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This paper concerns fixed point theorems for nonexpansive mappings in conjugate Banach spaces. An example shows that there exist fixed-point-free affine isometries on weak* compact convex sets. Asymptotic centers of decreasing net of founded sets in l^1 are shown to be compact and a common fixed point theorem for left reversible topological semigroup of non-expansive mappings in l^1 is given.

1. Introduction. Let K be a nonempty weakly compact convex subset of a Banach space and $T: K \to K$ a nonexpansive mapping, i.e., $\|Tx - Ty\| \leqq \|x - y\|$, $x, y \in K$. A theorem of Kirk [10] (see also Browder [1], Gödhe [6]) states that if K has normal structure then T has a fixed point. Whether the condition of normal structure is essential remains an open problem, although Schöneberg [13] has shown that some weakenings of normal structure suffice. With a slight modification of normal structure, Kirk's proof of his theorem also yields the following theorem in conjugate Banach spaces.

THEOREM 1 (*Kirk*). *Let K be a nonempty weak* compact convex subset of a conjugate Banach space and assume that K possesses weak* normal structure* (*see Definition* 1 *in* §3). *Then every nonexpansive selfmapping of K has a fixed point.*

One major observation presented in this note is that the condition of weak* normal structure in Theorem 1 is essential, even for affine isometries. We also derive a sufficient condition for a conjugate Banach space to have weak* normal structure. In particular, we show that l_1 possesses weak* normal structure. Asymptotic centers of decreasing nets of bounded subsets in l_1 are shown to form a normcompact nonempty subset and an application of this result is made to obtain a common fixed point theorem for families of nonexpansive mappings in l_1.

2. A counterexample. Let c_0 be the space of null sequences, equipped with the sup norm $\| \ \|_\infty$, $\|x\|_\infty = \sup_{i\geqq 1}|x_i|$, and l_1 the space of absolutely summable sequences equipped with the norm $\| \ \|_1$, $\|x\|_1 = \sum_{i=1}^\infty |x_i|$. For each sequence x, let x^+ and x^- be the positive and negative part of x, respectively. Renorm c_0 by the new norm defined by

$$|x| = \|x^+\|_\infty + \|x^-\|_\infty .$$

$|\cdot|$ is equivalent to $\|\cdot\|_\infty$ since $\|x\|_\infty \leqq |x| \leqq 2\|x\|_\infty$. This method of renorming was used by Bynum [4] to renorm l_p, $1 < p < \infty$.

LEMMA 1. *The dual of* $(c_0, |\cdot|)$ *is isometrically isomorphic to* $(l_1, \|\cdot\|)$ *with the norm* $\|\cdot\|$ *defined by*

$$\|x\| = \max(\|x^+\|_1, \|x^-\|_1) .$$

Proof. Since $|\cdot|$ is equivalent to $\|\cdot\|_\infty$, the dual of $(c_0, |\cdot|)$ is representable by l_1. It suffices to show that

$$\max(\|f^+\|_1, \|f^-\|_1) = \sup \left\{\sum_{i=1}^{\infty} x_i f_i \colon x \in c_0, \|x^+\|_\infty + \|x^-\|_\infty \leqq 1\right\}$$

for each $f = (f_i) \in l_1$. Note that the supremum on the right can be taken over x satisfying the further requirment that $x_i f_i \geqq 0$ for all i. (If $x_i f_i < 0$, replace x by another one with $x_i = 0$.) It then follows that

$$\sum_{i=1}^{\infty} x_i f_i \leqq \|x^+\|_\infty \|f^+\|_1 + \|x^-\|_\infty \|f^-\|_1$$
$$\leqq \max(\|f^+\|_1, \|f^-\|_1) .$$

For the reverse inequality, note that one can approximate $\|f^+\|_1$ $(\|f^-\|_1)$ by $\sum_{i=1}^{\infty} x_i f_i$ by suitably choosing $x_i = 1$ or 0 $(-1$ or $0)$.

