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A Hölder estimate for solutions to $\bar{\partial}u=\alpha$ in weakly pseudoconvex domains is obtained when the restriction of α to the boundary vanishes near the set of degeneracy of the Levi form. Applications are given to holomorphic approximation and division problems.

1. Introduction. Hölder estimates for the equation $\bar{\partial}u = \alpha$ in a strictly pseudoconvex domain D were first obtained by Kerzman [8] and Lieb [11], and were later sharpened by Romanov and Henkin [16]. In the weakly pseudoconvex case, analogous results were obtained by Range [15] by assuming that the support of is α bounded away from the set of boundary points were the Levi form degenerates. The argument used there requires the additional hypothesis of existence of a Stein neighborhood basis for $\bar{D}$, which in general may not exist (see Diederich and Fornaess [3]). The hypothesis of existence of Stein neighborhoods was removed by the author [1] by using the results of Kohn [10] concerning boundary regularity of $\bar{\partial}$ in weakly pseudoconvex domains to construct a global kernel of the Grauert-Lieb type (see [4]). In the present work this approach is refined in order to relax the support condition on α. In particular, we obtain a Hölder estimate for solutions to $\bar{\partial}u = \alpha$ whenever $\alpha|_{\partial D}$ vanishes near the set of degeneracy of the Levi-form. We remark that a simpler proof based on local solution operators is possible if one imposes a more stringent support condition on α (see Beatrous and Range [2]).

To facilitate the formulation of the main theorem, we introduce the following notation. If D is a smooth, bounded, pseudoconvex domain, we denote the set of strictly pseudoconvex boundary points by $S(D)$, and we set $W(D) = \partial D \backslash S(D)$. If N is a neighborhood of $W(D)$, let $Z_N^q(D)$ denote the set of $\bar{\partial}$ closed $(0, q)$ forms α of class C^1 on D which extend continuously to $\bar{D}$ with $\alpha|_{\partial D \cap N} = 0$. Set $Z^q(D) = \cup Z_N^q(D)$ where N runs over all neighborhoods of $W(D)$. Our main result is the following.

THEOREM 1.1. *Let D be a bounded, pseudoconvex domain in $\boldsymbol{C}^n$ with a smooth boundary. Then for each $q \geqq 1$ there is an operator $E_q\colon Z^q(D) \to Z^{q-1}(D)$ with $\bar{\partial}(E_q\alpha) = \alpha$. Moreover, for any neighborhood N of $W(D)$ there is a constant C_N such that the following estimate holds for $\alpha \in Z_N^q(D)$:*

$$|(E_q\alpha)(z') - (E_q\alpha)(z'')| \leqq C_N\|\alpha\|_D |z' - z''|^{1/2}$$

for all $z', z'' \in D$.

Here $\| \ \|_D$ denotes the sup norm on D.

The theorem will be proved by constructing an integral solution operator in § 2 and then estimating the kernel in § 3. In § 4 we give some applications concerning holomorphic functions with continuous boundary values.

2. The solution operator. The construction of a solution operator will require a special defining function for the hypersurface $S(D)$, which we now construct. Let τ be an arbitrary defining function for the domain D. The special defining function will be of the form

$$\rho(z) = \tau(z) \exp(\Phi(z)\tau(z))$$

where Φ is a smooth, positive function in a neighborhood of $S(D)$. Direct computation shows that for $z \in S(D)$ the Levi form of ρ has the form

$$\mathscr{L}_\rho(z; t) = \mathscr{L}_\tau(z; t) + 2\Phi(z)\left|\sum \frac{\partial\tau}{\partial z_j}(z)t_j\right|^2 . \tag{1}$$

Choose Φ to be a smooth, positive function on $S(D)$ which increases so rapidly as z approaches $W(D)$ that $\mathscr{L}_\rho(z; t) > 0$ for every $z \in S(D)$ and every $t \in C^n\backslash\{0\}$. This is possible since the first term in (1) is strictly positive where the second term vanishes. Now extend Φ to be a smooth, positive function in a neighborhood of $S(D)$. Then by continuity there is a neighborhood U of $S(D)$ on which ρ is strictly plurisubharmonic and $\nabla\rho \neq 0$.

