Pacific Journal of Mathematics

ANALYTIC FUNCTIONS IN TUBES WHICH ARE REPRESENTABLE BY FOURIER-LAPLACE INTEGRALS

RICHARD D. CARMICHAEL AND ELMER KINJI HAYASHI

Vol. 90, No. 1 September 1980

PACIFIC JOURNAL OF MATHEMATICS
Vol. 90, No. 1, 1980

ANALYTIC FUNCTIONS IN TUBES WHICH ARE REPRESENTABLE BY FOURIER-LAPLACE INTEGRALS

RICHARD D. CARMICHAEL AND ELMER K. HAYASHI

Spaces of analytic functions in tubes in C^n which generalize the Hardy H^p spaces are defined and studied. In addition Cauchy and Poisson integrals of distributions in $\mathscr{D}'_{L^p}$ are analyzed.

1. Introduction. Bochner ([1] and [2]) has defined the Hardy $H^2(T^C)$ spaces for tubes $T^C = \boldsymbol{R}^n + iC$ in $\boldsymbol{C}^n$ where $C \subset \boldsymbol{R}^n$ is an open convex cone. Stein and Weiss [11] have studied the $H^p(T^B)$ spaces for arbitrary $p > 0$ and with respect to tubes T^B, B being an open proper subset of $\boldsymbol{R}^n$ [11, pp. 90-91]. Vladimirov [12, §§ 25.3-25.4] has considered analytic functions in T^C, C being an open connected cone, which satisfy the growth [12, p. 224, (64)]. Vladimirov has stated [12, p. 227, lines 4-5] that the growth which defines the H^2 functions of Bochner is more restrictive than [12, p. 224, (64)]. We show in this paper that the H^2 growth is not more restrictive than [12, p. 224, (64)] by showing that the functions of Vladimirov are exactly the H^2 functions. However, Vladimirov's growth has led us to define new spaces of analytic functions in tubes which have growth estimates that are more general than that of the $H^p(T^B)$ spaces, and we analyze these new spaces in this paper. Further, we study Cauchy and Poisson integrals of distributions in $\mathscr{D}'_{L^p}$.

The n-dimensional notation in this paper is described in [7, p. 386]. The definitions of a cone in $\boldsymbol{R}^n$, projection of a cone pr(C), compact subcone, and dual cone $C^* = \{t \in \boldsymbol{R}^n: \langle t, y \rangle \geqq 0, y \in C\}$ of a cone C are given in [12, p. 218]. Terminology concerning distributions is that of Schwartz [10]. The support of a distribution or function g is denoted supp(g). Definitions, properties, and relevant topologies of the function spaces $\mathscr{S}$, $\mathscr{D}_{L^p}$, $\mathscr{B} = \mathscr{D}_{L^\infty}$, and $\dot{\mathscr{B}}$ and of the distribution spaces $\mathscr{S}'$ and $\mathscr{D}'_{L^p}$ are in [10]. The L^1 and $\mathscr{S}'$ Fourier and inverse Fourier transforms are defined in [7, pp. 387-388] and [10, p. 250], respectively. The limit in the mean Fourier and inverse Fourier transforms of functions in L^p, $1 < p \leqq 2$, and L^q, $(1/p) + (1/q) = 1$, are in [8] and [3]. $\mathscr{F}[\phi(t); x]$ ($\mathscr{F}^{-1}[\phi(x); t]$) denotes the Fourier (inverse Fourier) transform of a function in the relevant sense. If $V \in \mathscr{S}'$ we denote its Fourier (inverse Fourier) transform by $\mathscr{F}[V] = \hat{V}$ ($\mathscr{F}^{-1}[V]$). For $\phi \in L^p$, $1 < p \leqq 2$, the Parseval inequality is

$$(1.1)\qquad \|\mathscr{F}[\phi(t); x]\|_{L^q} \leqq \|\phi\|_{L^p}, \quad (1/p) + (1/q) = 1\,,$$

with equality if $p = 2$, the Parseval equality.

2. The Cauchy and Poisson kernel functions and technical results. Let C be an open connected cone, C^* be the dual cone of C, and $0(C)$ be the convex envelope (hull) of C. The Cauchy kernel function [6, p. 201] is

$$(2.1)\quad K(z-t) = \int_{C^*} \exp(2\pi i\langle z-t, \eta\rangle)d\eta,\ z \in T^{O(C)} = \boldsymbol{R}^n + iO(C),\ t \in \boldsymbol{R}^n\,.$$

