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AN ESTIMATE OF INFINITE CYCLIC COVERINGS AND KNOT THEORY

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In this paper we estimate the homology torsion module of an infinite cyclic covering space of an n-manifold by the homology of a Poincaré duality space of dimension $n-1$. To be concrete, we apply it to knot theory. In particular, it follows that any ribbon n-knot $K \subset S^{n+2}$ ($n \geqq 3$) is unknotted if $\pi_1(S^{n+2}-K) \cong \boldsymbol{Z}$. We add also in this paper a somewhat geometric proof to this unknotting criterion.

1. Statements of results. Let X be a compact, connected and smooth, piecewise-linear or topological n-manifold with nonzero 1st Betti number, i.e., $H^1(X; \boldsymbol{Z}) \neq 0$. Let $\tilde{X}$ be an infinite cyclic connected cover of X, that is, the cover of X associated with an indivisible element of $H^1(X; \boldsymbol{Z})$. We denote by $\langle t \rangle$ the covering transformation group of $\tilde{X}$ with a specified generator t. Let F be a field and $F\langle t \rangle$ be the group algebra of $\langle t \rangle$ over F. For $H_* = H_*(\tilde{X}; F)$ or $H_*(\tilde{X}, \partial\tilde{X}; F)$, H_* is canonically regarded as an $F\langle t \rangle$-module. We define $T_* = \mathrm{Tor}_{F\langle t \rangle} H_*$ and $T^* = \mathrm{Hom}_F[T_*, F]$. We assume $\tilde{X}$ is F-orientable. Note that $T_0(\tilde{X}; F) = H_0(\tilde{X}; F) \cong F$ and $T_{n-1}(\tilde{X}, \partial\tilde{X}; F) \cong F$. (Cf. [5, Duality Theorem (II) and Remark 1.3].) Let M be a connected Poincaré duality space with boundary ∂M of dimension $n-1$ over F.

THEOREM. *Suppose there is a map $f\colon (M, \partial M) \to (\tilde{X}, \partial\tilde{X})$ such that $f_* H_{n-1}(M, \partial M; F) = T_{n-1}(\tilde{X}, \partial\tilde{X}; F)$. Then*

$$\dim_F H_q(M; F) \geqq \dim_F T_q(\tilde{X}; F)$$

for all q. Further, if $f_* H_q(M; F) \subset T_q(\tilde{X}; F)$ for some q, then $f_* H_q(M; F) = T_q(\tilde{X}; F)$. In particular, if $T_q(\tilde{X}; F) = H_q(\tilde{X}; F)$ (e.g., $H_q(X; F) \cong H_q(S^1; F)$) *for some q, then the homomorphism*

$$f_*\colon H_q(M; F) \longrightarrow H_q(\tilde{X}; F)$$

is onto.

Note 1. Our proof will imply also that

$$\dim_F H_{n-q-1}(M, \partial M; F) \geqq \dim_F T_{n-q-1}(\tilde{X}, \partial\tilde{X}; F)$$

for all q and, if $f_* H_{n-q-1}(M, \partial M; F) \subset T_{n-q-1}(\tilde{X}, \partial\tilde{X}; F)$ for some q, then $f_* H_{n-q-1}(M, \partial M; F) = T_{n-q-1}(\tilde{X}, \partial\tilde{X}; F)$.