EXAMPLE 1. Let $K = \{(x_i) \in l_1 \colon x_i \geqq 0, \sum_{i=1}^{\infty} x_i \leqq 1\}$. K is a weak* compact convex set in $(l_1, \|\cdot\|)$ since it is the intersection of the unit ball and the weak* closed set $\{(x_i) \colon x_i \geqq 0\}$. Let $T \colon K \to K$ be the mapping defined by the equation

$$Tx = \left(1 - \sum_{i=1}^{\infty} x_i, x_1, x_2, \cdots, x_n, \cdots\right)$$

for $x = (x_i) \in K$. We show that T is an isometry. Let $x, y \in K$ and let $I = \{i \in \boldsymbol{Z}^+ \colon x_i - y_i \geqq 0\}$ and $J = \{j \in \boldsymbol{Z}^+ \colon x_j - y_j < 0\}$. Assume that $\sum_{i \in I} x_i - y_i \geqq \sum_{j \in J} y_j - x_j$. Then $\|x - y\| = \sum_{i \in I} x_i - y_i$ and

$$\begin{aligned}\|Tx - Ty\| &= \left\|\sum_{i=1}^{\infty} (y_i - x_i), x_1 - y_1, \cdots, x_n - y_n, \cdots\right\| \\ &= \|\sum_{j \in J} (y_j - x_j) - \sum_{i \in I} (x_i - y_i), x_1 - y_1, \cdots, x_n - y_n, \cdots\| \\ &= \max(\sum_{i \in I} x_i - y_i, \sum_{i \in I} x_i - y_i) \\ &= \sum_{i \in I} x_i - y_i = \|x - y\| .\end{aligned}$$

Similarly, we also have $\|Tx - Ty\| = \|x - y\|$ in case $\sum_{i \in I} x_i - y_i \leqq \sum_{j \in J} y_j - x_j$. Hence T is an isometry. T is clearly affine and fixed point free. Further properties of K and T are listed in the following:

(1) $\lim \|y - T^n x\| = \mathrm{Diam}(K) = 1$, $y, x \in K$.

(2) K does not possess weak* normal structure. This is necessarily true by Theorem 1 and the above demonstration.

(3) $T^n x$ converges weakly* to zero for each $x \in K$.

(4) K itself is a minimal T-invariant weak* compact convex set. Indeed every T-invariant weak* comyact convex subset C of K must contain 0 by (3). Hence $T^n(0) = e_n \in C$ for all n. Therefore $K = \overline{\mathrm{Co}}(\{e_n\} \cup \{0\}) \subseteqq C$ and $C = K$.

The above example shows that the condition of weak* normal structure cannot be removed from Theorem 1 even if the nonexpansive mapping is an affine isometry. In contrast, every affine nonexpansive selfmapping of a weakly compact convex set always has a fixed point.

3. Conjugate Banach spaces having weak normal structure.

In this section we derive a condition for a conjugate Banach space to have weak* normal structure.

DEFINITION 1. A weak* closed convex subset C of a conjugate Banach space is said to have weak* normal structure if every weak* compact convex subset K of C containing more than one point contains a point x_0 such that

$$\sup\{\|x_0 - y\| : y \in K\} < \mathrm{diam}\, K .$$

In the following theorem, $\boldsymbol{R}^+ = \{r \in \boldsymbol{R} : r \geqq 0\}$ and the notation $x_n \xrightarrow{*} y$ will denote the weak* convergence of x_n to y.