The defining function ρ will now be used to construct a smooth family of holomorphic support functions for $S(D)$ (cf. Henkin [6], Lemma 2.4). Let $F(\zeta, z)$ denote the Levi polynomial associated with ρ, i.e.,

$$F(\zeta, z) = -2\sum_i \frac{\partial\rho}{\partial z_j}(\zeta)(z_i - \zeta_i)$$
$$- \sum_{i,j} \frac{\partial^2\rho}{\partial z_i \partial z_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j) .$$

THEOREM 2.1. *Let k be a positive integer. There are a neighborhood U of $S(D)$, a smooth, positive function r on U, and a $\mathscr{C}^k$ function Φ on $U \times \bar{D}$ with the following properties:*

(i) *For each $\zeta \in U$, $\Phi(\zeta, \cdot) \in A^k(D) = \mathscr{C}^k(\bar{D}) \cap \mathscr{O}(D)$;*

(ii) $G(\zeta, z) = \Phi(\zeta, z)/F(\zeta, z)$ *is a nonvanishing* $\mathscr{C}^k$ *function on* $\{(\zeta, z) \in U \times \bar{D}: |\zeta - z| \leqq r(\zeta)\}$;

(iii) $\Phi(\zeta, z) \neq 0$ *if* $|\zeta - z| \geqq r(\zeta)$;

(iv) $\operatorname{Re} F(\zeta, z) > \rho(\zeta) - \rho(z) + r(\zeta)|\zeta - z|^2$ *if* $|\zeta - z| \leqq r(\zeta)$.

Proof. By expanding ρ in a Taylor series about $\zeta \in U$, and using the fact that ρ is strictly plurisubharmonic, one obtains a smooth, positive function δ on U such that $\operatorname{Re} F(\zeta, z) > \rho(\zeta) - \rho(z) + \delta(\zeta)|\zeta - z|^2$ whenever $\zeta \in U$ and $0 < |\zeta - z| < \delta(\zeta)$. (Here U denotes the neighborhood of $S(D)$ on which ρ is defined.)

Choose a smooth function $\chi(\zeta, z)$ on $U \times C^n$ with $0 \leqq \chi \leqq 1$ such that $\chi(\zeta, \cdot) = 0$ outside of $B_{\delta(\zeta)}(\zeta)$ and $\chi(\zeta, \cdot) = 1$ on $B_{1/2\delta(\zeta)}(\zeta)$. For each $\zeta \in U$, define a (0, 1) form α_ζ on $\bar{D}$ by setting $\alpha_\zeta(z) = \bar{\partial}_z(\chi(\zeta, z)(F(\zeta, z))^{-1})$ if $F(\zeta, z) \neq 0$, and $\alpha_\zeta(z) = 0$ if $F_\zeta(z) = 0$. Then, after shrinking U if necessary, the map $\zeta \mapsto \alpha_\zeta$ is a $\mathscr{C}^\infty$ map of U into $\mathscr{C}^\infty_{(0,1)}(\bar{D})$, and clearly $\bar{\partial}_z \alpha_\zeta(z) = 0$ for each $\zeta \in U$. Thus we can find solutions in D to the equation $\bar{\partial}_z u_\zeta(z) = \alpha_\zeta(z)$. Moreover, by using the solution operator for the $\bar{\partial}$-Neumann problem (with an appropriate weight function) and the Sobolev estimates of Kohn [10], the solution u_ζ can be chosen so that $u_\zeta \in \mathscr{C}^k(\bar{D})$ for each ζ and the map $(\zeta \mapsto u_\zeta)$: $U \to \mathscr{C}^k(\bar{D})$ is of class $\mathscr{C}^\infty$.

Define meromorphic functions m_ζ on D by setting $m_\zeta(z) = \chi(\zeta, z)F(\zeta, z)^{-1} - u_\zeta(z)$, and choose a smooth, real valued function τ on U such that $\operatorname{Re} m_\zeta(z) > \tau(\zeta)$ for $z \in \bar{D} \backslash B_{\delta(\zeta)/2}(\zeta)$ and $\operatorname{Re}(m_\zeta(z) - F(\zeta, z)^{-1}) > \tau(\zeta)$ for $z \in \bar{D} \cap B_{\delta(\zeta)}(\zeta)$. Set $\Phi(\zeta, z) = (m_\zeta(z) - \tau(\zeta))^{-1}$. Then, after shrinking U once again, Φ is a $\mathscr{C}^k$ function on $U \times \bar{D}$. Moreover, writing

$$\Phi(\zeta, z) = \frac{F(\zeta, z)}{1 + F(\zeta, z)(m_\zeta(z) - F(\zeta, z)^{-1} - \tau(\zeta))}$$

for z near ζ, one sees that Φ satifies (i)-(iv) for an appropriately chosen positive function r.