To avoid the triviality of $K(z-t) = 0$ we assume in this section that $\overline{O(C)}$ does not contain an entire straight line [12, p. 222, Lemma 1]. In [6, Theorem 1] one of us proved $K(z-t) \in \mathscr{D}_{L^q}$ for all q, $(1/p) + (1/q) = 1$, $1 < p \leqq 2$, as a function of $t \in \boldsymbol{R}^n$ for fixed $z \in T^{O(C)}$. But $\mathscr{D}_{L^q} \subset \dot{\mathscr{B}} \subset \mathscr{D}_{L^\infty}$ for every q, $1 \leqq q < \infty$, by [10, pp. 199-200]. We thus have

LEMMA 2.1. *Let $z \in T^{O(C)}$. As a function of $t \in \boldsymbol{R}^n$,*

$$(2.2)\quad K(z-t) \in \dot{\mathscr{B}} \cap \mathscr{D}_{L^q}\ \textit{for all}\ q,\ (1/p) + (1/q) = 1,\ 1 \leqq p \leqq 2\,.$$

For an open connected cone C the Poisson kernel function [6, p. 204] is

$$(2.3)\quad Q(z; t) = \frac{K(z-t)\overline{K(z-t)}}{K(2iy)}\,, \quad z = x + iy \in T^{O(C)},\quad t \in \boldsymbol{R}^n\,.$$

LEMMA 2.2. *$Q(z; t) \in \dot{\mathscr{B}} \cap \mathscr{D}_{L^q}$ for all q, $1 \leqq q \leqq \infty$, as a function of $t \in \boldsymbol{R}^n$ for arbitrary $z \in T^{O(C)}$.*

Proof. Let α be any n-tuple of nonnegative integers. By the Leibnitz rule

$$(2.4)\quad D_t^\alpha(Q(z; t)) = \frac{1}{K(2iy)} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D_t^\beta(K(z-t)) D_t^\gamma(\overline{K(z-t)})\,,$$

$$z = x + iy \in T^{O(C)}\,.$$

By (2.2)$D_t^\beta(K(z-t))$ and $D_t^\gamma(\overline{K(z-t)})$ are in $L^2 \cap L^\infty$ as functions of $t \in \boldsymbol{R}^n$. Thus $D_t^\alpha(Q(z; t)) \in L^1 \cap L^\infty \subseteq L^q$, $1 \leqq q \leqq \infty$. Hence $Q(z; t) \in \mathscr{D}_{L^q}$ $1 \leqq q \leqq \infty$; and $Q(z; t) \in \dot{\mathscr{B}}$ also since $\mathscr{D}_{L^q} \subset \dot{\mathscr{B}}$, $1 \leqq q < \infty$.

As a function of $x = \operatorname{Re}(z) \in \boldsymbol{R}^n$ for $y \in O(C)$ arbitrary we also have

$$(2.5)\quad Q(x; y) = \frac{K(x+iy)\overline{K(x+iy)}}{K(2iy)} \in \dot{\mathscr{B}} \cap \mathscr{D}_{L^q}\ \text{for all}\ q,\ 1 \leqq q \leqq \infty\,.$$

We conclude this section with two important and useful theorems.

THEOREM 2.1. *Let B be an open connected subset of $\boldsymbol{R}^n$. Let $1 \leqq p < \infty$ and $A \geqq 0$. Let $g(t)$ be a measurable function on $\boldsymbol{R}^n$ which satisfies*

$$\int_{R^n} |g(t)|^p e^{-2\pi p\langle y,t\rangle} dt \leqq M_{A,g}^p e^{2\pi pA|y|} , \quad y \in B , \tag{2.6}$$

where the constant $M_{A,g}$ depends only on A and $g(t)$ and not on $y \in B$. Then

$$F(z) = \int_{R^n} g(t) e^{2\pi i\langle z,t\rangle} dt, \; z \in T^B , \tag{2.7}$$

is an analytic function of $z \in T^B$ and has an analytic extension to $T^{O(B)}$.

Proof. For arbitrary $y_0 \in B$ there is an open neighborhood of y_0, $N(y_0) \subset B$, and a $\delta > 0$ such that $\{y: |y - y_0| = \delta\} \subset N(y_0)$. There are k cones Γ_j, $j = 1, \cdots, k$, having the properties as in [11, p. 92, lines 12–15] and such that whenever two points v and w are in a Γ_j then $\langle v, w\rangle \geqq (\sqrt{2}/2)|v||w|$. For each $j = 1, \cdots, k$ choose y_j such that $(y_0 - y_j) \in \Gamma_j$ and $|y_j - y_0| = \delta$. Then for each p, $1 \leqq p < \infty$, and all $t \in \Gamma_j$, $j = 1, \cdots, k$, we have $(-2\pi p\langle y_j - y_0, t\rangle) \geqq \varepsilon |t|$ where $\varepsilon = \sqrt{2}\pi p\delta > 0$. Using this fact, (2.6), and analysis as in [11, pp. 92–93] we have that the function

$$G(t) = g(t) \exp(\varepsilon|t|/2p) \exp(-2\pi\langle y_0, t\rangle), \; t \in \boldsymbol{R}^n, \; 1 \leqq p < \infty ,$$

is an L^1 function. If $y = \operatorname{Im}(z)$ is restricted so that $|y - y_0| < (\varepsilon/4\pi p)$ then

$$|g(t)e^{2\pi i\langle z,t\rangle}| \leqq |G(t)| , \quad t \in \boldsymbol{R}^n , \quad x = \operatorname{Re}(z) \in \boldsymbol{R}^n .$$

