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VECTOR VALUED DISTRIBUTIONS HAVING A SMOOTH CONVOLUTION INVERSE

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Let E, X be complex Banach spaces, (E, X) the space of linear operators from E into X equipped with its usual norm. We denote by $\mathscr{D}'(E)$ the space of E-valued distributions defined in $-\infty<t<\infty$ and by $\mathscr{D}_0(E)$ the subspace thereof consisting of distributions with support in $t\geq 0$. A distribution $P\in\mathscr{D}_0'((X; E))$ is said to have a *convolution inverse* (in symbols, $P\in \mathscr{D}_0'((E; X))^{-1}$ or simply $P\in\mathscr{D}_0'^{-1}$) if there exists $S\in\mathscr{D}_0'((E; X))$ such that

$$P*S=\delta\otimes I,\quad S*P=\delta\otimes J \tag{1}$$

where δ is the Dirac measure and I (resp. J) denotes the identity operator in E (resp. X). We examine the problem of characterizing those P which possess a convolution inverse $S=P^{-1}$ being smooth in various senses: infinitely differentiable, in a quasi-analytic class, analytic, etc.

1. Introduction. This paper continues the investigations in [6] on convolution inverses of vector valued distributions: for the necessary definitions and results see [14] or [15]. In particular, we use the definition of convolution in [15] as follows: if E, F, G, are Banach spaces and $U\in\mathscr{D}_0'((F; G))$, $V\in\mathscr{D}_0'((E; F))$ we understand by $U*V$ not the convolution in [15], which takes values in $(F; G)\hat{\otimes}(E; F)$ but rather its "composition" (in the sense of [15]) with the linear map from $(F; G)\hat{\otimes}(E; F)\to(E; G)$ induced by the (bilinear) product map from $(F, G)\times(E; F)$ into $(E; G)$; when U (resp. V) is, say a continuous $(F; G)$-valued (resp. $(E; F)$-valued) function, this definition coincides with the usual one. Necessary and sufficient conditions for a distribution $P\in\mathscr{D}_0'((X; E))$ to belong to $\mathscr{D}_0'^{-1}$ have been given in [6]. When $P\in\mathscr{S}_0'(X; E))$ (the space of all tempered, $(X; E)$-valued distributions with support in $t\geqq 0$) the Laplace transform $\mathfrak{P}(\lambda)=\mathscr{L}P(\lambda)=P(\exp(-\lambda(\cdot)))$ exists in the right half-plane, is analytic there, and grows no more than a polynomial as $|\lambda|\to\infty$. Denote by $\pi(P)$ the largest connected subset of the complex plane (containing the right half-plane) to which $\mathfrak{P}(\lambda)^{-1}$ can be extended as an analytic function. It follows from analyticity of $\mathfrak{P}(\lambda)$ that $\mathfrak{P}(\lambda)^{-1}$ exists in an open set $\rho(P)\subseteqq\pi(P)$ (perhaps empty) called the resolvent set of P and is analytic there. We have

THEOREM 1.1. *Let $P\in\mathscr{S}_0'((X; E))$. Then $P\in\mathscr{D}_0'^{-1}$ if and only* *if $\rho(P)$ contains a logarithmic region*

$$\Lambda(\alpha, \beta, \omega) = \{\lambda; \operatorname{Re} \lambda \geqq \max(\alpha \log|\lambda| + \beta, \omega)\}$$

where $\alpha, \beta \geqq 0$, $-\infty < \omega < \infty$, and

$$\|\mathfrak{P}(\lambda)^{-1}\|_{(E;X)} \leqq C(1 + |\lambda|)^m (\lambda \in \Lambda(\alpha, \beta, \omega)) .$$

For a proof see [6], Theorem 2.5. The case where $\exp(-\omega t)P \in \mathscr{S}_0'$ for sufficiently large ω can be reduced to the previous one noticing that

$$P \in \mathscr{D}_0'^{-1} \text{ if and only if } \exp(-\omega t)P \in \mathscr{D}_0'^{-1} ,$$

the inverses connected by the equation $\exp(-\omega t)P^{-1} = (\exp(-\omega t)P)^{-1}$. The problem of existence of P^{-1} can be localized, as the following result shows; in it, $\mathscr{D}_0'((-\infty, a); (E; X))$ denotes the space of $(E; X)$-valued distributions defined in $-\infty < t < a$ $(a > 0)$ with support in $t \geqq 0$.

THEOREM 1.2. *Assume that, for every $a > 0$ there exists $S_a \in \mathscr{D}_0'((-\infty; a); (E; X))$ satisfying* (1) *in $t < a$. Then $P \in \mathscr{D}_0'^{-1}$.*

The rather obvious proof can be seen in [6], p. 349; here $S = P^{-1}$ is defined locally by setting $S = S_a$ in $t < a$. This allows us to identify fully the class $\mathscr{D}_0' \cap \mathscr{D}_0'^{-1}$ (note that $\mathscr{L}P(\lambda)$ may not exist in this case). For this purpose and for future use we introduce the class $\mathscr{H}_a (a > 0)$ consisting of all test functions $\varphi \in \mathscr{D}$ such that $\varphi(t) = 1$ for $t \leqq a$, $\varphi(t) = 0$ for $t \geqq 2a$ and we set $\mathscr{H} = \bigcup_{a>0} \mathscr{H}_a$.