THEOREM 2. *Let X be a the conjugate space of a separable Banach space. Suppose that there exists a function $\delta: \boldsymbol{R}^+ \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ satisfying the following conditions.*

(i) *For each fixed s, $\delta(r, s)$ is continuous and strictly increasing in r,*

(ii) *$\delta(s, s) > s$ for every $s > 0$,*

(iii) *if $x_n \xrightarrow{*} 0$ and $\lim \|x_n\| = s > 0$, then*

$$\lim \|y - x_n\| = \delta(\|y\|, s) \quad \textit{for every } y \in K .$$

Then every weak closed convex subset of X has weak* normal structure.*

Proof. Suppose on the contrary that X contains a weak* closed convex subset C which does not have weak* normal structure. Then there exists a weak* compact convex subset K of C with Card $K > 1$ and for every $x \in K$

$$\sup\{\|x - y\|: y \in K\} = \operatorname{diam} K = d > 0 .$$

By a method of Brodskii-Milman [3], there exists a sequence $\{x_n\} \subset K$ such that $\lim d(x_{n+1}, \mathrm{Co}(x_i)_{i \leqq n}) = d$. Since subsequences of $\{x_n\}$ share the same property, we may assume that $x_n \xrightarrow{*} x_0$ for some $x_0 \in K$ and $\lim \|x_n - x_0\| = s$. Clearly, $s > 0$. For each fixed m, we have $\lim_n \|x_m - x_n\| = d$. Therefore, by (iii)

$$\mathrm{d} = \lim_n \|(x_n - x_0) - (x_m - x_0)\| = \delta(\|x_m - x_0\|, s) .$$

Using (i), $d = \delta(s, s)$. Using (ii), we have $s < d$. We shall show that $\sup\{\|x_0 - y\|: y \in K\} \leqq s$. Suppose not, then there exists $z \in K$ with $\|z - x_0\| > s$. Then

$$\begin{aligned}\lim \|z - x_n\| &= \lim \|(z - x_0) - (x_n - x_0)\| \\ &= \delta(\|z - x_0\|, s) \\ &> \delta(s, s) = d\end{aligned}$$

by (iii) and (i). This is impossible. Therefore, $\sup\{\|x_0 - y\|: y \in K\} \leqq s < d$, which again contradicts our initial assumption. Hence C has weak* normal structure.

The next proposition shows that the spaces l_p, $p \geqq 1$ satisfy the condition in Theorem 2 with $\delta(r, s) = (r^p + s^p)^{1/p}$.

PROPOSITION 1. *In* l_p, *if* $x_n \xrightarrow{*} x$, *then for every* $y \in l_p$,

$$(1) \qquad \limsup \|x_n - y\|^p = \limsup \|x_n - x\|^p + \|x - y\|^p .$$

In particular, if $\lim \|x_n - x\|$ *exists, we have*

$$\lim \|x_n - y\| = (\lim \|x_n - x\|^p + \|x - y\|^p)^{1/p} .$$

Proof. For $p = 1$, the equality is a special case of a more general equality given in Proposition 2; see Corollary 3. For $p > 1$, let $J: l_p \to l_q$, $1/q + 1/p = 1$, be the duality mapping defined by

$$Jx = (|x_1|^{p-1} \operatorname{sgn} x_1, \cdots, |x_n|^{p-1} \operatorname{sgn} x_n, \cdots) .$$

J is weakly continuous and $\langle Jx, x\rangle = \|x\|^p$, see [2]. Since J is the subdifferential of the convex function $f(x) = 1/p\,\|x\|^p$, we have

$$\frac{1}{p}\|x_n - y\|^p = \frac{1}{p}\|x_n - x\|^p + \int_0^1 \langle J(x_n - x + t(x - y)), x - y\rangle dt$$

(Gossez-Lami-Dozo [7]). Therefore

$$\begin{aligned}\limsup \|x_n - y\|^p &= \limsup \|x_n - x\|^p + p\int_0^1 t^{p-1}\|x - y\|^p dt \\ &= \limsup \|x_n - x\|^p + \|x - y\|^p .\end{aligned}$$

Proposition 1 and Theorem 2 implies that every weak* closed convex subset of l_1 has weak* normal structure. Note that such a set may not possess normal structure. For a simple example, let C be the unit ball and $K = \{(x_i): x_i \geqq 0, \sum_{i=1}^{\infty} x_i = 1\}$. Then K is closed convex and $\sup\{\|x - y\|: y \in K\} = \operatorname{diam} K = 2$ for every $x \in K$. Combining this result with Theorem 1 we have the following result of Karlovitz [9].

