AN ESTIMATE OF INFINITE CYCLIC COVERINGS AND KNOT THEORY

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In this paper we estimate the homology torsion module of an infinite cyclic covering space of an \(n\)-manifold by the homology of a Poincaré duality space of dimension \(n-1\). To be concrete, we apply it to knot theory. In particular, it follows that any ribbon \(n\)-knot \(K \subset S^{n+2} (n \geq 3)\) is unknotted if \(\pi_1(S^{n+2} - K) \cong \mathbb{Z}\). We add also in this paper a somewhat geometric proof to this unknotting criterion.

1. Statements of results. Let \(X\) be a compact, connected and smooth, piecewise-linear or topological \(n\)-manifold with nonzero 1st Betti number, i.e., \(H^1(X; \mathbb{Z}) \neq 0\). Let \(\tilde{X}\) be an infinite cyclic connected cover of \(X\), that is, the cover of \(X\) associated with an invisible element of \(H^1(X; \mathbb{Z})\). We denote by \(\langle t \rangle\) the covering transformation group of \(\tilde{X}\) with a specified generator \(t\). Let \(F\) be a field and \(F\langle t \rangle\) be the group algebra of \(\langle t \rangle\) over \(F\). For \(H_* = H_*(\tilde{X}; F)\) or \(H_*(\tilde{X}, \partial \tilde{X}; F)\), \(H_*\) is canonically regarded as an \(F\langle t \rangle\)-module. We define \(T_* = \text{Tor}_{F\langle t \rangle} H_*\) and \(T^* = \text{Hom}_F[T_*, F]\). We assume \(\tilde{X}\) is \(F\)-orientable. Note that \(T_0(\tilde{X}; F) = H_0(\tilde{X}; F) \cong F\) and \(T_{n-1}(\tilde{X}, \partial \tilde{X}; F) \cong F\). (Cf. [5, Duality Theorem (II) and Remark 1.3].)

Let \(M\) be a connected Poincaré duality space with boundary \(\partial M\) of dimension \(n-1\) over \(F\).

**Theorem.** Suppose there is a map \(f: (M, \partial M) \to (\tilde{X}, \partial \tilde{X})\) such that \(f_*H_{n-1}(M, \partial M; F) = T_{n-1}(\tilde{X}, \partial \tilde{X}; F)\). Then

\[
\dim_F H_q(M; F) \geq \dim_F T_q(\tilde{X}; F)
\]

for all \(q\). Further, if \(f_*H_q(M; F) \subset T_q(\tilde{X}; F)\) for some \(q\), then \(f_*H_q(M; F) = T_q(\tilde{X}; F)\). In particular, if \(T_q(\tilde{X}; F) = H_q(\tilde{X}; F)\) (e.g., \(H_q(X; F) \cong H_q(S^1; F)\)) for some \(q\), then the homomorphism

\[
f_q: H_q(M; F) \longrightarrow H_q(\tilde{X}; F)
\]

is onto.

**Note 1.** Our proof will imply also that

\[
\dim_F H_{n-q-1}(M, \partial M; F) \geq \dim_F T_{n-q-1}(\tilde{X}, \partial \tilde{X}; F)
\]

for all \(q\) and, if \(f_*H_{n-q-1}(M, \partial M; F) \subset T_{n-q-1}(\tilde{X}, \partial \tilde{X}; F)\) for some \(q\), then \(f_*H_{n-q-1}(M, \partial M; F) = T_{n-q-1}(\tilde{X}, \partial \tilde{X}; F)\).