THEOREM 1.3. *Let $P \in \mathscr{D}_0'((X; E))$. Then $P \in \mathscr{D}_0'^{-1}$ if and only if $\varphi P \in \mathscr{D}_0'^{-1}$ for all $\varphi \in \mathscr{H}$.*

The proof (an easy consequence of Theorem 1.2) can be found in [6], p. 353. Note that, since $\varphi P \in \mathscr{S}_0'$, whether or not φP belongs to $\mathscr{D}_0'^{-1}$ can be checked by means of Theorem 1.1. An improvement (of one half) of Theorem 1.3 will be seen in § 5.

We refer the reader to [6] for an account of the connections of the present theory with differential and hereditary equations; suffice it to say here that when $P = \delta' \otimes I - \delta \otimes A$ (A a closed operator with domain $D(A) \subseteq E$) then $P \in \mathscr{D}_0'((E; X))^{-1}$, $X = D(A)$ endowed with the graph norm, if and only if A generates a regular distribution semigroup ([9]); generators of the classical semigroups in [7] are encountered if diverse continuity or summability assumptions are imposed on $S = P^{-1}$. Likewise, autonomous functional differential equations are included in our treatment.

2. The abstract parabolic case. For many applications, the information that a distribution P belongs to $\mathscr{D}_0'^{-1}$ is insufficient, and smoothness conditions in $t \geqq 0$ or $t > 0$ are required. Let $\mathscr{F}$ be a space of (E, X)-valued functions defined in $t > 0$. We write $P \in (\mathscr{D}_0' \cap \mathscr{F})^{-1}$ to indicate that $P \in \mathscr{D}_0'^{-1}$ and that $S = P^{-1}$ coincides with a function in $\mathscr{F}$ for $t > 0$. The most useful spaces in this connection are $\mathscr{C}_+^m((E, X))$ or simply $\mathscr{C}_+^m$ (m a nonnegative integer) consisting of all $(E: X)$-valued functions f defined and m times continuously differentiable in $t > 0$ (infinitely differentiable if $m = \infty$). A distribution $P \in (\mathscr{D}_0' \cap \mathscr{C}_+^\infty)^{-1}$ is called *abstract parabolic*. In case $P \in \mathscr{E}_0'$, the subspace of $\mathscr{D}_0'$ consisting of distributions with compact support, abstract parabolic distributions can be characterized as follows: (note that we have here $\pi(P) = C$).

THEOREM 2.1. *Let $P \in \mathscr{E}_0'$. Then P is abstract parabolic if and only if for every $\alpha > 0$ there exists $\beta = \beta(\alpha)$, $\omega = \omega(\alpha) > 0$ such that $\rho(P)$ contains the reversed logarithmic region*

$$\Omega(\alpha, \beta, \omega) = \{\lambda; \operatorname{Re} \lambda \geqq \min(\beta - \alpha \log |\lambda|, \omega)\} \tag{2.1}$$

and

$$||\mathfrak{P}(\lambda)^{-1}||_{(E:X)} \leqq C(1 + |\lambda|)^m (\lambda \in \Omega(\alpha, \beta, \omega)) \tag{2.2}$$

where C, but not m, may depend on α.

The proof can be found in [6], p. 359. The following inverse Laplace transform representation holds for $S = P^{-1}$ and its derivatives in $t > 0$: if $n = 0, 1, 2, \cdots, \alpha > 0, \Gamma$ is the boundary of the region $\Omega(\alpha, \beta, \omega)$ and $t > (m + n + 1)/\alpha$ then

$$S^{(n)}(t) = \frac{1}{2\pi i} \int_\Gamma \lambda^n e^{\lambda t} \mathfrak{P}(\lambda)^{-1} d\lambda \, . \tag{2.3}$$

This formula will be of interest in § 3. One of its consequences is the exponential growth of S and all its derivatives at ∞: we can find constants $\omega_n > 0$ such that, for every $n = 0, 1, \cdots$ and every $\varepsilon > 0$ there exists a constant $C_{n,\varepsilon}$ such that

$$||S^{(n)}(t)||_{(E:X)} \leqq C_{n,\varepsilon} e^{\omega_n t} (t \geqq \varepsilon) \, . \tag{2.4}$$

We point out that the sufficiency part of Theorem 2.1, as well as its consequences (2.3) and (2.4) hold under the sole assumption that $P \in \mathscr{S}_0'$.

