the convergence (6.2) in each of the connected components \( O(C_\lambda), \lambda \in \Lambda \). The restriction of \( z \in T^{0}(C) \setminus \{z : y = \text{Im}(z) \in O(C), y_j = 0 \text{ for any } j = 1, \cdots, n\} \) in [6, Theorem 16] is unnecessary.

Now let \( U \in \mathcal{D}'_{L^p}, 1 \leq p \leq \infty \), and \( C \) be an open connected cone. By Lemma 2.2 the generalized Poisson integral of \( U \)

\[
P(U; z) = \left< U, Q(z; t) \right>, \quad z \in T^{0}(C),
\]

is a well defined function of \( z \in T^{0}(C) \). In general \( P(U; z) \) is not analytic. However, if \( z \) is in a generalized half plane in \( C^* \) then \( P(U; z) \) is \( n \)-harmonic by a proof as in [5, Theorem 7].

We now extend and generalize slightly [6, Lemma 8]. The proof is the same for all \( p, 1 \leq p \leq \infty \), and for \( \phi \in \mathcal{D}_{L^{1}} \) as that indicated for [6, Lemma 8].

**Lemma 6.1.** Let \( U \in \mathcal{D}'_{L^p}, 1 \leq p \leq \infty \), and \( z \in T^{0}(C), C \) being an open connected cone. For \( y \in O(C) \) we have
\[ \langle P(U); x + iy \rangle, \phi(x) \rangle = \langle U, \langle Q(x + iy; t), \phi(x) \rangle \rangle, \phi \in \mathcal{D}_{L^1}. \]

**Lemma 6.2.** Let \( C \) be an open connected cone and \( z = x + iy \in T^\circ (C) \). We have
\[ \lim_{y \to 0} \int_{\mathbb{R}^n} Q(x + iy; t)\phi(x) dx = \phi(t), \quad \phi \in \mathcal{D}_{L^1} \]
in the topology of \( \mathcal{D}_{L^q} \) for all \( q, 1 \leq q \leq \infty \), and in the topology of \( \mathcal{B} \).

**Proof.** For \( y \in \mathcal{O}(C) \) and any \( n \)-tuple \( \alpha \) of nonnegative integers
\[ D^\alpha_t (Q(x + iy; t), \phi(x)) = \int_{\mathbb{R}^n} D^\alpha_t (\phi(x + t))Q(x; y) dx, \quad \phi \in \mathcal{D}_{L^2}, \]
where \( Q(x; y) \) is defined in (2.5). \( \phi \in \mathcal{D}_{L^1} \) implies \( \psi^\alpha(t) = D^\alpha_t (\phi(t)) \in \mathcal{D}_{L^1} \subseteq \mathcal{D}_{L^q} \) for all \( q, 1 \leq q \leq \infty \). Using [6, Lemma 6, (50)], (6.8), and the analysis of [6, p. 214, (55)] and [6, Lemma 7] we have for any \( q, 1 \leq q < \infty \),
\[ \lim_{y \to 0} \left\| D^\alpha_t \left( \int_{\mathbb{R}^n} Q(x + iy; t)\phi(x) dx - D^\alpha_t (\phi(t)) \right) \right\|_{L^q} = 0 \]
which proves (6.7) in the topology of \( \mathcal{D}_{L^q} \) for all \( q, 1 \leq q < \infty \). Now \( \phi \in \mathcal{D}_{L^1} \subseteq \mathcal{B} \subseteq \mathcal{D}_{L^\infty} \) implies \( \psi^\alpha(t) = D^\alpha_t (\phi(t)) \in \mathcal{D}_{L^1} \subseteq \mathcal{B} \subseteq \mathcal{D}_{L^\infty} \). The definition of \( \mathcal{B} \) implies that \( \psi^\alpha(t) \) is uniformly continuous and bounded on \( \mathbb{R}^n \); hence the proof of [9, Proposition 3, (b)] yields
\[ \lim_{y \to 0} \int_{\mathbb{R}^n} \psi^\alpha(x + t)Q(x; y) dx = \psi^\alpha(t) \]
uniformly for \( t \in \mathbb{R}^n \). Because of this, (6.9) holds also for \( q = \infty \) which proves (6.7) in the topology of \( \mathcal{B} \) and in the topology of \( \mathcal{D}_{L^\infty} = \mathcal{B} \).

We now extend and generalize [6, Theorem 14].

**Theorem 6.2.** Let \( U \in \mathcal{D}'_{L^p}, 1 \leq p \leq \infty \). Let \( C \) be an open connected cone and \( z = x + iy \in T^\circ (C) \). We have
\[ \lim_{y \to 0} \langle P(U); x + iy \rangle, \phi(x) \rangle = \langle U, \phi \rangle, \quad \phi \in \mathcal{D}_{L^1}. \]

**Proof.** The proof follows by (6.6), (6.7), and the continuity of \( U \).
Using Theorem 6.2, [6, Theorem 17] can be extended and gen-
eralized for $U \in \mathcal{D}'_{L^p}$, $1 \leq p \leq \infty$, where $O(C)$ contains no entire straight line. One concludes the existence of $P(U; z)$, $z \in T^{0}(C)$, and the convergence (6.10) as $y \to 0$, $y \in O(C)$, $\lambda \in \Lambda$. The restriction of $z \in T^{0}(C) \setminus \{z: y = \text{Im}(z) \in O(C), y_j = 0 \text{ for any } j = 1, \ldots, n\}$ in [6, Theorem 17] is unnecessary.

7. Acknowledgment. One of us, R. D.C., expresses his ap-
preciation to the Department of Mathematics of Iowa State Uni-
versity for the opportunity of serving as Visiting Associate Pro-
fessor during 1978–1979. The authors thank the referee for his
comments and suggestions concerning this paper.

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Received October 19, 1978 and in revised form June 22, 1979.

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