For the kernel construction, it will be necessary to express the function Φ in the form

$$\Phi(\zeta, z) = \sum P_j(\zeta, z)(\zeta_j - z_j) \tag{2}$$

where $P_1, \cdots, P_n$ are sufficiently smooth on $U \times \bar{D}$ and holomorphic in z. One first observes that by property (ii) of Theorem (2.1), this division problem can be solved locally in z and ζ. Next, one uses once again the result of Kohn [10] on boundary regularity of $\bar{\partial}$ to pass from local to global in z. This step is rather technical, and, since the argument is virtually identical to that used by

Øvrelid in [14], it will not be given here. Finally, since we only require smoothness in ζ, we can pass from local to global via a partition of unity.

We are now ready to construct the solution operator. We will use the formalism of Harvey and Polking [5]. Let $B(\zeta, z)$ denote the Bochner-Martinelli kernel:

$$B(\zeta, z) = (2\pi i)^{-n}\frac{(\bar{\zeta} - \bar{z})\cdot d\zeta}{|\zeta - z|^2} \wedge \left[\frac{d(\bar{\zeta} - \bar{z})\cdot d\zeta}{|\zeta - z|^2}\right]^{n-1}$$

where $(\bar{\zeta} - \bar{z})\cdot d\zeta = \sum (\zeta_j - z_j)d\zeta_j$ and $d(\bar{\zeta} - \bar{z})\cdot d\zeta = \sum d(\zeta_j - z_j) \wedge d\zeta_j$. Then for any $(0, q)$ form α of class $\mathscr{C}^1$ on $\bar{D}$ we have (see [5] or [13]) the Bochner-Martinelli formula

$$\alpha(z) = \bar{\partial}\int_D B(\zeta, z) \wedge \alpha(\zeta) + \int_D B(\zeta, z) \wedge \bar{\partial}\alpha(\zeta) + \int_{\partial D} B(\zeta, z) \wedge \alpha(\zeta) ,$$

where it is to be understood that all differentials involving the z variable are to be moved to the right before performing the integration.

Set $P = (P_1, \cdots, P_n)$ where $P_1, \cdots, P_n$ are the functions from (2). Then the Henkin kernel (see [5]) is

$$H(\zeta, z) = (2\pi i)^{-n}\frac{P\cdot d\zeta}{\Phi} \wedge \frac{(\bar{\zeta} - \bar{z})\cdot d\zeta}{|\zeta - z|^2} \wedge \sum_{p=1}^{n-1}\left[\frac{\bar{\partial}P\cdot d\zeta}{\Phi}\right]^{p-1} \wedge \left[\frac{d(\bar{\zeta} - \bar{z})\cdot d\zeta}{|\zeta - z|^2}\right]^{n-p-1} \tag{3}$$

where $\bar{\partial}$ acts coordinatewise on $P = (P_1, \cdots, P_n)$. Our solution operator for the $\bar{\partial}$ equation is then defined by

$$(E_q\alpha)(z) = \int_D B(\zeta, z) \wedge \alpha(\zeta) + \int_{\partial D} H(\zeta, z) \wedge \alpha(\zeta) \tag{4}$$

where $\alpha \in Z^q(D)$ for some $q \geqq 1$.