Section 3 of this paper is devoted to versions of Theorem 2.1 for the spaces $\mathscr{F} = \mathscr{C}(\mathscr{M})$ of infinitely differentiable functions introduced by Hadamard, defined by bounds on all derivatives of their elements. (Theorems 3.5 and 3.6) We follow here closely ideas of Barbu [1] for the case $P = \delta' \otimes I - \delta \otimes A$. In particular, analogues of Theorem 2.1 are obtained for the classes $\mathscr{A}$ of real analytic functions and $\mathscr{A}(\phi)$ of functions analytic in a sector, the last ones generalizing results of Da Prato and Mosco in [2] and [3]. We consider in § 4 the question of whether a distribution P having an inverse smooth in $t > 0$ must itself be smooth in $t > 0$; under the assumption that $P \in \mathscr{E}_0'$ (also used in § 3) we show that the answer to this question is in the affirmative when E and X are finite dimensional (Theorem 4.6) but not in general (see the comments at the beginning of § 4). We examine in § 5 "continuable" properties of $S = P^{-1}$ (that is, properties whose validity for all $t>0$ are deduced from corresponding properties in certain finite intervals). Finally, the possibility of extending the results in § 2, § 3 and § 4 to distributions in several variables is briefly examined in § 6. Throughout the rest of this section we discuss (mostly with counterexamples) the problem of extending the results pertaining to the abstract parabolic case to distributions $P \in \mathscr{S}_0'$ and $\mathscr{D}_0'$.

Distributions P which have compact support correspond, roughly speaking, to systems having finite memory (see [6] for more details). It is of great importance in applications to extend results like Theorem 2.1 to general distributions $P \in \mathscr{D}_0'$ (some restriction at infinity like, say, $P \in \mathscr{S}_0'$ is probably reasonable in view of the principle of fading memory: see [16]). Unfortunately, many likely extensions fail to work. Counterexamples may be constructed with the help of the parametrix method. Assume we can find an *approximate inverse* or *parametrix* for P, that is, a distribution $S_0 \in \mathscr{D}_0'((E, X))$ such that

$$P*S_0 = \delta \otimes I - \Phi, \quad S_0*P = \delta \otimes J - \psi \tag{2.5}$$

where $\Phi \in \mathscr{D}_0'((E; E))$ and $\psi \in \mathscr{D}_0'((X; X))$. Then we have, formally,

$$\begin{aligned} &P*\{S_0*(\delta \otimes I - \Phi)^{*-1}\} = \delta \otimes I\,, \\ &\{(\delta \otimes J - \psi)^{*-1}*S_0\}*P = \delta \otimes J\,, \end{aligned} \tag{2.6}$$

where the convolution inverses $(\delta \otimes I - \Phi)^{*-1}$ and $(\delta \otimes J - \psi)^{*-1}$ are defined by their Neumann series,

$$\begin{aligned} &(\delta \otimes I - \Phi)^{*-1} = \delta \otimes I + \Phi + \Phi*\Phi + \cdots\,, \\ &(\delta \otimes J - \psi)^{-1} = \delta \otimes J + \psi + \psi*\psi + \cdots\,. \end{aligned} \tag{2.7}$$

Accordingly, if we can justify convergence of these two series in $\mathscr{D}'$ (or more generally, of

$$
(2.8) \qquad \begin{aligned} & S_0 + S_0 * \Phi + S_0 * \Phi^{*2} + \cdots , \\ & S_0 + \psi * S_0 + \psi^{*2} * S_0 + \cdots , \end{aligned}
$$

we can construct a two-sided inverse of P and thus show that $P \in \mathscr{D}'^{-1}$; if S_0, Φ, ψ are smooth enough and convergence occurs in a sufficiently strong topology for $t > 0$, the inverse thus constructed will be smooth in $t > 0$.

In what follows, we shall write sometimes distributions in "functional" notation; as in Schwartz [14] we indicate by $f(\hat{t})$ the function $t \to f(\hat{t})$ (or the distribution it defines) and we use the same rule for distributions: for instance, $\delta(\hat{t} - 2)$ indicates the Dirac measure centered at $t = 2$. Derivatives with respect to t will be indicated by D and its powers.

LEMMA 2.2. *Let $P \in \mathscr{D}_0'((X; E))$, S_0 a parametrix for P in $\mathscr{D}_0'((E; X)) \cap \mathscr{C}_+^\infty((E; X))$ such that Φ and ψ are infinitely differentiable in $t \geqq 0$ (in the sense of the norms of $(E; E)$ and $(X; X)$ respectively). Then $P \in (\mathscr{D}_0' \cap \mathscr{C}_+^\infty)^{-1}$.*

Proof. Define, for $\alpha > 0$

$$
Y_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha) \quad (t > 0), \quad Y_\alpha(t) = 0 \quad (t < 0) .
$$

