

Pacific Journal of Mathematics

PLANAR CONTINUA WITH RESTRICTED LIMIT DIRECTIONS

CHARLES L. BELNA, MICHAEL JON EVANS AND PAUL HUMKE

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C. L. BELNA, M. J. EVANS, AND P. D. HUMKE

An affirmative answer is given to a question of D. M. Campbell and J. Lamoreaux concerning minimal conditions on the set of limit directions of a planar continuum that guarantee it is a line segment.

Throughout we let E denote a planar continuum. The set E is said to have a *limit direction* $e^{i\alpha}$ at the point z in E if there is a sequence of points z_n in $E - \{z\}$ with $z_n \rightarrow z$ and $(z_n - z)/|z_n - z| \rightarrow e^{i\alpha}$; this limit direction is called a *right limit direction* if we also have $\operatorname{Re}(z_n) \geq \operatorname{Re}(z)$ for each z_n . The set of all [right] limit directions of E at z is denoted by $\mathcal{D}(z)[\mathcal{D}_R(z)]$ and is called the *contingent* of E at z in the older terminology of Saks [2].

D. M. Campbell and J. Lamoreaux [1] proved: *Let K be a subset of E such that $\mathcal{D}(z) \cap \{e^{i\theta}: 0 < |\theta| \leq \pi/2\} = \emptyset$ for each z in $E - K$. If the projection of K on the y -axis has measure zero, then E is a horizontal line segment.* Then they asked whether this theorem remains true when the condition on $\mathcal{D}(z)$ is replaced by the condition $\mathcal{D}_R(z) \subseteq \{1\}$. We now show this to be the case.

THEOREM. *Let K be a subset of E such that $\mathcal{D}_R(z) \subseteq \{1\}$ for each z in $E - K$. If the projection of K on the y -axis has measure zero, then E is a horizontal line segment.*

Proof. To prove this theorem we show that the projection of E on the y -axis is of measure zero.

One observes $\mathcal{D}_R(z) \subseteq \{1\}$ implies $\mathcal{D}(z) \cap \{e^{i\theta}: 0 < |\theta| < \pi/2\} = \emptyset$ and therefore for every point of $E - K$ the set $\mathcal{D}(z)$ cannot be the entire circle $\{e^{i\theta}: 0 \leq \theta \leq 2\pi\}$. By the first fundamental theorem on contingents of plane sets ([2], p. 266), at every point of $E - K$, except those of a set L of linear measure zero, the set $\mathcal{D}(z)$ is either a doubleton $\{e^{i\alpha}, -e^{i\alpha}\}$ or a semicircle $\{e^{i\theta}: \alpha \leq \theta \leq \alpha + \pi\}$. Since $\mathcal{D}_R(z) \subseteq \{1\}$ on $E - K$, it follows that for each z in $E - (K \cup L)$, the set $\mathcal{D}(z)$ is either the doubleton $\{i, -i\}$, the doubleton $\{1, -1\}$, or the arc $\{e^{i\theta}: \pi/2 \leq \theta \leq 3\pi/2\}$.

The second fundamental theorem on contingents of plane sets ([2], p. 267) asserts that $M \equiv \{z \in E - (K \cup L): \mathcal{D}(z) = \{1, -1\}\}$ has a projection on the y -axis of measure zero. Thus, to complete the proof we now show that the set $N \equiv E - (K \cup L \cup M)$ is countable.

For each $z \in N$, $\mathcal{D}_R(z) = \emptyset$ and hence there is a rational number $r(z)$ and a corresponding closed half-disk

$$D(z, r(z)) \equiv \{\zeta: -\pi/2 \leq \arg(\zeta - z) \leq \pi/2 \text{ and } |\zeta - z| \leq r(z)\}$$

such that $D(z, r(z)) \cap E = \{z\}$. Also, for each rational number r the set $N_r \equiv \{z \in N: r(z) = r\}$ is an isolated set, and the countability of N is established.

In closing we note that in view of its proof, the theorem above remains true when the hypothesis $\mathcal{D}_R(z) \subseteq \{1\}$ is replaced by any condition which guarantees that if $z \in E - K$, then either (i) $\mathcal{D}_R(z) = \emptyset$ or (ii) $1 \in \mathcal{D}(z)$ and $\mathcal{D}(z)$ is a subset of either $\{e^{i\theta}: 0 \leq \theta \leq \pi\}$ or $\{e^{i\theta}: \pi \leq \theta \leq 2\pi\}$.

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Frank Hayne Beatrous, Jr., <i>Hölder estimates for the $\bar{\partial}$ equation with a support condition</i>	249
Charles L. Belna, Michael Jon Evans and Paul Humke, <i>Planar continua with restricted limit directions</i>	259
Leon Brown and Takashi Ito, <i>Classes of Banach spaces with unique isometric preduals</i>	261
V. K. Deshpande, <i>Completions of Noetherian hereditary prime rings</i>	285
Deepak Dhar, <i>Asymptotic enumeration of partially ordered sets</i>	299
Zeev Ditzian, <i>On interpolation of $L_p[a, b]$ and weighted Sobolev spaces</i>	307
Andrew George Earnest, <i>Congruence conditions on integers represented by ternary quadratic forms</i>	325
Melvin Faierman, <i>Bounds for the eigenfunctions of a two-parameter system of ordinary differential equations of the second order</i>	335
Hector O. Fattorini, <i>Vector-valued distributions having a smooth convolution inverse</i>	347
Howard D. Fegan, <i>The spectrum of the Laplacian on forms over a Lie group</i>	373
Gerald Leonard Gordon, <i>On the degeneracy of a spectral sequence associated to normal crossings</i>	389
S. Madhavan, <i>On bisimple weakly inverse semigroups</i>	397
Françoise Mathot, <i>On the decomposition of states of some $*$-algebras</i>	411
Roger McCann, <i>Embedding asymptotically stable dynamical systems into radial flows in l_2</i>	425
Michael L. Mihalik, <i>Ends of fundamental groups in shape and proper homotopy</i>	431
Samuel Murray Rankin, III, <i>Boundary value problems for partial functional differential equations</i>	459
Randy Tuler, <i>Arithmetic sums that determine linear characters on $\Gamma(N)$</i>	469
Jeffrey D. Vaaler, <i>On linear forms and Diophantine approximation</i>	475
G. P. Wene, <i>Alternative rings whose symmetric elements are nilpotent or a right multiple is a symmetric idempotent</i>	483