COROLLARY 1 [9]. *Let K be a weak* compact convex nonempty subset of l_1 and $T: K \to K$ be a nonexpansive mapping. Then T has a fixed point.*

4. Asymptotic centers in l_1.

DEFINITION 2 [12]. Let C be a nonempty subset of a Banach space X and $\{B_\alpha: \alpha \in \Lambda\}$ a decreasing net of bounded nonempty subsets of X. For each $x \in C$ and $\alpha \in \Lambda$, let

$$r_\alpha(x) = \sup\{\|x - y\|: y \in B_\alpha\} ,$$
$$r(x) = \lim_\alpha r_\alpha(x) = \inf_\alpha r_\alpha(x) ,$$

and

$$r = \inf\{r(x): x \in C\} .$$

The set (possibly empty) $\mathscr{AC}$ $(\{B_\alpha: \alpha \in \Lambda\}, C) = \{x \in C: r(x) = r\}$ and the number r will be called, respectively, the asymptotic center of $\{B_\alpha: \alpha \in \Lambda\}$ w.r.t. C and the asymptotic radius of $\{B_\alpha: \alpha \in \Lambda\}$ w.r.t. C.

PROPOSITION 2. *Let $\{B_\alpha: \alpha \in \Lambda\}$ be a decreasing net of bounded subsets of l_1 and y_n a weak* convergent sequence with weak* limit y. Then*

$$\begin{aligned}\limsup_\alpha \{\|y - x\|: x \in B_\alpha\} &+ \limsup_n \|y_n - y\| \\ &= \limsup_n \limsup_\alpha \{\|y_n - x\|: x \in B_\alpha\} .\end{aligned} \tag{2}$$

Proof. For $x \in l_1$, we shall denote by $x^{(i)}$ the ith coordinate of x.

By the triangle inequality, we clearly have the inequality $\geqq$ in (2). By a simple diagonal process, we may assume that $\{B_\alpha: \alpha \in \Lambda\}$

is a decreasing sequence $\{B_n: n \geqq 1\}$ of bounded sets. Choose $x_n \in B_n$ such that $\limsup \|y - x_n\| = \limsup_n\{\|y - x\|: x \in B_n\}$. It follows that it suffices to prove the following inequality:

$$\limsup_n \|y - x_n\| + \limsup_m \|y_m - y\| \leqq \limsup_m \limsup_n \|y_m - x_n\| .$$

We may also assume, without loss of generality, that $y = 0$, and that $\lim \|x_n\|$, $\lim \|y_m\|$, and $\lim_m \limsup_n \|y_m - x_n\|$ exist.

Let $r = \lim_m \limsup_n \|y_m - x_n\|$ and $k = \lim \|y_m\|$. Suppose, on the contrary that $\lim \|x_n\| = r - k + p$ for some $p > 0$. Let $p > \varepsilon > 0$. Let m_1, N_1 and M_1 (N_1 and M_1 depend on m_1) be sufficiently large integers such that

$$\|y_{m_1}\| \geqq k - \frac{\varepsilon}{4},$$

$$\sum_{N_1+1}^{\infty} |y_{m_1}^{(i)}| \leqq \frac{\varepsilon}{8}$$

$$\|x_n - y_{m_1}\| \leqq r + \frac{\varepsilon}{4},$$

and

$$\|x_n\| \geqq r - k + p - \frac{\varepsilon}{4}, \quad \text{for all} \quad n \geqq M_1 .$$

Then for $n \geqq M_1$, we have

$$\begin{aligned} r + \frac{\varepsilon}{4} \geqq \|x_n - y_{m_1}\| &= \sum_1^{N_1} |x_n^{(i)} - y_{m_1}^{(i)}| + \sum_{N_1+1}^{\infty} |x_n^{(i)} - y_{m_1}^{(i)}| \\ &\geqq \sum_1^{N_1} |y_{m_1}^{(i)}| - \sum_1^{N_1} |x_n^{(i)}| + \sum_{N_1+1}^{\infty} |x_n^{(i)}| - \sum_{N_1+1}^{\infty} |y_{m_1}^{(i)}| \\ &= \|y_{m_1}\| - 2\sum_{N_1+1}^{\infty} |y_{m_1}^{(i)}| + \|x_n\| - 2\sum_1^{N_1} |x_n^{(i)}| \\ &\geqq k - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} + r - k + p - \frac{\varepsilon}{4} - 2\sum_1^{N_1} |x_n^{(i)}| . \end{aligned}$$