Then Y_α defines a distribution in $\mathscr{S}_0'$ that can be continued analytically to all complex α (see Schwartz [13], especially p. 174) and satisfies $Y_\alpha * Y_\beta = Y_{\alpha+\beta}$, $Y_\alpha' = Y_{\alpha-1}$, $Y_{-m} = \delta^{(m)}$ for $m = 0, 1, 2, \cdots$. When $\alpha > -1$, convolution by Y_α is the classical Riemann-Liouville operator; if m is a positive integer Y_m is simply the operator of integration from 0 to t iterated m times. Keeping in mind that any distribution in $\mathscr{D}'$ is (locally) a derivative of sufficiently high order of a continuous function, and that S_0 is smooth in $t > 0$, $Y_m * S_0$ is continuous in $-\infty < t < \infty$ for m large enough while remaining infinitely differentiable in $t > 0$. Since $\|\Phi(t)\|$ must be bounded in bounded subsets of $t \geqq 0$, the following estimate is rather obvious:

$$
(2.9) \qquad \|\Phi^{*n}(t)\| \leqq \frac{C}{(n-1)!} t^n \qquad (0 \leqq t \leqq a,\, n = 1, 2, \cdots)
$$

where $a > 0$ is arbitrary and the constant C may depend on a. Also, $\Phi^{*n}(0) = \cdots = D^{n-1}\Phi^{*n}(0) = 0$. It is easy to see that each of the terms after the first in the first series (2.7) is infinitely differentiable in $t \geqq 0$; moreover, if $n \geqq k+1$ we have $D^k \Phi^{*n} = D^k(\Phi^{*(n-k-1)} * \Phi^{*(k+1)}) = \Phi^{*(n-k-1)} * D^k \Phi^{*(k+1)}$, thus the estimate (2.9) implies convergence of $\sum \Phi^{*n}$ and of all its term-by-term differentiations, uniformly on compacts of $t \geqq 0$. It follows that $(\delta \otimes I - \Phi)^{*-1} = \delta \otimes I + \mathscr{M}(\hat{t})$, where $\mathscr{M}$ is an $(E; E)$-valued function infinitely differentiable in

$t \geqq 0$. Consider then $S_0*(\delta \otimes I - \Phi)^{*-1} = S_0 + S_0*\mathscr{M}$: by hypothesis, the first term belongs to $\mathscr{C}_+^\infty$. As for the second, we have

$$\begin{aligned} D^k(S_0*\mathscr{M}) &= D^{k+m}((Y_m*S_0)*\mathscr{M}) \\ &= (Y_m*S_0)*\left(\sum_{j=0}^{k+m-1} \delta^{(j)} \otimes D^{k+m-1-j}\mathscr{M}(0) + D^{k+m}\mathscr{M}\right) \\ &= \sum_{j=0}^{k+m-1} D^j(Y_m*S_0)D^{k+m-1-j}\mathscr{M}(0) + (Y_m*S_0)*D^{k+m}\mathscr{M} \end{aligned}$$

which is obviously infinitely differentiable in $t > 0$. Entirely similar manipulations with ψ produce a left inverse for P which must then coincide with the right inverse just constructed thus closing the argument. Note, incidentally, that smoothness assumptions on both Φ and ψ are not actually necessary: if Φ is a assumed smooth as above, we only need conditions on ψ assuring convergence of the second series (2.7) in $\mathscr{D}'$; this suffices to produce a left inverse in $\mathscr{D}_0'$, which is all that is needed here.

Although Lemma 2.2 has some interest in its own right, we shall use it only to produce a counterexample illustrating the failure of Theorem 2.1 in the case $P \in \mathscr{S}_0'$.

REMARK 2.3. Consider the distribution $P_0 = Y_1 \in \mathscr{S}(E = X = C)$. Then $P_0 \in \mathscr{D}_0'^{-1}$ and $P_0'^{-1} = Y_{-1} = \delta'$. Let g be an infinitely differentiable function in $t \geqq 0$ such that $g(0) = 0$. Then $(P_0 - g)*\delta' = \delta - g'$ hence δ' is a parametrix for $P_0 - g$ satisfying the assumptions in Lemma 2.2 and $f = P_0 - g \in (\mathscr{D}_0' \cap \mathscr{C}_+^\infty)^{-1}$. It follows that any function f, infinitely differentiable in $t \geqq 0$ and such that $f(0) = 1$ (in fact, such that $f(0) \neq 0$) belongs to $(\mathscr{D}_0' \cap \mathscr{C}_+^\infty)^{-1}$. Let now

$$f(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{-(n+m)} \exp\left\{ i\left(n + \frac{1}{m}\right)t \right\} .$$

It is easy to see that f is infinitely differentiable in $t \geqq 0$ and $f(0) = 1$. Since f is bounded in $t \geqq 0$ its Laplace transform exists in $\operatorname{Re}\lambda > 0$ and

$$\mathscr{L}f(\lambda) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{-(n+m)}\left(\lambda - i\left(n + \frac{1}{m}\right)\right)^{-1} .$$

If $P = f(\hat{t}) \in (\mathscr{D}_0' \cap \mathscr{C}_+^\infty)^{-1}$ satisfies the conditions in Theorem 2.1, $1/\mathscr{L}f(\lambda)$ has an extension to some reversed logarithmic region $\Omega(\alpha, \beta, \omega)$; since it must vanish at all points $n + 1/m$ lying in this region, it must vanish identically, which results in a contradiction. Note that here $\pi(P)$ coincides with the right half-plane.