THEOREM 2.2. *For any $\alpha \in Z^q(D)$ with $q \geqq 1$ we have $\bar{\partial}E_q\alpha = \alpha$ in D.*

Proof. By the Bochner-Martinelli formula we have

$$\begin{aligned} \bar{\partial}(E_q\alpha)(z) &= \alpha(z) - \int_{\partial D} B(\zeta, z) \wedge \alpha(\zeta) - \int_{\partial D} \bar{\partial}_z H(\zeta, z) \wedge \alpha(\zeta) \\ &= \alpha(z) - \int_{\partial D} B(\zeta, z) \wedge \alpha(\zeta) - \int_{\partial D} \bar{\partial} H(\zeta, z) \wedge \alpha(\zeta) \\ &\quad + \int_{\partial D} (\bar{\partial}_\zeta H(\zeta, z)) \wedge \alpha(\zeta) . \end{aligned} \tag{5}$$

For fixed $z \in D$, let $\varphi(\zeta)$ be a cut off function which vanishes near $W(D)$ and near the singularities of $H(\cdot, z)$, and which is identically 1 near $\operatorname{supp}(\alpha|_{\partial D})$. Then for the last term on the right in (5) we have

$$\int_{\partial D} \bar\partial_\zeta H(\zeta, z) \wedge \alpha(\zeta) = \int_{\partial D} \bar\partial_\zeta[\varphi(\zeta)H(\zeta, z) \wedge \alpha(\zeta)] = 0 .$$

The first equality follows since $\bar\partial\alpha = 0$, and the second from Stokes' formula. Moreover, one can compute from (3) (see [5]) that

$$\bar\partial H(\zeta, z) = -B(\zeta, z) + (2\pi i)^{-n}\frac{P(\zeta, z)\cdot d\zeta}{\Phi(\zeta, z)} \wedge \left[\frac{\bar\partial P(\zeta, z)\cdot d\zeta}{\Phi(\zeta, z)}\right]^{n-1} .$$

Since P is holomorphic in z, the second term on the right is of type $(n, n-1)$ in ζ. Thus we obtain the formula

$$\int_{\partial D} \bar\partial H(\zeta, z) \wedge \alpha(\zeta) = -\int_{\partial D} B(\zeta, z) \wedge \alpha(\zeta)$$

and it follows from (5) that

$$\bar\partial(E_q\alpha)(z) = \alpha(z) .$$

3. Estimates for the kernel. In this section we fix a neighborhood N of $W(D)$ and we restrict our attention to forms in $Z_N^q(D)$, $q \geqq 1$. First, we remark that Hölder $1-\varepsilon$ estimates for the Bocher-Martinelli kernel are well known (see Kerzman [8]), so in order to complete the proof of Theorem 1.1 it will suffice to estimate

$$v(z) = \int_{\partial D} H(\zeta, z) \wedge \alpha(\zeta) .$$

Thus, by Lemma 4 of Romanov and Henkin [16] it will suffice to obtain the estimate

$$|\nabla v(z)| \leqq C_N \|\alpha\|_D |\tau(z)|^{-1/2} \tag{6}$$

for all $z \in D$, where τ is some fixed defining function for D.

Choose neighborhoods N' and N'' of $W(D)$ with $N'' \subset\subset N' \subset\subset N$, and let τ be a defining function for D which agrees with ρ near $\partial D \backslash N''$ (where ρ is the defining function for $S(D)$ from the preceding section). Let U, r and F be as in Theorem 2.1 and assume, by shrinking U, that $\tau|_{U\backslash N''} = \rho|_{U\backslash N''}$. Set $U' = (U \cap \bar D)\backslash N'$. Then for $\zeta \in U'$, $r(\zeta)$ is bounded below by a positive number ε. Moreover, by shrinking U' and choosing ε sufficiently small we can choose a positive constant γ such that (c.f. [16] Lemma 1)

(7) $|G(\zeta, z)| \geqq \gamma$ for $\zeta \in U'$ and $|\zeta - z| \leqq \varepsilon$;

(8) $|\Phi(\zeta, z)| \geqq \gamma$ for $\zeta \in U'$ and $|\zeta - z| \geqq \varepsilon$;

(9) $\operatorname{Re} F(\zeta, z) \geqq \rho(\zeta) - \rho(z) + \gamma|\zeta - z|^2$ for $\zeta \in U'$ and $|\zeta - z| \leqq \varepsilon$.

The estimate (6) is now easy if z is bounded away from $\partial D \backslash N$.

LEMMA 3.1. *For any $\sigma > 0$ there is a positive constant C_σ such that $|\nabla v(z)| \leqq C_\sigma \|\alpha\|_D$ whenever $z \in N'$ or* $\operatorname{dist}(z, \partial D) \geqq \sigma$.

