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**AN ESTIMATE OF INFINITE CYCLIC COVERINGS AND KNOT
THEORY**

AKIO KAWAUCHI AND TAKAO MATUMOTO

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In this paper we estimate the homology torsion module of an infinite cyclic covering space of an n -manifold by the homology of a Poincaré duality space of dimension $n-1$. To be concrete, we apply it to knot theory. In particular, it follows that any ribbon n -knot $K \subset S^{n+2}$ ($n \geq 3$) is unknotted if $\pi_1(S^{n+2} - K) \cong \mathbb{Z}$. We add also in this paper a somewhat geometric proof to this unknotting criterion.

1. **Statements of results.** Let X be a compact, connected and smooth, piecewise-linear or topological n -manifold with nonzero 1st Betti number, i.e., $H^1(X; \mathbb{Z}) \neq 0$. Let \tilde{X} be an infinite cyclic connected cover of X , that is, the cover of X associated with an indivisible element of $H^1(X; \mathbb{Z})$. We denote by $\langle t \rangle$ the covering transformation group of \tilde{X} with a specified generator t . Let F be a field and $F\langle t \rangle$ be the group algebra of $\langle t \rangle$ over F . For $H_* = H_*(\tilde{X}; F)$ or $H_*(\tilde{X}, \partial\tilde{X}; F)$, H_* is canonically regarded as an $F\langle t \rangle$ -module. We define $T_* = \text{Tor}_{F\langle t \rangle} H_*$ and $T^* = \text{Hom}_F[T_*, F]$. We assume \tilde{X} is F -orientable. Note that $T_0(\tilde{X}; F) = H_0(\tilde{X}; F) \cong F$ and $T_{n-1}(\tilde{X}, \partial\tilde{X}; F) \cong F$. (Cf. [5, Duality Theorem (II) and Remark 1.3].) Let M be a connected Poincaré duality space with boundary ∂M of dimension $n-1$ over F .

THEOREM. *Suppose there is a map $f: (M, \partial M) \rightarrow (\tilde{X}, \partial\tilde{X})$ such that $f_* H_{n-1}(M, \partial M; F) = T_{n-1}(\tilde{X}, \partial\tilde{X}; F)$. Then*

$$\dim_F H_q(M; F) \geq \dim_F T_q(\tilde{X}; F)$$

for all q . Further, if $f_ H_q(M; F) \subset T_q(\tilde{X}; F)$ for some q , then $f_* H_q(M; F) = T_q(\tilde{X}; F)$. In particular, if $T_q(\tilde{X}; F) = H_q(\tilde{X}; F)$ (e.g., $H_q(X; F) \cong H_q(S^1; F)$) for some q , then the homomorphism*

$$f_*: H_q(M; F) \longrightarrow H_q(\tilde{X}; F)$$

is onto.

Note 1. Our proof will imply also that

$$\dim_F H_{n-q-1}(M, \partial M; F) \geq \dim_F T_{n-q-1}(\tilde{X}, \partial\tilde{X}; F)$$

for all q and, if $f_* H_{n-q-1}(M, \partial M; F) \subset T_{n-q-1}(\tilde{X}, \partial\tilde{X}; F)$ for some q , then $f_* H_{n-q-1}(M, \partial M; F) = T_{n-q-1}(\tilde{X}, \partial\tilde{X}; F)$.

