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PLANAR CONTINUA WITH RESTRICTED LIMIT DIRECTIONS

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An affirmative answer is given to a question of D. M. Campbell and J. Lamoreaux concerning minimal conditions on the set of limit directions of a planar continuum that guarantee it is a line segment.

Throughout we let E denote a planar continuum. The set E is said to have a *limit direction* $e^{i\alpha}$ at the point z in E if there is a sequence of points z_n in $E - \{z\}$ with $z_n \to z$ and $(z_n - z)/|z_n - z| \to e^{i\alpha}$; this limit direction is called a *right limit direction* if we also have $\operatorname{Re}(z_n) \geqq \operatorname{Re}(z)$ for each z_n. The set of all [right] limit directions of E at z is denoted by $\mathscr{D}(z)[\mathscr{D}_R(z)]$ and is called the contingent of E at z in the older terminology of Saks [2].

D. M. Campbell and J. Lamoreaux [1] proved: *Let K be a subset of E such that $\mathscr{D}(z) \cap \{e^{i\theta}: 0 < |\theta| \leqq \pi/2\} = \emptyset$ for each z in $E - K$. If the projection of K on the y-axis has measure zero, then E is a horizontal line segment.* Then they asked whether this theorem remains true when the condition on $\mathscr{D}(z)$ is replaced by the conition $\mathscr{D}_R(z) \subseteqq \{1\}$. We now show this to be the case.

THEOREM. *Let K be a subset of E such that $\mathscr{D}_R(z) \subseteqq \{1\}$ for each z in $E - K$. If the projection of K on the y-axis has measure zero, then E is a horizontal line segment.*

Proof. To prove this theorem we show that the projection of E on the y-axis is of measure zero.

One observes $\mathscr{D}_R(z) \subseteqq \{1\}$ implies $\mathscr{D}(z) \cap \{e^{i\theta}: 0 < |\theta| < \pi/2\} = \emptyset$ and therefore for every point of $E - K$ the set $\mathscr{D}(z)$ cannot be the entire circle $\{e^{i\theta}: 0 \leqq \theta \leqq 2\pi\}$. By the first fundamental theorem on contingents of plane sets ([2], p. 266), at every point of $E - K$, except those of a set L of linear measure zero, the set $\mathscr{D}(z)$ is either a doubleton $\{e^{i\alpha}, -e^{i\alpha}\}$ or a semicircle $\{e^{i\theta}: \alpha \leqq \theta \leqq \alpha + \pi\}$. Since $\mathscr{D}_R(z) \subseteqq \{1\}$ on $E - K$, it follows that for each z in $E - (K \cup L)$, the set $\mathscr{D}(z)$ is either the doubleton $\{i, -i\}$, the doubleton $\{1, -1\}$, or the arc $\{e^{i\theta}: \pi/2 \leqq \theta \leqq 3\pi/2\}$.

The second fundamental theorem on contingents of plane sets ([2], p. 267) asserts that $M \equiv \{z \in E - (K \cup L): \mathscr{D}(z) = \{1, -1\}\}$ has a projection on the y-axis of measure zero. Thus, to complete the proof we now show that the set $N \equiv E - (K \cup L \cup M)$ is countable.

For each $z \in N$, $\mathscr{D}_R(z) = \emptyset$ and hence there is a rational number $r(z)$ and a corresponding closed half-disk

$$D(z, r(z)) \equiv \{\zeta: -\pi/2 \leqq \arg(\zeta - z) \leqq \pi/2 \text{ and } |\zeta - z| \leqq r(z)\}$$

such that $D(z, r(z)) \cap E = \{z\}$. Also, for each rational number r the set $N_r \equiv \{z \in N: r(z) = r\}$ is an isolated set, and the countability of N is established.

In closing we note that in view of its proof, the theorem above remains true when the hypothesis $\mathscr{D}_R(z) \subseteqq \{1\}$ is replaced by any condition which guarantees that if $z \in E - K$, then either (i) $\mathscr{D}_R(z) = \emptyset$ or (ii) $1 \in \mathscr{D}(z)$ and $\mathscr{D}(z)$ is a subset of either $\{e^{i\theta}: 0 \leqq \theta \leqq \pi\}$ or $\{e^{i\theta}: \pi \leqq \theta \leqq 2\pi\}$.

References

1. D. M. Campbell and J. Lamoreaux, *Continua in the plane with limit directions,* Pacific J. Math., 74 (1978), 37-46.
2. S. Saks, *Theory of the integral,* Monographie Matematyczne 7, Warszawa-Lwów, 1937.

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