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A NOTE ON THE KONHAUSER SETS OF BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

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A NOTE ON THE KONHAUSER SETS OF BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

H. M. SRIVASTAVA

Two new classes of bilateral generating functions are given for the Konhauser polynomials $Y_n^\alpha(x; k)$, which are biorthogonal to the Konhauser polynomials $Z_n^\alpha(x; k)$ with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0, \infty)$, $\alpha > -1$ and $k = 1, 2, 3, \cdots$. The bilateral generating functions (1) and (2) below would reduce, when $k = 1$, to similar results for the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$. Furthermore, for $k = 2$, these formulas yield the corresponding properties of the Preiser polynomials.

It is also shown how the bilateral generating function (2) can be applied to derive a new generating function for the product

$$Y_n^{\alpha - kn}(x; k) Z_n^\beta(y; l) ,$$

where $\alpha, \beta > -1$, $k, l = 1, 2, 3, \cdots$, and $n = 0, 1, 2, \cdots$.

1. **Introduction.** Recently, Joseph D. E. Konhauser discussed two polynomial sets $\{Y_n^\alpha(x; k)\}$ and $\{Z_n^\alpha(x; k)\}$, which are biorthogonal with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0, \infty)$, where $\alpha > -1$ and k is a positive integer. For the polynomials $Y_n^\alpha(x; k)$, the following bilateral generating functions are derived in this paper:

$$\begin{aligned} &\sum_{n=0}^{\infty} Y_{m+n}^{\alpha}(x; k) \zeta_n(\boldsymbol{y}; z) t^n \\ &\quad = (1 - t)^{-m-(\alpha+1)/\kappa} \exp\left(x[1 - (1 - t)^{-1/k}]\right) \\ &\qquad \cdot F[x(1 - t)^{-1/k}; \boldsymbol{y}; zt^q/(1 - t)^q] \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty} Y_{m+n}^{\alpha-kn}(x; k) \zeta_n(\boldsymbol{y}; z) t^n \\ &\quad = (1 + t)^{-1+(\alpha+1)/k} \exp\left(x[1 - (1 + t)^{1/k}]\right) \\ &\qquad \cdot G[x(1 + t)^{1/k}; \boldsymbol{y}; zt^q/(1 + t)^q] , \end{aligned} \tag{2}$$

where

$$F[x; \boldsymbol{y}; t] = \sum_{n=0}^{\infty} \lambda_n Y_{m+qn}^{\alpha}(x; k) \sigma_n(\boldsymbol{y}) t^n , \tag{3}$$

$$G[x; \boldsymbol{y}; t] = \sum_{n=0}^{\infty} \lambda_n Y_{m+qn}^{\alpha-kqn}(x; k) \sigma_n(\boldsymbol{y}) t^n , \tag{4}$$

$\lambda_n \neq 0$ are arbitrary constants, $m \geqq 0$ and $q \geqq 1$ are integers, $\sigma_n(\boldsymbol{y})$ is a nonvanishing function of N variables $y_1, \cdots, y_N$, $N \geqq 1$, and

$$\zeta_n(\boldsymbol{y}; z) = \sum_{r=0}^{[n/q]} \binom{m+n}{n-qr} \lambda_r \sigma_r(\boldsymbol{y}) z^r . \tag{5}$$

For $k = 2$, our formulas (1) and (2) yield the corresponding properties of the polynomials introduced earlier by Preiser [5]. When $k = 1$, both (1) and (2) would reduce to similar bilateral generating functions for the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, since (cf. [2] and [3])

$$Y_n^{\alpha}(x; 1) \equiv L_n^{(\alpha)}(x), \quad \alpha > -1 . \tag{6}$$

Furthermore, in its special case when $m = 0$, $q = 1$ and $\sigma_n(\boldsymbol{y}) \equiv 1$, $n \geqq 0$, (1) would give us our earlier result [6, p. 485, Eq. (1)], while a similar special case of (2) is believed to be new.

2. **Proofs of the bilateral generating functions (1) and (2).** In our derivations of the bilateral generating functions (1) and (2), we shall require the following known results:

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha}(x; k) t^n = (1-t)^{-m-(\alpha+1)/k} \cdot \exp\left(x[1-(1-t)^{-1/k}]\right) Y_m^{\alpha}(x(1-t)^{-1/k}; k) \tag{7}$$

and

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha-kn}(x; k) t^n = (1+t)^{-1+(\alpha+1)/k} \cdot \exp\left(x[1-(1+t)^{1/k}]\right) Y_m^{\alpha}(x(1+t)^{1/k}; k) , \tag{8}$$

where m is an arbitrary nonnegative integer, and (by definition) $\alpha > -1$ and $k = 1, 2, 3, \cdots$. The generating-function relationship (7) was proven *independently* by Calvez *et* Génin [1, p. A42, Eq. (6)] and Prabhakar [4, p. 803, Eq. (3.3)], while (8) is due to Calvez *et* Génin [1, p. A42, Eq. (7)].