Since $y_0 \in B$ was arbitrary it follows that $F(z)$ is analytic in T^B and has an analytic extension to $T^{O(B)}$ by [4, p. 92, Theorem 9].

Note the indicatrix function $u_C(t)$ of a cone C defined in [12, p. 219]. $\overline{O(C)}$ may or may not contain an entire straight line in the next theorem.

THEOREM 2.2. *Let C be any open connected cone and $A \geqq 0$. Let $g(t) \in L^p$, $1 \leqq p < \infty$, such that*

$$\int_{R^n} |g(t)|^p e^{-2\pi p\langle y,t\rangle} dt \leqq M_{A,\varepsilon,g}^p \exp(2\pi p(A + \varepsilon)|y|) , \quad y \in C , \tag{2.8}$$

for all $\varepsilon > 0$ where the constant $M_{A,\varepsilon,g}$ depends on A, ε, and $g(t)$

and not on $y \in C$. *Then* $\operatorname{supp}(g) \subseteq S_A = \{t: u_C(t) \leqq A\}$ *almost everywhere* (a.e.).

Proof. Assume $g(t) \neq 0$ on a set of positive measure in $S^A = R^n \backslash S_A = \{t: u_C(t) > A\}$, an open set. Then there exists $t_0 \in S^A$ such that $g(t) \neq 0$ on a set of positive measure in any open neighborhood of t_0. Using $t_0 \in S^A$ and the continuity of the inner product, there is a point $y_0 \in \operatorname{pr}(C) \subset C$, a fixed number $\sigma > 0$, and a fixed open neighborhood $N_\eta(t_0)$ of t_0 such that $(-\langle y_0, t\rangle) > (A + \sigma) > 0$ for all $t \in N_\eta(t_0)$. Then

$$(2.9) \qquad -\langle \lambda y_0, t\rangle = -\lambda\langle y_0, t\rangle > \lambda A + \lambda\sigma > 0\ , \quad t \in N_\eta(t_0)\ , \quad \lambda > 0\ .$$

Since $y_0 \in \operatorname{pr}(C) \subset C$ and C is a cone then $\lambda y_0 \in C$ for all $\lambda > 0$ and $|y_0| = 1$. Using (2.9) and then (2.8) with $y = \lambda y_0$ we have for all $\lambda > 0$ that

$$(2.10)\ \exp(2\pi p(\lambda A + \lambda\sigma)) \int_{N_\eta(t_0)} |g(t)|^p dt \leqq M^p_{A,\varepsilon,g} \exp(2\pi p\lambda(A + \varepsilon))$$

and hence

$$(2.11) \qquad \exp(2\pi p\lambda(\sigma - \varepsilon)) \int_{N_\eta(t_0)} |g(t)|^p dt \leqq M^p_{A,\varepsilon,g}$$

for all $\varepsilon > 0$. By fixing $\varepsilon > 0$ such that $\sigma > \varepsilon > 0$ and letting $\lambda \to \infty$ in (2.11) we obtain a contradiction. The conclusion follows by noting that S_A is a closed set.

3. The analytic functions.

The base B of the tube $T^B = R^n + iB$ is an open proper subset of R^n in this section.

Let $p > 0$ and $A \geqq 0$. $V^p_A = V^p_A(T^B)$ is the space of all functions $f(z)$ which are analytic in $z \in T^B$ and which satisfy

$$(3.1)\ \ \|f(x + iy)\|_{L^p} = \left(\int_{R^n} |f(x + iy)|^p dx\right)^{1/p} \leqq M_{A,f} e^{2\pi A|y|}\ , \quad y \in B\ ,$$

where the constant $M_{A,f}$ depends on $A \geqq 0$ and f and does not depend on $y \in B$.

$V^p = V^p(T^B)$, $p > 0$, is the space of all functions $f(z)$ which are analytic in T^B and which satisfy

$$(3.2) \quad \|f(x + iy)\|_{L^p} = \left(\int_{R^n} |f(x + iy)|^p dx\right)^{1/p} \leqq M_{\varepsilon,f} e^{2\pi\varepsilon|y|}, y \in B\ ,$$

for every $\varepsilon > 0$ where the constant $M_{\varepsilon,f}$ depends on the arbitrary $\varepsilon > 0$ and on f and does not depend on $y \in B$.