In case X is oriented and piecewise-linear and $\tilde{X}$ is obtained from a piecewise-linear map $g\colon X \to S^1$, the preimage $X_1 = g^{-1}(p)$ is a bicollared, oriented, proper $(n-1)$-submanifold of $\tilde{X}$ for any non-vertex point p of S^1. Then, we see that the inclusion $i\colon (X_1, \partial X_1) \subset (\tilde{X}, \partial\tilde{X})$ sends the fundamental class of X_1 to a generator of $T_{n-1}(\tilde{X}, \partial\tilde{X}; F)$ for any F. [*Proof.* Let X' be a manifold obtained from X by splitting along X_1. X' is imbedded canonically in $\tilde{X}$ so that $\partial X' = X_1 \cup (X' \cap \partial X) \cup -tX_1$. This implies that $(1-t)[X_1] = [X_1] = t[X_1] = 0$ in $H_{n-1}(\tilde{X}, \partial\tilde{X}; F)$, i.e., $[X_1] \in T_{n-1}(\tilde{X}, \partial\tilde{X}; F)$. $[X_1] \neq 0$ in $H_{n-1}(X, \partial X; F)$ and hence in $H_{n-1}(\tilde{X}, \partial\tilde{X}; F)$, since it is the Poincaré dual of $g^*[S^1] \in H^1(X; F)$. Thus, $[X_1]$ generates $T_{n-1}(\tilde{X}, \partial\tilde{X}; F) \cong F$.] Let $\hat{X}_1$ be the interior oriented connected sum of the components of X_1. Since $\tilde{X}$ is connected, we can construct from i a map $\hat{i}\colon (\hat{X}_1, \partial\hat{X}_1) \to (\tilde{X}, \partial\tilde{X})$ such that $\hat{i}_* H_{n-1}(\hat{X}_1, \partial\hat{X}_1; F) = T_{n-1}(\tilde{X}, \partial\tilde{X}; F)$. From this observation and the theorem, we see the following:

COROLLARY 1. *$\dim_F H_q(X_1; F) \geqq \dim_F T_q(\tilde{X}; F)$ for all q and F. If $i_* H_q(X_1; F) \subset T_q(\tilde{X}; F)$ for some q and some F, then $i_* H_q(X_1; F) = T_q(\tilde{X}; F)$.*

In knot theory this corollary gives a general relation between the homology of a Seifert manifold of a knot (or link) and its knot (or link) module (associated with an infinite cyclic covering). For a classical knot (i.e., 1-knot) k, this has been recognized as (the genus of k) $\geqq (1/2)\cdot$(the degree of the knot polynomial of k). (Cf. H. Seifert [9].)

Next, suppose X is orientable and $H_1(X; \boldsymbol{Z}) \cong \boldsymbol{Z}$. Such a manifold occurs, for example, as the complement of an open regular neighborhood of a closed connected orientable $(n-2)$-manifold imbedded piecewise-linearly in S^{n+2}. By Poincaré duality $H_{n-1}(X, \partial X; \boldsymbol{Z}) \cong \boldsymbol{Z}$.

COROLLARY 2. *If there is a map $f\colon (M, \partial M) \to (X, \partial X)$ inducing an isomorphism $f_*\colon H_{n-1}(M, \partial M; \boldsymbol{Z}) \cong H_{n-1}(X, \partial X; \boldsymbol{Z})$ and a 0-map $f_* = 0\colon H_1(M; \boldsymbol{Z}) \to H_1(X; \boldsymbol{Z})$, then*

$$\dim_F H_q(M; F) \geqq \dim_F T_q(\tilde{X}; F)$$

for all q and F.

To see this, note that $H_{n-1}(\tilde{X}, \partial\tilde{X}; \boldsymbol{Z}) \cong \boldsymbol{Z}$ and t acts trivially on it and the covering projection $\tilde{X} \to X$ induces an isomorphism $H_{n-1}(\tilde{X}, \partial\tilde{X}; \boldsymbol{Z}) \cong H_{n-1}(X, \partial X; \boldsymbol{Z})$. This follows from [3, Theorem 2.3], (or its topological version [4]) and the Wang exact sequence. So, it suffices to show that f has a lifting to $\tilde{X}$. This is clear, however, by the assumption that $f_*: H_1(M; \boldsymbol{Z}) \to (X; \boldsymbol{Z})$ is a 0-map.

For the following application, spaces and maps are considered in the piecewise-linear category. Let L be a trivial n-link in S^{n+2} of some $r+1$ components and a collection $\{B_1, \cdots, B_r\}$ of r $(n+1)$-balls imbedded locally-flatly and mutually disjointly in S^{n+2} such that for each i B_i spans L as 1-handle i.e., $B_i \cap L = (\partial B_i) \cap L =$ the disjoint union of two n-balls. An n-knot K in S^{n+2} is called a *ribbon n-knot* if it is obtained from such an L and a $\{B_1, \cdots, B_r\}$ by doing an imbedded surgery. (Cf. T. Yanagawa [12], R. Hitt [1].) The knot K is often said to be a *fusion of the link L along 1-handles* $\{B_1, \cdots, B_r\}$.