Proof. It follows from (7), (8) and (9) that $|\Phi(\zeta, z)|$ is bounded below by a positive constant whenever z is as in the statement of the lemma and $\zeta \in \partial D \backslash N$. Thus, since α vanishes on $\partial D \cap N$, the estimate follows by differentiation under the integral sign.

Choose $\sigma \in (0, \varepsilon]$ sufficiently small that $\zeta \in U'$ whenever $\zeta \in D \backslash N'$ and $\operatorname{dist}(\zeta, \partial D) < 2\sigma$. Then by the preceding lemma it suffices to verify (6) when $z \in U_\sigma = \{z \in D \backslash N' : \operatorname{dist}(z, \partial D) < \sigma\}$. Following Romanov and Henkin, we set $\tilde{\Phi}(\zeta, z) = (F(\zeta, z) - 2\rho(\zeta))G(\zeta, z)$ for $z \in U_\sigma$ and $|\zeta - z| \leqq \sigma$. Let $\tilde{H}(\zeta, z)$ be the kernel defined by replacing Φ with $\tilde{\Phi}$ in (3). Using (7)-(9), one easily verifies that for $z \in U_\sigma$

$$|\nabla v(z)| \leqq \text{const.} \|\alpha\|_D + |\nabla v_1(z)| \tag{10}$$

where

$$v_1(z) = \int_{\partial(D \cap B_\sigma(z))} \tilde{H}(\zeta, z) \wedge \alpha(\zeta) .$$

Thus, to complete the proof, it will suffice to estimate $|\nabla v_1(z)|$. By Stokes' formula we have (since $\bar{\partial}\alpha = 0$)

$$v_1(z) = \int_{D \cap B_\sigma(z)} \bar{\partial}_\zeta \tilde{H}(\zeta, z) \wedge \alpha(\zeta) .$$

By direct computation, one finds that each coefficient of $\bar{\partial}_\zeta \tilde{H}$ has one of the following forms:

$$\frac{(\bar{\zeta}_j - \bar{z}_j)a(\zeta, z)}{\tilde{\Phi}(\zeta, z)^p |\zeta - z|^{2n-2p}}, \quad \frac{(\bar{\zeta}_j - \bar{z}_j)a(\zeta, z)}{\tilde{\Phi}(\zeta, z)^{p+1} |\zeta - z|^{2n-2p}}, \quad \frac{a(\zeta, z)}{\tilde{\Phi}(\zeta, z)^p |\zeta - z|^{2n-2p}},$$

or

$$\frac{(\bar{\zeta}_i - \bar{z}_i)(\zeta_j - z_j)a(\zeta, z)}{\tilde{\Phi}(\zeta, z)^p |\zeta - z|^{2n-2p+2}} .$$

Here $1 \leqq i, j \leqq n$, $1 \leqq p \leqq n - 1$, and $a(\zeta, z)$ denotes some smooth function of ζ and z.

Differentiating the coefficients of $\bar{\partial}_\zeta \tilde{H}(\zeta, z)$ with respect to z, one finds that for $z \in U_\sigma$

$$|\nabla v_1^{\backslash}(z)| \leqq \text{const.}\ \|\alpha\|_D \sum_{p=1}^{n-1} I_1^p + I_2^p + I_3^p \tag{11}$$

where

$$I_1^p = \int_{D \cap B_\sigma(z)} \frac{dV}{|\tilde{\Phi}(\zeta, z)|^p |\zeta - z|^{2n-2p+1}} ,$$

$$I_2^p = \int_{D \cap B_\sigma(z)} \frac{dV}{|\tilde{\Phi}(\zeta - z)|^{p+1} |\zeta - z|^{2n-2p}} ,$$

and

$$I_3^p = \int_{D \cap B_\sigma(z)} \frac{dV}{|\tilde{\Phi}(\zeta, z)|^{p+2} |\zeta - z|^{2n-2p-1}} .$$

The estimate

$$I_j^p \leqq \text{const.}\ |\rho(z)|^{-1/2}$$

can now be obtained from (7)–(9) as in Romanov and Henkin [16]. Thus, combining (10) and (11) we have the estimate (6). This completes the proof of Theorem 1.1.

4. Applications.

In this section we give two applications of Theorem 1.1 which generalize certain well known results for strictly pseudoconvex domains.