It is interesting to note that, although $P = f$ does not satisfy the assumptions of Theorem 2.1 the estimates (2.4) hold. In fact, $S^{(n)} = (P^{-1})^{(n)} = \delta^{(n+1)}*(\delta + g + g*g + \cdots)$. Using the fact that all derivatives of $g = 1 - f$ are bounded in $t \geqq 0$ and a combination of the ideas of Lemma 2.2 and Theorem 3.2 in [6] we can easily show that (2.4) holds for all n, ε.

REMARK 2.4. The example above shows that the characterization of the class $(\mathscr{E}_0 \cap \mathscr{C}_+^\infty)^{-1}$ in Theorem 2.1 does not extend to $(\mathscr{S}_0 \cap \mathscr{C}_+^\infty)^{-1}$ (although the sufficiency condition obviously does). We may then ask whether a local characterization of the type of Theorem 1.3 will work in the $\mathscr{C}_+^\infty$ case. Regrettably, the answer is in the negative. On the one hand, it is easy to see that a distribution $P \in \mathscr{D}_0'$ such that $\varphi P \in (\mathscr{E}_0' \cap \mathscr{C}_+^\infty)^{-1}$ for all $\varphi \in \mathscr{H}$ must belong to $(\mathscr{D}_0' \cap \mathscr{C}_+^\infty)^{-1}$. On the other hand, however, the converse is false even if $P \in \mathscr{E}_0$. To see this, we use an example already exploited in [6] for a different purpose. Let $\{T(t);\, t \geqq 0\}$ be a strongly continuous semigroup (or, more generally, a regular distribution semigroup) in the Banach space X with infinitesimal generator A. Set $E = D(A)$ with its graph norm. Then $P = T \in (\mathscr{D}_0' \cap \mathscr{C}_+^\infty)^{-1}$ with $S = P^{-1} = \delta' \otimes I - \delta \otimes A$. If $\varphi \in \mathscr{H}_a$ we have $\varphi P*(\delta' \otimes I - \delta \otimes A) = \delta \otimes I + \varphi' T$, $(\delta' \otimes I - \delta \otimes A)*\varphi P = \delta \otimes J + \varphi' T$. We can then construct the inverse $S_\varphi = (\varphi P)^{-1}$ by the method outlined earlier:

$$\begin{aligned} S_\varphi &= (\delta' \otimes I - \delta \otimes A)*(\delta \otimes I - \varphi' T)^{*-1} \\ &= \delta' \otimes I - \delta \otimes A - \varphi'' T + \mathscr{R} \end{aligned}$$

where $\mathscr{R}$ has support in $t \geqq 2a$ (note that $\Phi = \varphi' T$ has support in $t \geqq a$, hence Φ^{*n} has support in $t \geqq na$ and the first series (2.7) is locally a finite sum, therefore converges in $\mathscr{D}'$). It follows that $S_\varphi = \varphi'' T$ in $a < t < 2a$, thus S_φ shall not be even $(E; X)$-continuous there if T is not, as will be the case, say, if T is a group or distribution group with unbounded generator. The situation is not improved if we require that $P \in \mathscr{E}_0$, since there exist semigroups T which vanish outside of a finite interval (see [7], p. 537) and exhibit no smoothness properties beyond strong continuity.

The present example is of interest also in that it produces a distribution P having an inverse smooth in $t > 0$ but failing itself to be smooth there. Note that the smoothness of S can be improved (and that of P worsened) by taking $P_m = Y_{-m}*P = P^{(m)}$, in which case $S_m = P_m^{-1} = Y_{m-1} \otimes I - Y_m \otimes A$. As we shall see in § 4), this kind of situation may not arise when E and (or) X are finite dimensional, at least when P has compact support.

3. Quasi-analytic classes. Let $\mathscr{M} = \{M_n\}$ be a sequence of positive numbers, I an open interval in the real line, F a Banach space. We denote by $\mathscr{C}(I, \mathscr{M}; F)$ (or simply $\mathscr{C}(I, \mathscr{M})$) the linear space of all F-valued functions f defined and infinitely differentiable in I and such that

$$\|f^{(n)}(t)\| \leqq C^{n+1}M_n \quad (t \in I,\ n = 0, 1, \cdots) \tag{3.1}$$

where C may depend on f. Diverse assumptions will be imposed on the sequence $\mathscr{M}$ in what follows; since not all these hypotheses are used in all the results, we shall specify each time the ones needed.

If $\mathscr{M}$ satisfies

$$M_0 = 1,\ M_n^2 \leqq M_{n-1}M_{n+1} \quad (n = 1, 2, \cdots) \tag{3.2}$$

(the second requirement simply means that the sequence $\{\log M_n\}$ is convex) we easily obtain that $M_jM_{n-j} \leqq M_0M_n = M_n$ $(n = 0, 1, \cdots, 0 \leqq j \leqq n)$, thus it follows from the formula of Leibniz that for any $f \in \mathscr{C}(I, \mathscr{M}; F)$ and $\varphi \in \mathscr{C}(I, \mathscr{M}, C)$ we have $\varphi f \in \mathscr{C}(I, \mathscr{M}; F)$. On the other hand, if

$$M_{n+1} \leqq K^n M_n \qquad (n = 0, 1, \cdots) \tag{3.3}$$

for some constant K, any class $\mathscr{C}(I, \mathscr{M})$ is closed under (iterated) differentiation.