The spaces defined above have been motivated by the growth [12, p. 224, (64)] of Vladimirov; we have denoted them as V^p_A and

V^p accordingly. Notice that $V^p = \bigcap_{\varepsilon>0} V^p_\varepsilon$, $p > 0$; hence $V^p \subseteqq V^p_A$, $A > 0$, $p > 0$. The Hardy spaces $H^p(T^B) = V^p_0(T^B)$, $p > 0$, [11, pp. 90–91] satisfy $H^p \subseteqq V^p$, $p > 0$; hence $H^p \subseteqq V^p_A$, $p > 0$, $A \geqq 0$. There are tubes T^B and values of p such that H^p, V^p, and V^p_A contain nonzero functions and such that V^p_A contains functions which are not in H^p or V^p.

4. Representations of the analytic functions. Analysis as in [11, p. 99, Lemma 2.12], the L^p Fourier transform theory, $1 < p \leqq 2$, and a proof similar to that in [11, pp. 100–101] yield

LEMMA 4.1. *Let B be an open connected subset of $\boldsymbol{R}^n$ and $B' \subset B$ such that $\inf\{|y_1 - y_2|: y_1 \in B', y_2 \in B\} \geqq \delta$ for some $\delta > 0$. Let $f(z) \in V^p_A(T^B)$, $p > 0$, $A \geqq 0$. There exists a constant K which does not depend on $z \in T^{B'}$ such that*

$$|f(z)| \leqq Ke^{2\pi A|y|}, \quad z = x + iy \in T^{B'} . \tag{4.1}$$

If $1 < p \leqq 2$, then

$$e^{2\pi\langle y,t\rangle}h_y(t) = e^{2\pi\langle y',t\rangle}h_{y'}(t) \tag{4.2}$$

for all y and y' in B and for almost every $t \in \boldsymbol{R}^n$ where

$$h_y(t) = \mathscr{F}^{-1}[f(x + iy); t], \quad y \in B , \tag{4.3}$$

is the L^q, $(1/p) + (1/q) = 1$, inverse Fourier transform of $f(x + iy)$, $y \in B$.

We now represent some $V^p_A(T^B)$ spaces using Fourier-Laplace integrals.

THEOREM 4.1. *Let B be an open connected subset of $\boldsymbol{R}^n$. Let $f(z) \in V^p_A(T^B)$, $1 < p \leqq 2$, $A \geqq 0$. There exists a measurable function $g(t)$, $t \in \boldsymbol{R}^n$, such that*

$$(e^{-2\pi\langle y,t\rangle}g(t)) \in L^q , \quad (1/p) + (1/q) = 1 , \tag{4.4}$$

for all $y \in B$,

$$\int_{R^n} |g(t)|^q e^{-2\pi q\langle y,t\rangle}dt \leqq M^q_{A,f}e^{2\pi qA|y|} , \quad y \in B , \tag{4.5}$$

where the constant $M_{A,f}$ depends on A and on f but not on $z \in T^B$, and

$$f(z) = \int_{R^n} g(t)e^{2\pi i\langle z,t\rangle}dt , \quad z \in T^B . \tag{4.6}$$

Proof. Define $h_y(t)$ as in (4.3) and put

$$g(t) = e^{2\pi\langle y,t\rangle}h_y(t)\,, \quad y \in B\,. \tag{4.7}$$

By (4.2) $g(t)$ is independent of $y \in B$. From (4.3) and (4.7) we have

$$e^{-2\pi\langle y,t\rangle}g(t) = \mathscr{F}^{-1}[f(x+iy); t]\,, \quad y \in B\,; \tag{4.8}$$

hence (4.4) holds by the Fourier transform theory. Since $f(z) \in V_A^p(T^B)$, $1 < p \leqq 2$, (1.1) holds for $\mathscr{F}^{-1}[f(x+iy); t]$; and by (4.8) and (1.1) we have

$$||e^{-2\pi\langle y,t\rangle}g(t)||_{L^q} \leqq ||f(x+iy)||_{L^p} \leqq M_{A,f}e^{2\pi A|y|}\,, \quad y \in B\,, \tag{4.9}$$

from which (4.5) follows. The Fourier transform theory and (4.8) yield

$$f(z) = \mathscr{F}[e^{-2\pi\langle y,t\rangle}g(t); x]\,, \quad z = x+iy \in T^B\,. \tag{4.10}$$

By Theorem 2.1 the integral on the right of (4.6) is analytic in T^B and is the L^1 Fourier transform of $(\exp(-2\pi\langle y, t\rangle)g(t)) \in L^1$, $y \in B$. (4.6) now follows by the Fourier transform theory and (4.10).