In case \(X\) is oriented and piecewise-linear and \(\tilde{X}\) is obtained
from a piecewise-linear map \( g: X \to S^1 \), the preimage \( X_\lambda = g^{-1}(\lambda) \) is a bicollared, oriented, proper \((n-1)\)-submanifold of \( \tilde{X} \) for any non-vertex point \( \lambda \) of \( S^1 \). Then, we see that the inclusion \( i: (X_\lambda, \partial X_\lambda) \subset (\tilde{X}, \partial \tilde{X}) \) sends the fundamental class of \( X_\lambda \) to a generator of \( T_{n-1}(\tilde{X}, \partial \tilde{X}; F) \) for any \( F \). [Proof. Let \( X' \) be a manifold obtained from \( X \) by splitting along \( X_\lambda \) so \( X' \) is imbedded canonically in \( \tilde{X} \) so that \( \partial X' = X_\lambda \cup (X' \cap \partial X) \cup -tX_\lambda \). This implies that \((1-t)[X_\lambda] = t[X_\lambda] = 0 \) in \( H_{n-1}(\tilde{X}, \partial \tilde{X}; F) \), i.e., \([X_\lambda] \in T_{n-1}(\tilde{X}, \partial \tilde{X}; F) \). \([X_\lambda] \neq 0 \) in \( H_{n-1}(X, \partial X; F) \) and hence in \( H_{n-1}(\tilde{X}, \partial \tilde{X}; F) \), since it is the Poincare dual of \( g^*[S^1] \in H^1(X; F) \). Thus, \([X_\lambda] \) generates \( T_{n-1}(\tilde{X}, \partial \tilde{X}; F) \) for any \( F \).] Let \( \tilde{X}_\lambda \) be the interior oriented connected sum of the components of \( X_\lambda \). Since \( \tilde{X} \) is connected, we can construct from \( i \) a map \( \hat{i}: (\tilde{X}_\lambda, \partial \tilde{X}_\lambda) \to (\tilde{X}, \partial \tilde{X}) \) such that \( \hat{i}_* T_{n-1}(\tilde{X}_\lambda, \partial \tilde{X}_\lambda; F) = T_{n-1}(\tilde{X}, \partial \tilde{X}; F) \). From this observation and the theorem, we see the following:

\textbf{COROLLARY 1.} \( \dim_F H_q(X_\lambda; F) \leq \dim_F T_q(\tilde{X}; F) \) for all \( q \) and \( F \). If \( i_* H_q(X_\lambda; F) \subset T_q(\tilde{X}; F) \) for some \( q \) and some \( F \), then \( i_* H_q(X_\lambda; F) = T_q(\tilde{X}; F) \).

In knot theory this corollary gives a general relation between the homology of a Seifert manifold of a knot (or link) and its knot (or link) module (associated with an infinite cyclic covering). For a classical knot (i.e., 1-knot) \( k \), this has been recognized as (the genus of \( k \)) \( \geq (1/2) \cdot (\text{the degree of the knot polynomial of } k) \). (Cf. H. Seifert [9].)

Next, suppose \( X \) is orientable and \( H_1(X; Z) \cong Z \). Such a manifold occurs, for example, as the complement of an open regular neighborhood of a closed connected orientable \((n-2)\)-manifold imbedded piecewise-linearly in \( S^{n+2} \). By Poincaré duality \( H_{n-1}(X, \partial X; Z) \cong Z \).

\textbf{COROLLARY 2.} \textit{If there is a map} \( f: (M, \partial M) \to (X, \partial X) \) \textit{inducing an isomorphism} \( f_*: H_{n-1}(M, \partial M; Z) \cong H_{n-1}(X, \partial X; Z) \) \textit{and a 0-map} \( f_* = 0: H_1(M; Z) \to H_1(X; Z) \), \textit{then}

\[ \dim_F H_q(M; F) \geq \dim_F T_q(\tilde{X}; F) \]

\textit{for all} \( q \) \textit{and} \( F \).

To see this, note that \( H_{n-1}(\tilde{X}, \partial \tilde{X}; Z) \cong Z \) and \( t \) acts trivially on it and the covering projection \( \tilde{X} \to X \) induces an isomorphism \( H_{n-1}(\tilde{X}, \partial \tilde{X}; Z) \cong H_{n-1}(X, \partial X; Z) \). This follows from [3, Theorem 2.3], (or its topological version [4]) and the Wang exact sequence. So, it suffices to show that \( f \) has a lifting to \( \tilde{X} \). This is clear,
however, by the assumption that $f_*: H_*(M; \mathbb{Z}) \to (X; \mathbb{Z})$ is a 0-map.