Hence

$$\sum_1^{N_1} |x_n^{(i)}| \geqq \frac{1}{2}(p - \varepsilon), \quad n \geqq M_1 .$$

Since $y_m \xrightarrow{*} 0$ there exist m_2, $N_2 > N_1$ and $M_2 > M_1$ (N_2 and M_2 depend on m_2) such that

$$\sum_1^{N_1} |y_{m_2}^{(i)}| \leqq \frac{\varepsilon}{10},$$

$$\|y_{m_2}\| \geqq k - \frac{\varepsilon}{5},$$

$$\sum_{N_2+1}^{\infty} |y_{m_2}^{(i)}| \leqq \frac{\varepsilon}{10},$$

$$||x_n - y_{m_2}|| \leqq r + \frac{\varepsilon}{5},$$

and

$$||x_n|| \geqq r - k + p - \frac{\varepsilon}{5}, \quad \text{for} \quad n \geqq M_2 .$$

Then for $n \geqq M_2$, we have

$$\begin{aligned} r + \frac{\varepsilon}{5} \geqq ||x_n - y_{m_2}|| &= \sum_1^{N_1} |x_{n_1}^{(i)} - y_{m_2}^{(i)}| + \sum_{N_1+1}^{N_2} |x_n^{(i)} - y_{m_2}^{(i)}| + \sum_{N_2+1}^{\infty} |x_n^{(i)} - y_{m_2}^{(i)}| \\ &\geqq \sum_1^{N_1} |x_n^{(i)}| - \sum_1^{N_1} |y_{m_2}^{(i)}| + \sum_{N_1+1}^{N_2} |y_{m_2}^{(i)}| - \sum_{N_1+1}^{N_2} |x_n^{(i)}| \\ &\qquad + \sum_{N_2+1}^{\infty} |x_n^{(i)}| - \sum_{N_2+1}^{\infty} |y_{m_2}^{(i)}| \\ &= ||y_{m_2}|| - 2\sum_1^{N_1} |y_{m_2}^{(i)}| - 2 \sum_{N_2+1}^{\infty} |y_{m_2}^{(i)}| \\ &\qquad + ||x_n|| - 2 \sum_{N_1+1}^{N_2} |x_n^{(i)}| \\ &\geqq k - \frac{\varepsilon}{5} - \frac{\varepsilon}{5} - \frac{\varepsilon}{5} + r - k + p - \frac{\varepsilon}{5} - 2 \sum_{N_1+1}^{N_2} |x_n^{(i)}| . \end{aligned}$$

Hence

$$\sum_{N_1+1}^{N_2} |x_n^{(i)}| \geqq \frac{1}{2}(p - \varepsilon) \quad \text{for} \quad n \geqq M_2 .$$

Continuing in this way, we obtain two sequences $M_1 < M_2 < \cdots$ and $N_1 < N_2 < \cdots$ such that for $n \geqq M_k$,

$$\sum_{N_{k-1}+1}^{N_k} |x_n^{(i)}| \geqq \frac{1}{2}(p - \varepsilon) , \quad N_0 = 0 .$$

Thus for $n \geqq M_k$, $||x_n|| \geqq \sum_1^{N_k} |x_n^{(i)}| \geqq k\cdot 1/2(p - \varepsilon)$. This contradicts the boundedness of the sequence x_n.