COROLLARY 4.1. *Let C be an open connected cone. Let* $f(z) \in V_A^p(T^C)$, $1 < p \leqq 2$, $A \geqq 0$. *There exists a function* $g(t) \in L^q$, $(1/p)+(1/q) = 1$, *with* $\operatorname{supp}(g) \subseteqq \{t: u_C(t) \leqq A\}$ a.e. *such that* (4.4), (4.5), *and* (4.6) *hold.*

Proof. The existence of a measurable function $g(t)$ such that (4.4), (4.5), and (4.6) hold corresponding to C follows from Theorem 4.1. Let $k > 0$ be arbitrary. For any $y \in C$

$$\begin{aligned}\int_{|t|\leqq k} |g(t)|^q dt &\leqq \int_{|t|\leqq k} |g(t)|^q e^{-2\pi q\langle y,t\rangle} e^{2\pi q|y||t|}dt \\ &\leqq M_{A,f}^q \exp(2\pi q(A+k)|y|)\end{aligned} \tag{4.11}$$

since $g(t)$ satisfies (4.5). Choose $y_k = (y_0)/(A+k)$, $y_0 \in \operatorname{pr}(C)$, the projection of C. Then $y_k \in C$, $k > 0$, since C is a cone and $A \geqq 0$. By (4.11) with $y = y_k$

$$\int_{|t|\leqq k} |g(t)|^q dt \leqq M_{A,f}^q \exp(2\pi q(A+k)|y_k|) = M_{A,f}^q e^{2\pi q} \tag{4.12}$$

since $y_0 \in \operatorname{pr}(C)$. From Theorem 4.1 $g(t)$ is independent of $y \in C$, and the right side of (4.12) is independent of the arbitrary $k > 0$. Hence (4.12) proves $g(t) \in L^q$. Theorem 2.2 now yields $\operatorname{supp}(g) \subseteqq \{t: u_C(t) \leqq A\}$ a.e.

The next result follows by the techniques used to prove Theorem 4.1 and Corollary 4.1 together with the facts that $\{t: u_C(t) \leqq 0\} = C^*$ and measure $(C^*) = 0$ if $\overline{O(C)}$ contains an entire straight line [12, p. 222, Lemma 1].

COROLLARY 4.2. *Let C be an open connected cone. Let $f(z) \in V^p(T^C)$, $1 < p \leqq 2$. There exists a function $g(t) \in L^q$, $(1/p) + (1/q) = 1$, with* $\mathrm{supp}(g) \subseteq C^*$ a.e. *such that*

$$(4.13) \qquad \int_{R^n} |g(t)|^q e^{-2\pi q \langle y, t \rangle} dt \leqq M_{\varepsilon,f}^q e^{2\pi q \varepsilon |y|}, \quad y \in C,$$

for every $\varepsilon > 0$ where the constant $M_{\varepsilon,f}$ depends at most on ε and f; and (4.6) *holds for $z \in T^C$. Further, if $\overline{O(C)}$ contains an entire straight line then $f(z) = 0$, $z \in T^C$.*

If we assumed that $g(t) \in L^q$ in Corollary 4.2 satisfies $g(t) = \mathscr{F}^{-1}[h(\eta); t]$ for some $h \in L^p$ then we can prove

$$(4.14) \qquad f(z) = \int_{R^n} g(t) e^{2\pi i \langle z, t \rangle} dt = \int_{R^n} h(\eta) K(z - \eta) d\eta, \quad z \in T^C,$$

in Corollary 4.2. If $p = 2$ the assumption of such a function $h \in L^2$ is redundant [3].

Since $H^p(T^B) \subseteq V^p(T^B)$, $p > 0$, and $H^p(T^B) \subseteq V_A^p(T^B)$, $p > 0$, $A \geqq 0$, Theorem 4.1 and Corollaries 4.1 and 4.2 hold for $f(z) \in H^p(T^B)$, $1 < p \leqq 2$.

COROLLARY 4.3. *Let C be an open connected cone. We have $V^2(T^C) = H^2(T^C)$.*

Proof. Given $f(z) \in V^2(T^C)$, Corollary 4.2 yields $g(t) \in L^2$ with $\mathrm{supp}(g) \subseteq C^*$ a.e. such that (4.13) and (4.6) hold. The Parseval equality (1.1) for $p = 2$ yields

$$||f(x + iy)||_{L^2} = ||g(t)e^{-2\pi \langle y, t \rangle}||_{L^2} \leqq ||g||_{L^2};$$

hence $f(z) \in H^2(T^C)$. The proof is complete since $H^p(T^C) \subseteq V^p(T^C)$, $p > 0$.