In case X is oriented and piecewise-linear and \tilde{X} is obtained

from a piecewise-linear map $g: X \rightarrow S^1$, the preimage $X_1 = g^{-1}(p)$ is a bicollared, oriented, proper $(n-1)$ -submanifold of \tilde{X} for any non-vertex point p of S^1 . Then, we see that the inclusion $i: (X_1, \partial X_1) \subset (\tilde{X}, \partial \tilde{X})$ sends the fundamental class of X_1 to a generator of $T_{n-1}(\tilde{X}, \partial \tilde{X}; F)$ for any F . [*Proof.* Let X' be a manifold obtained from X by splitting along X_1 . X' is imbedded canonically in \tilde{X} so that $\partial X' = X_1 \cup (X' \cap \partial X) \cup -tX_1$. This implies that $(1-t)[X_1] = [X_1] = t[X_1] = 0$ in $H_{n-1}(\tilde{X}, \partial \tilde{X}; F)$, i.e., $[X_1] \in T_{n-1}(\tilde{X}, \partial \tilde{X}; F)$. $[X_1] \neq 0$ in $H_{n-1}(X, \partial X; F)$ and hence in $H_{n-1}(\tilde{X}, \partial \tilde{X}; F)$, since it is the Poincaré dual of $g^*[S^1] \in H^1(X; F)$. Thus, $[X_1]$ generates $T_{n-1}(\tilde{X}, \partial \tilde{X}; F) \cong F$.] Let \hat{X}_1 be the interior oriented connected sum of the components of X_1 . Since \tilde{X} is connected, we can construct from i a map $\hat{i}: (\hat{X}_1, \partial \hat{X}_1) \rightarrow (\tilde{X}, \partial \tilde{X})$ such that $\hat{i}_*H_{n-1}(\hat{X}_1, \partial \hat{X}_1; F) = T_{n-1}(\tilde{X}, \partial \tilde{X}; F)$. From this observation and the theorem, we see the following:

COROLLARY 1. $\dim_F H_q(X_1; F) \geq \dim_F T_q(\tilde{X}; F)$ for all q and F . If $i_*H_q(X_1; F) \subset T_q(\tilde{X}; F)$ for some q and some F , then $i_*H_q(X_1; F) = T_q(\tilde{X}; F)$.

In knot theory this corollary gives a general relation between the homology of a Seifert manifold of a knot (or link) and its knot (or link) module (associated with an infinite cyclic covering). For a classical knot (i.e., 1-knot) k , this has been recognized as (the genus of k) $\geq (1/2) \cdot$ (the degree of the knot polynomial of k). (Cf. H. Seifert [9].)

Next, suppose X is orientable and $H_1(X; \mathbf{Z}) \cong \mathbf{Z}$. Such a manifold occurs, for example, as the complement of an open regular neighborhood of a closed connected orientable $(n-2)$ -manifold imbedded piecewise-linearly in S^{n+2} . By Poincaré duality $H_{n-1}(X, \partial X; \mathbf{Z}) \cong \mathbf{Z}$.

COROLLARY 2. If there is a map $f: (M, \partial M) \rightarrow (X, \partial X)$ inducing an isomorphism $f_*: H_{n-1}(M, \partial M; \mathbf{Z}) \cong H_{n-1}(X, \partial X; \mathbf{Z})$ and a 0-map $f_* = 0: H_1(M; \mathbf{Z}) \rightarrow H_1(X; \mathbf{Z})$, then

$$\dim_F H_q(M; F) \geq \dim_F T_q(\tilde{X}; F)$$

for all q and F .

To see this, note that $H_{n-1}(\tilde{X}, \partial \tilde{X}; \mathbf{Z}) \cong \mathbf{Z}$ and t acts trivially on it and the covering projection $\tilde{X} \rightarrow X$ induces an isomorphism $H_{n-1}(\tilde{X}, \partial \tilde{X}; \mathbf{Z}) \cong H_{n-1}(X, \partial X; \mathbf{Z})$. This follows from [3, Theorem 2.3], (or its topological version [4]) and the Wang exact sequence. So, it suffices to show that f has a lifting to \tilde{X} . This is clear,

however, by the assumption that $f_*: H_1(M; \mathbf{Z}) \rightarrow (X; \mathbf{Z})$ is a 0-map.

For the following application, spaces and maps are considered in the piecewise-linear category. Let L be a trivial n -link in S^{n+2} of some $r + 1$ components and a collection $\{B_1, \dots, B_r\}$ of r $(n + 1)$ -balls imbedded locally-flatly and mutually disjointly in S^{n+2} such that for each i B_i spans L as 1-handle i.e., $B_i \cap L = (\partial B_i) \cap L =$ the disjoint union of two n -balls. An n -knot K in S^{n+2} is called a *ribbon n -knot* if it is obtained from such an L and a $\{B_1, \dots, B_r\}$ by doing an imbedded surgery. (Cf. T. Yanagawa [12], R. Hitt [1].) The knot K is often said to be a *fusion of the link L along 1-handles $\{B_1, \dots, B_r\}$* .