The problem which originally motivated this investigation was that of approximating a given function in $A(D) = \mathscr{C}(\bar{D}) \cap \mathscr{O}(D)$ by functions which extend holomorphically across the boundary. An example due to Diederich and Fornaess [3] shows that this may not be possible if D is only assumed to be weakly pseudoconvex. However, Theorem 1.1 implies the following generalization of the classical result for strictly pseudoconvex domains.

THEOREM 4.1. *Suppose that D is a smooth, bounded pseudoconvex domain in C^n and that $f \in A(D)$. Then f can be uniformly approximated on D by functions in $A(D)$ which extend holomorphically across $S(D)$.*

This result has appeared previously in Beatrous [1] and Beatrous and Range [2], and the proof will not be repeated here. The same result had been proved earlier by Range [15] under the additional hypothesis of existence of a Stein neighborhood basis for $\bar{D}$, but this earlier result did not apply to the example of Diederich and Fornaess [3].

For the second application we consider the following division problem. Let D be a domain in C^n and let p be a point in D. Denote by $M_p(D)$ the maximal ideal of $A(D)$ consisting of functions which vanish at p. We wish to show that $M_p(D)$ is generated (algebraically) by $\{z_1 - p_1, \cdots, z_n - p_n\}$. In the strictly psudoconvex case this problem was solved by Kerzman and Nagel when $n = 2$ and by Lieb [12] and Øvrelid [14] in higher dimensions. If D is a weakly pseudoconvex domain in C^2 we obtain the following result.

THEOREM 4.2. *Suppose that D is a smooth, bounded, pseudoconvex domain in C^2 and that p is a point in D. If there is a complex hyperplane through p which does not meet $W(D)$ then $M_p(D)$ is generated by $\{z_1 - p_1, z_2 - p_2\}$.*

Proof. By translation and rotation of coordinates we may assume that $p = 0$ and that $W(D) \cap \{z_1 = 0\} = \varnothing$. Let f be a function in $A(D)$ satisfying $f(0) = 0$. We must construct functions f_1 and f_2 in $A(D)$ such that $f = z_1 f_1 + z_2 f_2$.

Choose a small polydisc U_0 of radius ε about 0 with $\bar{U}_0 \cap W(D) = \varnothing$ and holomorphic functions f_1^0 and f_2^0 on U_0 with $f = z_1 f_1^0 + z_2 f_2^0$ in U_0. For $j = 1, 2$, set $U_j = \{|z_j| > \varepsilon/2\}$, and define $f_i^j = \delta_i^j z_j^{-1}$ on U_j. Then $f = z_1 f_1^j + z_2 f_2^j$ on U_j. For $0 \leqq i, j \leqq 2$ we have $(f_1^i - f_1^j)z_1 + (f_2^i + f_2^j)z_2 = 0$ on $U_i \cap U_j$. Set $g_{ij} = (f_2^i - f_2^j)z_1^{-1} = (f_1^j - f_1^i)z_2^{-1}$ on $U_i \cap U_j$.

Choose smooth functions χ_j with compact support in U_j such that $0 \leqq \chi_j \leqq 1$, $\chi_0 + \chi_1 + \chi_2 = 1$ on $\bar{D}$, and $\chi_1 = 1$ in a neighborhood of $W(D)$. Set $g_j = \sum_i \chi_i g_{ji}$ on U_j. Then $g_i - g_j = g_{ij}$ on $U_i \cap U_j$, so we can define a (0, 1) form α on $\bar{D}$ by setting $\alpha = \bar{\partial} g_j$ on U_j. Moreover, since $\chi_1 = 1$ near $W(D)$, the support of α is bounded away from $W(D)$. Thus, by Theorem 1.1, there is a function $u \in \mathscr{C}^\infty(D) \cap \mathscr{C}(\bar{D})$ with $\bar{\partial} u = \alpha$. We can now define the functions f_1 and f_2 in $U_i \cap \bar{D}$ by

$$f_1 = f_1^i + (g_i - u)z_2$$

and

$$f_2 = f_2^i - (g_i - u)z_1 .$$

One checks easily that f_1 and f_2 are well defined functions in $A(D)$ and that $z_1 f_1 + z_2 f_2 = f$.

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