The proof of the preceding corollary combined with the representation [12, p. 225, (67)] and the properties obtained for $g(t)$ there show that the analytic functions of Vladimirov in [12, §§ 25.3–25.4] are exactly the $H^2(T^C) = V^2(T^C)$ functions.

5. Converse and dual theorems.

We now prove a dual result to Theorem 4.1.

THEOREM 5.1. *Let B be an open connected subset of $\boldsymbol{R}^n$. Let $1 < p \leqq 2$ and $A \geqq 0$. Let $g(t)$ be a measurable function on $\boldsymbol{R}^n$ which satisfies* (2.6). *Then the function $F(z)$, $z \in T^B$, defined by* (2.7) *is an element of $V_A^q(T^B)$, $(1/p) + (1/q) = 1$.*

Proof. $F(z)$ is analytic in T^B by Theorem 2.1, which also implies $(\exp(-2\pi\langle y, t\rangle)g(t)) \in L^1$, $y \in B$; and by (2.6) this function is in L^p also, $y \in B$. Thus (1.1) and (2.6) yield

$$||F(x + iy)||_{L^q} \leqq ||e^{-2\pi\langle y,t\rangle}g(t)||_{L^p} \leqq M_{A,g}e^{2\pi A|y|}\ , \quad y \in B\ ,$$

and $F(z) \in V_A^q(T^B)$ as desired.

COROLLARY 5.1. *Let C be an open connected cone. Let $1<p\leqq 2$ and $A \geqq 0$. Let $g(t)$ be a measurable function on $\boldsymbol{R}^n$ which satisfies* (2.6) *for every $y \in C$. Then $g(t) \in L^p$,* $\text{supp}(g)\subseteq\{t: u_C(t) \leqq A\}$ a.e., *and the function $F(z)$, $z \in T^C$, defined by* (2.7) *is an element of $V_A^q(T^C)$,* $(1/p) + (1/q) = 1$.

Proof. Theorem 5.1, the proof of Corollary 4.1, and Theorem 2.2 yield the results.

If $p = 2$, Theorem 5.1 and Corollary 5.1 are converses of Theorem 4.1 and Corollary 4.1, respectively. Similarly the next corollary is a converse of Corollaries 4.2 and 4.3 together with (4.14) for $p = 2$.

COROLLARY 5.2. *Let C be an open connected cone. Let $1<p\leqq 2$. Let $g(t)$ be a measurable function on $\boldsymbol{R}^n$ such that* (4.13) *holds with q replaced by p and $M_{\varepsilon,f}$ replaced by $M_{\varepsilon,g}$. Then $g(t) \in L^p$;* $\text{supp}(g) \subseteq C^*$ a.e.; *the function $F(z)$, $z \in T^C$, defined by* (2.7) *is an element of $H^q(T^C)$,* $(1/p) + (1/q) = 1$; *and there exists a function $h \in L^q$ such that $F(x + iy) \to h(x)$ in L^q as $y \to 0$, $y \in C$, with this boundary value being obtained independently of how $y \to 0$, $y \in C$. Further, if $p = 2$ then $F(z)$ has the representation* (4.14); *and if $\overline{O(C)}$ contains an entire straight line then* $F(z) = 0$, $z \in T^C$.

Proof. Because of previous analysis the only new idea is the boundary value property. Since $g \in L^p$ there exists $h \in L^q$ such that $h(x) = \mathscr{F}[g(t); x]$ in L^q. Then $(F(x + iy) - h(x)) = \mathscr{F}[(\exp(-2\pi\langle y, t\rangle) g(t)) - g(t); x]$ in L^q, $y \in C$. Using (1.1) and the Lebesgue dominated convergence theorem the proof is completed.

6. Generalized Cauchy and Poisson integrals.

Throughout this section C is an open connected cone such that $\overline{O(C)}$ does not contain an entire straight line.

Let $U \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq 2$. By Lemma 2.1, the generalized Cauchy integral of U

$$C(U; z) = \langle U, K(z - t)\rangle,\ z \in T^{O(C)}\ , \tag{6.1}$$

is a well defined function of $z \in T^{O(C)}$.

Using similar proofs we see that [6, Lemma 4] holds for $p = 1$, and the convergence in [6, Lemma 5] holds in the topology of $\dot{\mathscr{B}}$. The analysis used to prove [6, Theorems 2, 9, and 10] can be adapted where necessary to show that these results hold also for $p = 1$, and we have the following extension of these results.