For the following application, spaces and maps are considered in the piecewise-linear category. Let $L$ be a trivial $n$-link in $S^{n+2}$ of some $r + 1$ components and a collection $\{B_1, \cdots, B_r\}$ of $r (n + 1)$-balls imbedded locally-flatly and mutually disjointly in $S^{n+2}$ such that for each $i$ $B_i$ spans $L$ as 1-handle i.e., $B_i \cap L = (\partial B_i) \cap L = \text{the disjoint union of two } n \text{-balls}$. An $n$-knot $K$ in $S^{n+2}$ is called a ribbon $n$-knot if it is obtained from such an $L$ and a $\{B_1, \cdots, B_r\}$ by doing an imbedded surgery. (Cf. T. Yanagawa [12], R. Hitt [1].) The knot $K$ is often said to be a fusion of the link $L$ along 1-handles $\{B_1, \cdots, B_r\}$.

**Corollary 3.** Let $n \geq 3$. A ribbon $n$-knot $K$ is unknotted, if $\pi_1(S^{n+2} - k) \cong \mathbb{Z}$.

To see this, note that any ribbon $n$-knot has a Seifert $(n + 1)$-manifold $M$, homeomorphic to a manifold of the form $\#^n S^1 \times S^n$. Int $B^{n+1}(B^{n+1})$ is an $(n+1)$-ball. ([12], [1]). Let $X = S^{n+2}$-Int $N(K)$, $N(K)$ being a regular neighborhood of $K$ in $S^{n+2}$. The manifold $X \cap M(= M)$ gives a generator of $H_{n+1}(X, \partial X; \mathbb{Z}) = \mathbb{Z}$ and the inclusion $X \cap M \subset X$ induces a 0-map on $H_*$. By Corollary 2, $T_i(\tilde{X}; F) = 0$, $i \neq 0, 1, n$. (Of course, one can also apply Corollary 1 to obtain this.) But $T_*(\tilde{X}; F) = H_*(\tilde{X}; F)$. As a result, $\tilde{H}_*(\tilde{X}; F) = 0$ by using Milnor duality [8] or [5, Duality Theorem (II)], since $\tilde{X}$ is simply connected. Then by taking $F = Q$, we see that $\tilde{H}_*(\tilde{X}; \mathbb{Z})$ is a torsion group. Next, by taking $F = Z_p$, $p$ prime, and considering the universal coefficient theorem, the torsion product $\text{Tor}_2(H_{*-1}(\tilde{X}; Z), Z_p) = 0$. This shows that $\tilde{H}_*(\tilde{X}; Z) = 0$ and $X$ has the homotopy type of $S^1$. By [6], [10], [11], $K$ is unknotted for $n \geq 3$.

**Note 2.** For $n = 2$, a corresponding result is proved by Y. Marumoto [7] in the simplest case, that is, the case of $L$ having two components. However, a general case is unknown.

2. Proof of theorem. Let $i: T_* \subset H_*$. $i$ induces an epimorphism $i^*: H^* \to T^*$. Let $x \in H^q(\tilde{X}; F)$ such that $i^*(x) \neq 0$. By [5, Duality Theorem (II)], the cup product $H^q(\tilde{X}; F) \times H^{n-q-1}(\tilde{X}, \partial \tilde{X}; F) \cup H^{n-q-1}(\tilde{X}, \partial \tilde{X}; F)$ induces a nonsingular pairing $T^q(\tilde{X}; F) \times T^{n-q-1}(\tilde{X}, \partial \tilde{X}; F) \to T^{n-q-1}(\tilde{X}, \partial \tilde{X}; F)$, also denoted by $\cup$. Hence we find an element $y \in H^{n-q-1}(\tilde{X}, \partial \tilde{X}; F)$ such that $i^*(x) \cup i^*(y) = i^*(x \cup y) \neq 0$. By assumption, $f: (M, \partial M) \to (\tilde{X}, \partial \tilde{X})$ induces the following commutative triangle
and $f^*: T^n(-X, \partial X; F) \to H^n(M, \partial M; F)$ is an isomorphism. Thus, $f^*(x \cup y) = f^*(x) \cup f^*(y) \neq 0$, so that $f^*(x) \neq 0$. We obtain a (non-canonical) monomorphism $r: T^n(-X; F) \to H^n(M, \partial M; F)$. Hence, $\dim_F T^n(-X; F) = \dim_F T^n(X; F) \leq \dim_F H^n(M; F) = \dim_F H_q(M, F)$. If $f_*H_q(M; F) \subset T_q(-X; F)$, then we may replace $r$ by a canonical epimorphism $r': T^n(X; F) \to \text{Hom}_F [f_*H_q(M; F), F]$ composed with the natural inclusion into $H^n(M; F)$. Since $r'$ is an isomorphism, we see that $f_*H_q(M; F) = T_q(X; F)$. This completes the proof of the theorem.