COROLLARY 2. *Let x_n be a bounded sequence in l_1 and $y_n \xrightarrow{*} y$. Then*

$$\limsup_n ||x_n - y|| + \limsup_m ||y_m - y|| = \limsup_m \limsup_n ||x_n - y_m|| .$$

COROLLARY 3. *Proposition 1 for $p = 1$.*

THEOREM 3. *Let C be a weak* closed convex nonempty subset of l_1 and $\{B_\alpha : \alpha \in \Lambda\}$ a decreasing net of bounded nonempty subsets* *of C. Let the function $r(x)$ be defined as in Definition 2. Then for each $s \geqq 0$, $\{x \in C: r(x) \leqq s\}$ is weak* compact convex and the asymptotic center of $\{B_\alpha: \alpha \in \Lambda\}$ w.r.t. C is a nonempty (norm) compact convex subset of C.*

Proof. Let $K_s = \{x \in C: r(x) \leqq s\}$ and let K be the asymptotic center. Clearly, $\mathrm{diam}\,(K_s) \leqq 2s$. Since $r(\cdot)$ is a convex function, K_s is also convex. To show that K is weak* compact, it suffices to sohw that K_s is weak* closed. Let $y_n \in K_s$ and $y_n \xrightarrow{*} y$. By Proposition 2.

$$(3) \qquad r(y) = \limsup r(y_n) - \limsup \|y_n - y\| \leqq s\ .$$

Hence $y \in K_s$ and K_s is weak* closed. Suppose now that $s = r$, where r is the asymptotic radius of $\{B_\alpha: \alpha \in \Lambda\}$ w.r.t. C. If $r(y_n) = r$, then we must have $\limsup \|y_n - y\| = 0$ for otherwise $r(y) < r$, a contradiction to the definition of r. Therefore, for a sequence in K, weak* convergence implies norm convergence. Hence K is compact. Since $K = \bigcap\{K_s: K_s \neq \varnothing\}$ and each K_s is nonempty weak* compact, we have $K \neq \varnothing$.

COROLLARY 4. *Let C be a weak* closed convex subset of l_1 and D a nonempty bounded subset of C. Then the Chebyshev center of D w.r.t. C is nonempty compact convex. In particular, for any two points x and y, the set $\{z \in l_1: \|z - x\| = \|z - y\| = 1/2\|x - y\|\}$ is compact.*

Proof. If we let $B_\alpha = D$ for every $\alpha \in \Lambda$, the asymptotic center of $\{B_\alpha: \alpha \in \Lambda\}$ is the same as the Chebyshev center of D.

We conclude this section by giving an application of Theorem 3. Let K be a set and S a semigroup of selfmaps of K. S is said to be a topological semigroup if S is equipped with a Hausdorff topology such that for each $a \in S$, the two mappings from S into S defined by $s \to as$ and $s \to sa$ for all $s \in S$, are continuous. S is said to be left reversible if any two nonempty closed right ideals of S have nonempty intersection (cf. [5, p. 34]). If K is a topological space and S a left reversible topological semigroup of selfmappings of K suce that the mapping $(s, x) \to s(x)$ is separately continuous, then S becomes a directed set if we define $a \geqq b$ if and only if $aS \subseteqq \mathrm{cl}(bS)$. Moreover, if for a fixed element $u \in K$, we define $W_s = \mathrm{cl}(sS(u)))$ for all $s \in S$, then the family $\{W_s: s \in S\}$ is a decreasing net of subsets of K (see [8]).

THEOREM 4. *Let C be a weak* closed convex nonempty subset of l_1 and S a left reversible topological semigroup of nonexpansive selfmappings of C such that the mapping $(s, x) \to s(x)$ is separately continuous. If for some $x \in C$, $s \in S$, $sS(x)$ is bounded, then S has a common fixed point in C.*

Proof. Let W_s be defined as in the last paragraph. By Theorem 2 in [12], the asymptotic center K of $\{W_s: s \in S\}$ is a S-invariant subset of C. By Theorem 4, K is a nonempty compact convev set. Since a compact convex set has normal structure, by Theorem 3 in [12] or Corollary 1 in [8], S has a common fixed point in K.

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