THEOREM 6.1. *Let* $U \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq 2$, *and let* C *be an open connected cone.* $C(U; z)$ *is an analyic function of* $z \in T^{O(C)}$ *which satisfies* [6, p. 202, (8)] *for* $z \in T^{C'}$, C' *being any compact subcone of* $O(C)$. *For any* $\phi \in \mathscr{S}$ *we have*

$$\lim_{\substack{y \to 0 \\ y \in O(C)}} 0\langle C(U; x + iy), \phi(x)\rangle = \langle \mathscr{F}[I_{C^*}(\eta)\mathscr{F}^{-1}[U]], \phi(x)\rangle \tag{6.2}$$

with the transforms being in the $\mathscr{S}'$ *sense. If* $U = \hat{V}$ *where* $V \in \mathscr{S}'$ *with* $\mathrm{supp}(V) \subseteq C^*$, *then* $V = \sum_{|\alpha| \leqq m} t^\alpha h_\alpha(t)$, $h_\alpha(t) \in L^q$, $(1/p) + (1/q) = 1$, *for some nonnegative integer* m; *we have*

$$C(U; z) = \langle V, e^{2\pi i \langle z, t\rangle}\rangle\ , \quad z \in T^{O(C)}\ , \tag{6.3}$$

as elements of $\mathscr{S}'$; *and*

$$\lim_{\substack{y \to 0 \\ y \in C' \subset O(C)}} 0\langle C(U; x + iy), \phi(x)\rangle = \langle U, \phi\rangle\ , \quad \phi \in \mathscr{S}\ . \tag{6.4}$$

[6, Corollary 1, Theorems 11, 12, and 15] hold for $p = 1$ also. [6, Theorem 16] can now be extended to include $p = 1$ and to conclude the analyticity of $C(U; z)$ in $T^{O(C)}$, the growth [6, p. 202, (8)] for $z \in T^{C'}$, $C' \subset O(C)$, and the convergence (6.2) in each of the connected components $O(C_\lambda)$, $\lambda \in \Lambda$. The restriction of $z \in T^{O(C)} \backslash \{z: y = \mathrm{Im}(z) \in O(C), y_j = 0 \text{ for any } j = 1, \cdots, n\}$ in [6, Theorem 16] is unnecessary.

Now let $U \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq \infty$, and C be an open connected cone. By Lemma 2.2 the generalized Poisson integral of U

$$P(U; z) = \langle U, Q(z; t)\rangle\ , \quad z \in T^{O(C)}\ , \tag{6.5}$$

is a well defined function of $z \in T^{O(C)}$. In general $P(U; z)$ is not analytic. However, if z is in a generalized half plane in C^n then $P(U; z)$ is n-harmonic by a proof as in [5, Theorem 7].

We now extend and generalize slightly [6, Lemma 8]. The proof is the same for all p, $1 \leqq p \leqq \infty$, and for $\phi \in \mathscr{D}_{L^1}$ as that indicated for [6, Lemma 8].

LEMMA 6.1. *Let* $U \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq \infty$, *and* $z \in T^{O(C)}$, C *being an open connected cone. For* $y \in O(C)$ *we have*

(6.6) $\langle P(U; x+iy), \phi(x)\rangle = \langle U, \langle Q(x+iy; t), \phi(x)\rangle\rangle, \ \phi \in \mathscr{D}_{L^1}$.

LEMMA 6.2. *Let C be an open connected cone and $z = x + iy \in T^{O(C)}$. We have*

$$\lim_{\substack{y\to 0\\ y\in O(C)}} \int_{R^n} Q(x+iy; t)\phi(x)dx = \phi(t), \quad \phi \in \mathscr{D}_{L^1} \tag{6.7}$$

in the topology of $\mathscr{D}_{L^q}$ for all q, $1 \leqq q \leqq \infty$, and in the topology of $\dot{\mathscr{B}}$.

Proof. For $y \in O(C)$ and any n-tuple α of nonnegative integers

$$D_t^\alpha(\langle Q(x+iy; t), \phi(x)\rangle) = \int_{R^n} D_t^\alpha(\phi(x+t))Q(x; y)dx, \quad \phi \in \mathscr{D}_{L^2}, \tag{6.8}$$

where $Q(x; y)$ is defined in (2.5). $\phi \in \mathscr{D}_{L^1}$ implies $\psi_\alpha(t) = D_t^\alpha(\phi(t)) \in \mathscr{D}_{L^1} \subseteqq \mathscr{D}_{L^q}$ for all q, $1 \leqq q \leqq \infty$. Using [6, Lemma 6, (50)], (6.8), and the analysis of [6, p. 214, (55)] and [6, Lemma 7] we have for any q, $1 \leqq q < \infty$,