COROLLARY 3. *Let $n \geqq 3$. A ribbon n-knot K is unknotted, if $\pi_1(S^{n+2} - k) \cong \boldsymbol{Z}$.*

To see this, note that any ribbon n-knot has a Seifert $(n+1)$-manifold M, homeomorphic to a manifold of the form $\sharp^m S^1 \times S^n$-Int $B^{n+1}(B^{n+1}$ is an $(n+1)$-ball.) ([12], [1]). Let $X = S^{n+2}$-Int $N(K)$, $N(K)$ being a regular neighborhood of K in S^{n+2}. The manifold $X \cap M(\cong M)$ gives a generator of $H_{n+1}(X, \partial X; \boldsymbol{Z}) = \boldsymbol{Z}$ and the inclusion $X \cap M \subset X$ induces a 0-map on H_1. By Corollary 2, $T_i(\tilde{X}; F) = 0$, $i \neq 0, 1, n$. (Of course, one can also apply Corollary 1 to obtain this.) But $T_*(\tilde{X}; F) = H_*(\tilde{X}; F)$. As a result, $\tilde{H}_*(\tilde{X}; F) = 0$ by using Milnor duality [8] or [5, Duality Theorem (II)], since $\tilde{X}$ is simply connected. Then by taking $F = \boldsymbol{Q}$, we see that $\tilde{H}_*(\tilde{X}; \boldsymbol{Z})$ is a torsion group. Next, by taking $F = \boldsymbol{Z}_p$, p prime, and considering the universal coefficient theorem, the torsion product $\mathrm{Tor}_{\boldsymbol{Z}}(H_{*-1}(\tilde{X}; \boldsymbol{Z}), \boldsymbol{Z}_p) = 0$. This shows that $\tilde{H}_*(\tilde{X}; \boldsymbol{Z}) = 0$ and X has the homotopy type of S^1. By [6], [10], [11], K is unknotted for $n \geqq 3$.

Note 2. For $n = 2$, a corresponding result is proved by Y. Marumoto [7] in the simplest case, that is, the case of L having two components. However, a general case is unknown.

2. **Proof of theorem.** Let $i: T_* \subset H_*$. i induces an epimorphism $i^*: H^* \to T^*$. Let $x \in H^q(\tilde{X}; F)$ such that $i^*(x) \neq 0$. By [5, Duality Theorem (II)], the cup product $H^q(\tilde{X}; F) \times H^{n-q-1}(\tilde{X}, \partial\tilde{X}; F) \xrightarrow{\cup} H^{n-1}(\tilde{X}, \partial\tilde{X}; F)$ induces a nonsingular pairing $T^q(\tilde{X}; F) \times T^{n-q-1}(\tilde{X}, \partial\tilde{X}; F) \to T^{n-1}(\tilde{X}, \partial\tilde{X}; F)$, also denoted by $\cup$. Hence we find an element $y \in H^{n-q-1}(\tilde{X}, \partial\tilde{X}; F)$ such that $i^*(x) \cup i^*(y) = i^*(x \cup y) \neq 0$. By assumption, $f: (M, \partial M) \to (\tilde{X}, \partial\tilde{X})$ induces the following commutative triangle

$$\begin{tikzcd}[column sep=small] H^{n-1}(\tilde{X}, \partial\tilde{X}; F) \arrow[d, "i^*"'] \arrow[dr, "f^*"] & \\ & H^{n-1}(M, \partial M; F) \\ T^{n-1}(\tilde{X}, \partial\tilde{X}; F) \arrow[ur, "f^*"'] & \end{tikzcd}$$

and $f^*\colon T^{n-1}(\tilde{X}, \partial\tilde{X}; F) \to H^{n-1}(M, \partial M; F)$ is an isomorphism. Thus, $f^*(x \cup y) = f^*(x) \cup f^*(y) \neq 0$, so that $f^*(x) \neq 0$. We obtain a (non-canonical) monomorphism $r\colon T^q(\tilde{X}; F) \to H^q(M; F)$. Hence, $\dim_F T_q(\tilde{X}; F) = \dim_F T^q(\tilde{X}; F) \leqq \dim_F H^q(M; F) = \dim_F H_q(M. F)$. If $f_* H_q(M; F) \subset T_q(\tilde{X}; F)$, then we may replace r by a canonical epimorphism $r'\colon T^q(\tilde{X}; F) \to \mathrm{Hom}_F [f_* H_q(M; F), F]$ composed with the natural inclusion into $H^q(M; F)$. Since r' is an isomorphism, we see that $f_* H_q(M; F) = T_q(\tilde{X}; F)$. This completes the proof of the theorem.