The following result, valid for a general sequence $\mathscr{M}$ is essentially due to Mandelbrojt (see [10] and [12], p. 103).

LEMMA 3.1. *Let $\mathscr{M} = \{M_j\}$ be an arbitrary sequence of positive numbers with $M_0 = 1$ and let*

$$Q_n = \sum_{j=1}^{n} (M_{j-1}/M_j)\ . \tag{3.4}$$

Finally, let n be a positive integer. Then there exists a function $\psi_n \in \mathscr{D}$ with support in $|t| \leqq 1$ and such that

$$\psi_n(t) \geqq 0\ , \qquad \int \psi_n(t)dt = 1\ ,$$

$$|\psi_n^{(j)}(t)| \leqq (2Q_n + 4)^{j+1}M_j \quad (-\infty < t < \infty,\ 1 \leqq j \leqq n)\ . \tag{3.5}$$

Proof. Let $\lambda_j = M_{j-1}/M_j$ for $1 \leqq j \leqq n$; for $j > n$ select $\lambda_j > 0$ at will under the only condition that

$$\sum_{j=0}^{\infty} \lambda_j \leqq Q_n + 1\ . \tag{3.6}$$

Define

$$\psi_{nk} = \eta * \chi_1 * \cdots * \chi_k \tag{3.7}$$

where η is a nonnegative continuous function in $(-\infty, \infty)$ with support in $|t| \leqq 1$, integral 1 and such that $\eta(t) \leqq 1$ (say, $\eta(t) = \max(0, 1 - |t|)$) and $\chi_k(x) = (2\lambda_k)^{-1}$ for $|t| \leqq \lambda_k$, $\chi_k(t) = 0$ elsewhere. It follows from well known properties of convolutions that each ψ_{nk} is nonnegative, does not surpass 1, has support in $|t| \leqq 2(1 + \lambda_1 + \cdots + \lambda_k) \leqq 2Q_n + 4$ and integral 1. Also, each ψ_k is k times continuously differentiable, the successive derivatives given by the formulas

$$\psi'_{nk}(t) = \frac{1}{2\lambda_1}\{(\eta * \chi_2 * \cdots * \chi_k)(t + \lambda_1) - (\eta * \chi_2 * \cdots * \chi_k)(t - \lambda_1)\} ,$$

$$\begin{aligned}\psi''_{nk}(t) = \frac{1}{4\lambda_1\lambda_2}\{&(\eta * \chi_3 * \cdots * \chi_k)(t + \lambda_1 + \lambda_2) \\ &- (\eta * \chi_3 * \cdots * \chi_k)(t - \lambda_1 + \lambda_2) \\ &- (\eta * \chi_3 * \cdots * \chi_k)(t - \lambda_1 + \lambda_2) \\ &+ (\eta * \chi_3 * \cdots * \chi_k)(t - \lambda_1 - \lambda_2)\} ,\end{aligned}$$

etc. It results from these formulas that for every $k > j$,

$$|\psi_{nk}^{(j)}(t)| \leqq M_j \quad (-\infty < t < \infty,\ 1 \leqq j \leqq n,\ k \geqq 1) . \tag{3.8}$$

Finally, we observe that, if $j < k < l$, $j \leqq n$, $\psi_{nk}^{(j)} - \psi_{nl}^{(j)}$ can be expressed as the sum of 2^j terms; each of them, save by a constant depending only on j, is of the form

$$(\xi_{jk} * \chi_{k+1} * \cdots * \chi_l)(t + \nu_j) - \xi_{jk}(t + \nu_j) \tag{3.9}$$

where $\xi_{jk} = \eta * \chi_{j+1} * \cdots * \chi_k$ and $\nu_j = \pm\lambda_1 \pm \cdots \pm \lambda_j$ for some choice of signs. Note that ξ'_{jk} is uniformly bounded with respect to t (independently of k) thus ξ_{jk} is uniformly Lipschitz continuous. Write $\chi_{kl} = \chi_{k+1} * \cdots * \chi_l$. Then χ_{kl} is nonnegative, has integral 1 and support in $|t| \leqq 2(\lambda_{k+1} + \cdots + \lambda_l) = \mu_{kl} \to 0$ as $k, l \to \infty$. Hence we can estimate (3.9) in absolute value by

$$\begin{aligned}&\int |\xi_{jk}(t - s) - \xi_{jk}(t)|\chi_{kl}(s)ds \\ &\quad \leqq \max\{|\xi_{jk}(s) - \xi_{jk}(t)|; |s - t| \leqq \mu_{kl}\} \\ &\quad \leqq C\mu_{kl} \longrightarrow 0 .\end{aligned}$$

It follows then that for each j the sequence $\{\psi_{nk}^{(j)}\}$ is Cauchy in the uniform norm, hence $\{\psi_{nk}\}$ converges uniformly together with all its derivatives to an infinitely differentiable nonnegative function $\hat{\psi}_n$ having support in $|t| \leqq 2(1 + \lambda_1 + \lambda_2 + \cdots) \leqq 2Q_n + 4$ and (in view of (3.8)) satisfying $\hat{\psi}_n^{(j)}(t) \leqq M_j$, $1 \leqq j \leqq n$; therefore, the function $\psi_n(t) = (2Q_n + 4)\hat{\psi}_n((2Q_n + 4)t)$ possesses all the required properties.