COROLLARY 3. *Let $n \geq 3$. A ribbon n -knot K is unknotted, if $\pi_1(S^{n+2} - k) \cong \mathbf{Z}$.*

To see this, note that any ribbon n -knot has a Seifert $(n + 1)$ -manifold M , homeomorphic to a manifold of the form $\#^m S^1 \times S^n$ -Int B^{n+1} (B^{n+1} is an $(n + 1)$ -ball.) ([12], [1]). Let $X = S^{n+2}$ -Int $N(K)$, $N(K)$ being a regular neighborhood of K in S^{n+2} . The manifold $X \cap M (\cong M)$ gives a generator of $H_{n+1}(X, \partial X; \mathbf{Z}) = \mathbf{Z}$ and the inclusion $X \cap M \subset X$ induces a 0-map on H_1 . By Corollary 2, $T_i(\tilde{X}; F) = 0, i \neq 0, 1, n$. (Of course, one can also apply Corollary 1 to obtain this.) But $T_*(\tilde{X}; F) = H_*(\tilde{X}; F)$. As a result, $\tilde{H}_*(\tilde{X}; F) = 0$ by using Milnor duality [8] or [5, Duality Theorem (II)], since \tilde{X} is simply connected. Then by taking $F = \mathbf{Q}$, we see that $\tilde{H}_*(\tilde{X}; \mathbf{Z})$ is a torsion group. Next, by taking $F = \mathbf{Z}_p, p$ prime, and considering the universal coefficient theorem, the torsion product $\text{Tor}_{\mathbf{Z}}(H_{*-1}(\tilde{X}; \mathbf{Z}), \mathbf{Z}_p) = 0$. This shows that $\tilde{H}_*(\tilde{X}; \mathbf{Z}) = 0$ and X has the homotopy type of S^1 . By [6], [10], [11], K is unknotted for $n \geq 3$.

Note 2. For $n = 2$, a corresponding result is proved by Y. Marumoto [7] in the simplest case, that is, the case of L having two components. However, a general case is unknown.

2. Proof of theorem. Let $i: T_* \subset H_*$. i induces an epimorphism $i^*: H^* \rightarrow T^*$. Let $x \in H^q(\tilde{X}; F)$ such that $i^*(x) \neq 0$. By [5, Duality Theorem (II)], the cup product $H^q(\tilde{X}; F) \times H^{n-q-1}(\tilde{X}, \partial \tilde{X}; F) \xrightarrow{\cup} H^{n-1}(\tilde{X}, \partial \tilde{X}; F)$ induces a nonsingular pairing $T^q(\tilde{X}; F) \times T^{n-q-1}(\tilde{X}, \partial \tilde{X}; F) \rightarrow T^{n-1}(\tilde{X}, \partial \tilde{X}; F)$, also denoted by \cup . Hence we find an element $y \in H^{n-q-1}(\tilde{X}, \partial \tilde{X}; F)$ such that $i^*(x) \cup i^*(y) = i^*(x \cup y) \neq 0$. By assumption, $f: (M, \partial M) \rightarrow (\tilde{X}, \partial \tilde{X})$ induces the following commutative triangle

$$\begin{array}{ccc}
 H^{n-1}(\tilde{X}, \partial\tilde{X}; F) & & \\
 \downarrow i^* & \searrow f^* & \\
 & & H^{n-1}(M, \partial M; F) \\
 & \nearrow f^* & \\
 T^{n-1}(\tilde{X}, \partial\tilde{X}; F) & &
 \end{array}$$

and $f^*: T^{n-1}(\tilde{X}, \partial\tilde{X}; F) \rightarrow H^{n-1}(M, \partial M; F)$ is an isomorphism. Thus, $f^*(x \cup y) = f^*(x) \cup f^*(y) \neq 0$, so that $f^*(x) \neq 0$. We obtain a (non-canonical) monomorphism $r: T^q(\tilde{X}; F) \rightarrow H^q(M; F)$. Hence, $\dim_F T_q(\tilde{X}; F) = \dim_F T^q(\tilde{X}; F) \leq \dim_F H^q(M; F) = \dim_F H_q(M; F)$. If $f_*H_q(M; F) \subset T_q(\tilde{X}; F)$, then we may replace r by a canonical epimorphism $r': T^q(\tilde{X}; F) \rightarrow \text{Hom}_F[f_*H_q(M; F), F]$ composed with the natural inclusion into $H^q(M; F)$. Since r' is an isomorphism, we see that $f_*H_q(M; F) = T_q(\tilde{X}; F)$. This completes the proof of the theorem.