If we substitute for the coefficients $\zeta_n(\boldsymbol{y}; z)$ from (5) into the left-hand side of (1), we find that

$$\begin{aligned} &\sum_{n=0}^{\infty} Y_{m+n}^{\alpha}(x; k) \zeta_n(\boldsymbol{y}; z) t^n \\ &= \sum_{n=0}^{\infty} Y_{m+n}^{\alpha}(x; k) t^n \sum_{r=0}^{[n/q]} \binom{m+n}{n-qr} \lambda_r \sigma_r(\boldsymbol{y}) z^r \\ &= \sum_{r=0}^{\infty} \lambda_r \sigma_r(\boldsymbol{y}) z^r t^{qr} \sum_{n=0}^{\infty} \binom{M+n}{n} Y_{M+n}^{\alpha}(x; k) t^n , \end{aligned} \tag{9}$$

where, for convenience, $M = m + qr$, $r = 0, 1, 2, \cdots$.

The inner series can now be summed by applying the generating relation (7) with m replaced by M, and the bilateral generating function (1) would follow if we interpret the resulting expression by means of (3).

Formula (2) can be proven similarly by appealing to (8) in place of (7), and we omit the details involved.

3. Applications. By assigning suitable values to the arbitrary coefficients λ_n, it is fairly straightforward to derive, from our formulas (1) and (2), a large variety of bilateral generating functions for the polynomials $Y_n^{\alpha}(x; k)$ and $Y_n^{\alpha-kn}(x; k)$, respectively. On the other hand, in every situation when the multivariable function $\sigma_n(y_1, \cdots, y_N)$ can be expressed as a suitable product of several simpler functions, we shall be led to interesting classes of multilateral generating functions for the Konhauser polynomials considered.

As a simple illustration, if in (2), (4) and (5) we set $m = 0$, $q = 1$,

$$(10)\quad \lambda_n = \frac{(-1)^n}{\Gamma(\beta + ln + 1)}, \quad \sigma_n(\boldsymbol{y}) \equiv 1, \quad n \geqq 0, \quad \beta > -1, \quad l = 1, 2, 3, \cdots,$$

and make use of the definition [3, p. 304, Eq. (5)], we shall readily arrive at the bilateral generating function

$$(11)\quad \begin{aligned} &\sum_{n=0}^{\infty} \frac{n!}{\Gamma(\beta + ln + 1)} Y_n^{\alpha-kn}(x; k) Z_n^{\beta}(y; l) t^n \\ &\qquad = (1 + t)^{-1+(\alpha+1)/k} \exp\left(x[1 - (1 + t)^{1/k}]\right) \\ &\qquad\quad H[x(1 + t)^{1/k}, \ -y^l t/(1 + t)], \end{aligned}$$

where, for convenience,

$$(12)\quad H[x, t] = \sum_{n=0}^{\infty} Y_n^{\alpha-kn}(x; k) \frac{t^n}{\Gamma(\beta + ln + 1)}.$$

For $k = l = 1$, the generating relation (11) would evidently reduce to a familiar result for the generalized Laguerre polynomials; indeed, (11) may also be looked upon as being complementary to our earlier result [6, p. 491, Eq. (14)].

References

1. Léon-Claude Calvez *et* Robert Génin, *Applications des relations entre les fonctions génératrices et les formules de type Rodrigues*, C.R. Acad. Sci. Paris Sér. A-B, **270** (1970), A41-A44.

2. Joseph D. E. Konhauser, *Some properties of biorthogonal polynomials,* J. Math. Anal. Appl., **11** (1965), 242-260.
3. ———, *Biorthogonal polynomials suggested by the Laguerre polynomials,* Pacific J. Math., **21** (1967), 303-314.
4. T. R. Prabhakar, *On the other set of the biorthogonal polynomials suggested by the Laguerre polynomials,* Pacific J. Math., **37** (1971), 801-804.
5. S. Preiser, *An investigation of biorthogonal polynomials derivable from ordinary differential equations of the third order,* J. Math. Anal. Appl., **4** (1962), 38-64.
6. H. M. Srivastava, *On the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials,* Pacific J. Math., **49** (1973), 489-492.

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