$$\begin{aligned} &\lim_{\substack{y\to 0\\ y\in O(C)}} \left\| D_t^\alpha\left(\int_{R^n} Q(x+iy; t)\phi(x)dx\right) - D_t^\alpha(\phi(t))\right\|_{L^q} \\ &\quad = \lim_{\substack{y\to 0\\ y\in O(C)}} \left\| \int_{R^n} (\psi_\alpha(x+t) - \psi_\alpha(t))Q(x; y)dx\right\|_{L^q} = 0 \end{aligned} \tag{6.9}$$

which proves (6.7) in the topology of $\mathscr{D}_{L^q}$ for all q, $1 \leqq q < \infty$. Now $\phi \in \mathscr{D}_{L^1} \subset \dot{\mathscr{B}} \subset \mathscr{D}_{L^\infty}$ implies $\psi_\alpha(t) = D_t^\alpha(\phi(t)) \in \mathscr{D}_{L^1} \subset \dot{\mathscr{B}} \subset \mathscr{D}_{L^\infty}$. The definition of $\dot{\mathscr{B}}$ implies that $\psi_\alpha(t)$ is uniformly continuous and bounded on R^n; hence the proof of [9, Proposition 3, (b)] yields

$$\lim_{\substack{y\to 0\\ y\in O(C)}} \int_{R^n} \psi_\alpha(x+t)Q(x; y)dx = \psi_\alpha(t)$$

uniformly for $t \in R^n$. Because of this, (6.9) holds also for $q = \infty$ which proves (6.7) in the topology of $\dot{\mathscr{B}}$ and in the topology of $\mathscr{D}_{L^\infty} = \mathscr{B}$.

We now extend and generalize [6, Theorem 14].

THEOREM 6.2. *Let $U \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq \infty$. Let C be an open connected cone and $z = x + iy \in T^{O(C)}$. We have*

$$\lim_{\substack{y\to 0\\ y\in O(C)}} \langle P(U; x+iy), \phi(x)\rangle = \langle U, \phi\rangle, \quad \phi \in \mathscr{D}_{L^1}. \tag{6.10}$$

Proof. The proof follows by (6.6), (6.7), and the continuity of U.

Using Theorem 6.2, [6, Theorem 17] can be extended and generalized for $U \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq \infty$, where $\overline{O(C)}$ contains no entire straight line. One concludes the existence of $P(U; z)$, $z \in T^{O(C)}$, and the convergence (6.10) as $y \to 0$, $y \in O(C_\lambda)$, $\lambda \in \Lambda$. The restriction of $z \in T^{O(C)} \backslash \{z: y = \mathrm{Im}(z) \in O(C), y_j = 0$ for any $j = 1, \cdots, n\}$ in [6, Theorem 17] is unnecessary.

7. **Acknowledgment.** One of us, R. D. C., expresses his appreciation to the Department of Mathematics of Iowa State University for the opporunity of serving as Visiting Associate Professor during 1978-1979. The authors thank the referee for his comments and suggestions concerning this paper.

References

1. S. Bochner, *Bounded analytic functions in several variables and multiple Laplace integrals*, Amer. J. Math., **59** (1937), 732-738.
2. ———, *Group invariance of Cauchy's formula in several variables*, Ann. of Math., **45** (1944), 686-707.
3. S. Bochner and K. Chandrasekharan, *Fourier transforms*, Princeton University Press, Princeton, N. J., 1949.
4. S. Bochner and W. T. Martin, *Several complex variables*, Princeton University Press, Princeton, N. J., 1948.
5. Richard D. Carmichael, *Distributional boundary values in $\mathscr{D}'_{L^p}$. II*, Rend. Sem. Mat. Univ. Padova, **45** (1971), 249-277.
6. ———, *Generalized Cauchy and Poisson integrals and distributional boundary values*, SIAM J. Math. Anal., **4** (1973), 198-219.
7. Richard D. Carmichael and E. O. Milton, *Distributional boundary values in the dual spaces of spaces of type $\mathscr{S}$*, Pacific J. Math., **56** (1975), 385-422.
8. Y. Katznelson, *An Introduction to Harmonic Analysis*, John Wiley and Sons, Inc., New York, 1968.
9. Adam Korányi, *A Poisson integral for homogeneous wedge domains*, J. Analyse Math., **14** (1965), 275-284.
10. L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.
11. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, N. J., 1971.
12. V. S. Vladimirov, *Methods of theory of the functions of many complex variables*, M. I. T. Press, Cambridge, Mass., 1966.

Received October 19, 1978 and in revised form June 22, 1979.

IOWA STATE UNIVERSITY
AMES, IA 50011
WAKE FOREST UNIVERSITY
WINSTON-SALEM, NC 27109
AND
WAKE FOREST UNIVERSITY
WINSTON-SALEM, NC 27109

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 90, No. 1 September, 1980