REMARK 3.2. Assume $\mathscr{M}$ is such that $\lim Q_n = Q < \infty$ (i.e.,

$$\sum (M_{j-1}/M_j) < \infty) . \tag{3.10}$$

Then we can run the previous argument with $\lambda_n = M_{n-1}/M_n$ for all n; we obtain in this fashion a nonnegative function $\psi \in \mathscr{D}$ with support in $|t| \leqq 1$, integral 1 and such that

$$|\psi^{(j)}(t)| \leqq (2Q + 4)^{j+1} M_j \quad (-\infty < t < \infty,\ j = 0, 1, \cdots) . \tag{3.11}$$

This construction is impossible if (3.10) fails: in fact, if $Q_n \to \infty$ a function ψ in a class $\mathscr{C}(I, \mathscr{M})$ that vanishes at a point $t_0 \in I$ together with all its derivatives must vanish identically. A sequence $\mathscr{M}$ such that (3.10) does not hold is called *quasi-analytic*. For instance, $\mathscr{M} = \{n!\}$ is quasi-analytic; for each I $\mathscr{C}(I, \mathscr{M}, F) = \mathscr{A}(I; F)$ consists exactly of all functions real analytic in the closure of I. Under the logarithmic convexity assumption (3.2), quasi-analyticity is equivalent to the condition

$$\int_0^\infty \frac{\log\ \Theta(t)}{1 + t^2} dt = \infty \tag{3.12}$$

where

$$\Theta(t) = \sum_{n=0}^{\infty} t^j / M_j \tag{3.13}$$

and also to the condition $\sum M_n^{-1/n} = \infty$ (for these and other results see [10], [11] and especially [12]).

COROLLARY 3.3. *Let $\mathscr{M} = \{M_n\}$, $\{Q_n\}$ as in Lemma 3.1. Then, given a positive integer n there exists a function $\varphi_n \in \mathscr{D}$ with support in $|t| \leqq 2$, equal to 1 in $|t| \leqq 1$, not surpassing 1 anywhere and such that*

$$|\varphi_n^{(j)}(t)| \leqq 3(2Q_n + 4)^{j+1} M_j \quad (-\infty < t < \infty, 1 \leqq j \leqq n) . \tag{3.14}$$

If $\mathscr{M}$ is not quasi-analytic and $Q = \lim Q_n$ there exists $\varphi \in \mathscr{D}$ having support in $|t| \leqq 2$, equal to 1 in $|t| \leqq 1$, not surpassing 1 anywhere and such that

$$|\varphi^{(j)}(t)| \leqq 3(2Q + 4)^{j+1} M_j \quad (-\infty < t < \infty, 1 \leqq j \leqq \infty) . \tag{3.15}$$

Proof. Let χ be the characteristic function of $|t| \leqq 3/2$, ψ_n the function constructed in Lemma 3.1 (replaced by the function ψ

in Remark 3.2 in case $\mathscr{M}$ is not quasi-analytic). Then $\varphi_n = \chi * \psi_n$ ($\varphi = \chi * \psi$) does the required job.

We shall consider in what follows the classes $\mathscr{C}_+(\mathscr{M}; F)$ consisting of all F-valued functions infinitely often differentiable in $t > 0$ and belonging to $\mathscr{C}(I, \mathscr{M}; F)$ for every open interval I whose closure is contained in $(0, \infty)$. In the case $\mathscr{M} = \{n!\}$ we write $\mathscr{C}_+(\mathscr{M}; F) = \mathscr{C}_+(F)$; when confusion is not likely to occur we shall omit F from these labels. A sequence $\mathscr{M}$ is called *regular* if there exists another sequence $\tilde{\mathscr{M}} = \{\tilde{M}_n\}$ of positive numbers such that if $\tilde{Q}_n$ is defined from $\{\tilde{M}_n\}$ by (3.4) there exists a constant K independent of n such that

$$\tilde{Q}_n^j \tilde{M}_j M_{n-j} \leqq K^n M_n \quad (n = 0, 1, \cdots, 0 \leqq j \leqq n) . \tag{3.16}$$

A non quasi-analytic sequence $\mathscr{M}$ is always regular if (3.2) holds, as we see taking $\tilde{\mathscr{M}} = \mathscr{M}$ and $K = Q$. On the other hand some quasi-analytic sequences are regular, in particular the all important $\{n!\}$. This follows from:

LEMMA 3.4. *Let $\{M_n\} = \{(n!)\sigma_n\}$ with $\{\sigma_n\}$ positive and nondecreasing. Then $\{M_n\}$ is regular.*

Proof. We take $\tilde{M}_n = 1$ for all n. Then $\tilde{Q}_n = n$ and, by Stirling's formula,

$$\begin{aligned} \tilde{Q}_n^j \tilde{M}_j M_{n-j} = n^j \sigma_{n-j}(n-j)! &\leqq n^j (n-j)^{n-j} \sigma_{n+j} \\ &\leqq n^n \sigma_n \leqq e^{n+1}(n!)\sigma_n \end{aligned}$$

for n sufficiently large.