3. Alternative proof of Corollary 3. We now describe a different, somewhat geometric proof of Corollary 3. This method, as a matter of fact, has been earlier obtained and is near to the argument of [2]. Let $T(m)$ be an $n$-manifold homeomorphic to $\#^a S^1 \times S^{n-1}$ and imbedded locally-flatly in $S^{n+2}$. (The following four lemmas are true when $n \geq 2$.) For $m = 0$, $T(m)$ is an $n$-sphere, i.e., an $n$-knot. Such a $T(m)$ is unknotted if it bounds a manifold locally-flatly imbedded in $S^{n+2}$ and homeomorphic to a disk sum $\#^a S^1 \times B^n$. As an analogous argument to [2, Theorem 1.2], we have the following:

3.1 Any two unknotted $T(m)_1, T(m)_2$ are ambient isotopic.

Thus, the following is obtained:

3.2. If $T(m)$ is unknotted, $S^{n+2} - T(m)$ is homotopy equivalent to a bouquet $S^1 \vee S^2 \vee \cdots \vee S^2 \vee S^n \vee \cdots \vee S^n$ of one 1-sphere, $m$ 2-spheres and $m$ $n$-spheres. [Regard $T(m)$ as the common boundary of $\#^a S^1 \times B^n$ and $\#^a B^2 \times S^{n-1}$ whose union forms an unknotted $(n+1)$-sphere $S_{n+1}^n$ in $S^{n+2}$. Then, $S^{n+2} - T(m)$ is homotopy equivalent to the suspension of $S_{n+1}^n - T(m)$].

3.3. Let $T(m + 1)$ and $T(m + 1)'$ be the manifolds obtained from the same $T(m)$ by surgeries along 1-handles $B^{n+1}$ and $B'^{n+1}$ on $T(m)$ imbedded locally-flatly in $S^{n+2}$, respectively. If $\pi_1(S^{n+2} - T(m)) \cong \mathbb{Z}$, then $T(m + 1)$ and $T(m + 1)'$ are ambient isotopic.

This is proved easily as an analogy to [2, Lemma 2.7].

From 3.3 and the definition of ribbon knots, we see the following:
3.4. For any ribbon $n$-knot $K$ obtained from $(m + 1)$ balls and $m$ 1-handles, the surgery along some standard mutually disjoint $m$ 1-handles on $K$ imbedded locally-flatly in $S^{n+2}$ produces an unknotted $T(m)$. Further, if $\pi_1(S^{n+2} - K) \cong \mathbb{Z}$, then $T(m)$ is ambient isotopic to a knot sum $K \# T(m)'$ for some unknotted $T(m)'$.

Now assume $\pi_1(S^{n+2} - K) \cong \mathbb{Z}$. In 3.4, let $E = S^{n+2} - K$, $X = S^{n+2} - T(m)$ and $X' = S^{n+2} - T(m)'$. Take their infinite cyclic connected covers. We have $\tilde{H}_*(\tilde{E}; \mathbb{Z}) \oplus \tilde{H}_*(\tilde{X}'; \mathbb{Z}) \cong \tilde{H}_*(\tilde{X}; \mathbb{Z})$ as $\mathbb{Z}\langle t \rangle$-modules. By 3.2, $\tilde{H}_*(\tilde{X}'; \mathbb{Q})$ and $\tilde{H}_*(\tilde{X}; \mathbb{Q})$ are free $\mathbb{Q}\langle t \rangle$-modules of the same rank, so that $\tilde{H}_*(\tilde{E}; \mathbb{Q}) = 0$, i.e., $\tilde{H}_*(\tilde{E}; \mathbb{Z})$ is a torsion group. By 3.2 again, $\tilde{H}_*(\tilde{X}'; \mathbb{Z})$ and $\tilde{H}_*(\tilde{X}; \mathbb{Z})$ are free abelian, hence $\tilde{H}_*(\tilde{E}; \mathbb{Z}) = 0$ and $E$ has the homotopy type of $S^1$. By [6], [10], [11], $K$ is unknotted for $n \geq 3$.

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