3. Alternative proof of Corollary 3. We now describe a different, somewhat geometric proof of Corollary 3. This method, as a matter of fact, has been earlier obtained and is near to the argument of [2]. Let $T(m)$ be an n -manifold homeomorphic to $\sharp^m S^1 \times S^{n-1}$ and imbedded locally-flatly in S^{n+2} . (The following four lemmas are true when $n \geq 2$.) For $m = 0$, $T(m)$ is an n -sphere, i.e., an n -knot. Such a $T(m)$ is *unknotted* if it bounds a manifold locally-flatly imbedded in S^{n+2} and homeomorphic to a disk sum $\sharp^m S^1 \times B^n$. As an analogous argument to [2, Theorem 1.2], we have the following:

3.1 *Any two unknotted $T(m)_1, T(m)_2$ are ambient isotopic.*

Thus, the following is obtained:

3.2. *If $T(m)$ is unknotted, $S^{n+2} - T(m)$ is homotopy equivalent to a bouquet $S^1 \vee S^2 \vee \dots \vee S^2 \vee S^n \vee \dots \vee S^n$ of one 1-sphere, m 2-spheres and m n -spheres. [Regard $T(m)$ as the common boundary of $\sharp^m S^1 \times B^n$ and $\sharp^m B^2 \times S^{n-1}$ whose union forms an unknotted $(n+1)$ -sphere S_0^{n+1} in S^{n+2} . Then, $S^{n+2} - T(m)$ is homotopy equivalent to the suspension of $S_0^{n+1} - T(m)$.]*

3.3. *Let $T(m+1)$ and $T(m+1)'$ be the manifolds obtained from the same $T(m)$ by surgeries along 1-handles B^{n+1} and B'^{n+1} on $T(m)$ imbedded locally-flatly in S^{n+2} , respectively. If $\pi_1(S^{n+2} - T(m)) \cong \mathbf{Z}$, then $T(m+1)$ and $T(m+1)'$ are ambient isotopic.*

This is proved easily as an analogy to [2, Lemma 2.7].

From 3.3 and the definition of ribbon knots, we see the following:

3.4. For any ribbon n -knot K obtained from $(m + 1)$ balls and m 1-handles, the surgery along some standard mutually disjoint m 1-handles on K imbedded locally-flatly in S^{n+2} produces an unknotted $T(m)$. Further, if $\pi_1(S^{n+2} - K) \cong \mathbf{Z}$, then $T(m)$ is ambient isotopic to a knot sum $K\#T(m)'$ for some unknotted $T(m)'$.

Now assume $\pi_1(S^{n+2} - K) \cong \mathbf{Z}$. In 3.4, let $E = S^{n+2} - K$, $X = S^{n+2} - T(m)$ and $X' = S^{n+2} - T(m)'$. Take their infinite cyclic connected covers. We have $\tilde{H}_*(\tilde{E}; \mathbf{Z}) \oplus \tilde{H}_*(\tilde{X}'; \mathbf{Z}) \cong \tilde{H}_*(\tilde{X}; \mathbf{Z})$ as $\mathbf{Z}\langle t \rangle$ -modules. By 3.2, $\tilde{H}_*(\tilde{X}'; \mathbf{Q})$ and $\tilde{H}_*(\tilde{X}; \mathbf{Q})$ are free $\mathbf{Q}\langle t \rangle$ -modules of the same rank, so that $\tilde{H}_*(\tilde{E}; \mathbf{Q}) = 0$, i.e., $\tilde{H}_*(\tilde{E}; \mathbf{Z})$ is a torsion group. By 3.2 again, $\tilde{H}_*(\tilde{X}'; \mathbf{Z})$ and $\tilde{H}_*(\tilde{X}; \mathbf{Z})$ are free abelian, hence $\tilde{H}_*(\tilde{E}; \mathbf{Z}) = 0$ and E has the homotopy type of S^1 . By [6], [10], [11], K is unknotted for $n \geq 3$.

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