3. Alternative proof of Corollary 3. We now describe a different, somewhat geometric proof of Corollary 3. This method, as a matter of fact, has been earlier obtained and is near to the argument of [2]. Let $T(m)$ be an n-manifold homeomorphic to $\#^m S^1 \times S^{n-1}$ and imbedded locally-flatly in S^{n+2}. (The following four lemmas are true when $n \geqq 2$.) For $m = 0$, $T(m)$ is an n-sphere, i.e., an n-knot. Such a $T(m)$ is *unknotted* if it bounds a manifold locally-flatly imbedded in S^{n+2} and homeomorphic to a disk sum $\natural^m S^1 \times B^n$. As an analogous argument to [2, Theorem 1.2], we have the following:

3.1 *Any two unknotted $T(m)_1$, $T(m)_2$ are ambient isotopic.*

Thus, the following is obtained:

3.2. *If $T(m)$ is unknotted, $S^{n+2} - T(m)$ is homotopy equivalent to a bouquet $S^1 \vee S^2 \vee \cdots \vee S^2 \vee S^n \vee \cdots \vee S^n$ of one 1-sphere, m 2-spheres and m n-spheres.* [Regard $T(m)$ as the common boundary of $\natural^m S^1 \times B^n$ and $\natural^m B^2 \times S^{n-1}$ whose union forms an unknotted $(n+1)$-sphere S_0^{n+1} in S^{n+2}. Then, $S^{n+2} - T(m)$ is homotopy equivalent to the suspension of $S_0^{n+1} - T(m)$.]

3.3. *Let $T(m+1)$ and $T(m+1)'$ be the manifolds obtained from the same $T(m)$ by surgeries along 1-handles B^{n+1} and B'^{n+1} on $T(m)$ imbedded locally-flatly in S^{n+2}, respectively. If $\pi_1(S^{n+2} - T(m)) \cong \boldsymbol{Z}$, then $T(m+1)$ and $T(m+1)'$ are ambient isotopic.*

This is proved easily as an analogy to [2, Lemma 2.7].

From 3.3 and the definition of ribbon knots, we see the following:

3.4. *For any ribbon n-knot K obtained from $(m+1)$ balls and m 1-handles, the surgery along some standard mutually disjoint m 1-handles on K imbedded locally-flatly in S^{n+2} produces an unknotted $T(m)$. Further, if $\pi_1(S^{n+2}-K)\cong \boldsymbol{Z}$, then $T(m)$ is ambient isotopic to a knot sum $K\#T(m)'$ for some unknotted $T(m)'$.*

Now assume $\pi_1(S^{n+2}-K)\cong \boldsymbol{Z}$. In 3.4, let $E=S^{n+2}-K$, $X=S^{n+2}-T(m)$ and $X'=S^{n+2}-T(m)'$. Take their infinite cyclic connected covers. We have $\tilde{H}_*(\tilde{E};\boldsymbol{Z})\oplus\tilde{H}_*(\tilde{X}';\boldsymbol{Z})\cong\tilde{H}_*(\tilde{X};\boldsymbol{Z})$ as $\boldsymbol{Z}\langle t\rangle$-modules. By 3.2, $\tilde{H}_*(\tilde{X}';\boldsymbol{Q})$ and $\tilde{H}_*(\tilde{X};\boldsymbol{Q})$ are free $\boldsymbol{Q}\langle t\rangle$-modules of the same rank, so that $\tilde{H}_*(\tilde{E};\boldsymbol{Q})=0$, i.e., $\tilde{H}_*(\tilde{E};\boldsymbol{Z})$ is a torsion group. By 3.2 again, $\tilde{H}_*(\tilde{X}';\boldsymbol{Z})$ and $\tilde{H}_*(\tilde{X};\boldsymbol{Z})$ are free abelian, hence $\tilde{H}_*(\tilde{E};\boldsymbol{Z})=0$ and E has the homotopy type of S^1. By [6], [10], [11], K is unknotted for $n\geqq 3$.

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