We note that condition (3.16) implies a brisk rate of increase for $\mathscr{M}$, at least if $\tilde{M}_n \geqq m > 0$; in fact, if $\tilde{\mathscr{M}}$ is not quasi-analytic $\mathscr{M}$ itself will not be quasi-analytic as $M_n^{1/n} \geqq K^{-1}\tilde{Q}_n(M_0\tilde{M}_n)^{1/n}$, hence $\sum M_n^{-1/n} < \infty$. On the other hand, if $\tilde{\mathscr{M}}$ is quasi-analytic, $M_n^{-1/n} = o(\tilde{M}_n^{-1/n})$ since $\tilde{Q}_n \to \infty$.

THEOREM 3.5. *Let $\mathscr{M} = \{M_j\}$ be regular and satisfy (3.3). Assume $P \in \mathscr{E}_0'((X; E))$ belongs to $\{\mathscr{D}_0'((E; X)) \cap \mathscr{C}_+(\mathscr{M}; (E; X))\}^{-1}$. Then, for every $\varepsilon > 0$ there exist positive constants $\alpha = \alpha(\varepsilon)$, $\beta = \beta(\varepsilon)$ such that $\rho(P)$ contains the region*

$$\Theta(\alpha, \beta, \gamma) = \{\lambda; \operatorname{Re} \lambda \geqq -\gamma \log \Theta(\alpha|\lambda|) + \beta\} \tag{3.17}$$

and

$$\|\mathfrak{P}(\lambda)^{-1}\|_{(E; X)} \leqq C(1 + |\lambda|)^m e^{\varepsilon|\operatorname{Re} \lambda|} \tag{3.18}$$
$$(\lambda \in \Theta(\alpha, \beta, \gamma))$$

where C, but not m or γ, may depend on ε.

Proof. Let $n \geqq 0$ be, for the moment, fixed. We make use of the function φ_n in Corollary 3.3, this time constructed with respect to the sequence $\{\tilde{M}_j\}$ in (3.16). Given $\varepsilon > 0$, set $\varphi_{\varepsilon,n}(t) = \varphi_n(2t/\varepsilon)$. We have

$$P * \varphi_{\varepsilon,n} S = \delta \otimes I - \Phi_{\varepsilon,n}, \quad \varphi_{\varepsilon,n} S * P = \delta \otimes J - \psi_{\varepsilon,n} \tag{3.19}$$

where $\Phi_{\varepsilon,n} = P * (1 - \varphi_{\varepsilon,n})S = \delta \otimes I - P * \varphi_{\varepsilon,n} S$ is an infinitely differentiable $(E; E)$-valued function which vanishes for $t \leqq \varepsilon/2$ and for $t \geqq a + \varepsilon$ if the support of P is contained in $t \leqq a$; likewise, $\psi_{\varepsilon,n} = (1 - \varphi_{\varepsilon,n})S * P = \delta \otimes J - \varphi_{\varepsilon,n} S * P$ is an infinitely differentiable $(X; X)$-valued function enjoying the same properties as $\Phi_{\varepsilon,n}$. Take now m so large that $Y_m * P = f$ is continuous and vanishes in $t \leqq 0$; since $\Phi_{\varepsilon,n} = P * (1 - \varphi_{\varepsilon,n} S) = (Y_m * P) * (Y_{-m} * (1 - \varphi_{\varepsilon,n})S) = f * D^m(1 - \varphi_{\varepsilon,n})S$, we have

$$\Phi^{(n)}_{\varepsilon\, n}(t) = \int_0^t f(t - s)\{1 - \varphi_{\varepsilon,n}(s))S(s)\}^{(m+n)} ds\ . \tag{3.20}$$

Taking into account that $S \in \mathscr{C}((\varepsilon/2, a + \varepsilon); \mathscr{M})$, and the inequalities (3.14) for the derivatives of φ_n we obtain

$$\begin{aligned} &\|\Phi^{(n)}_{\varepsilon\, n}(t)\|_{(E;\, E)} \\ &\leqq C_1 \sum_{j=0}^{m=n} \binom{m+n}{j} (2\tilde{Q}_n + 4)^{j+1} (2\varepsilon)^{-j} \tilde{M}_j C_2^{m+n-j} M_{m+n-j} \end{aligned} \tag{3.21}$$

where the constants C_1 and C_2 are independent of n (note that it follows from (3.3) and (3.16) for $j = 1$ that $\tilde{Q}_n \leqq C_3^n$ for some constant C_3). Making use of (3.3) we deduce the existence of a constant C, depending on ε but not on n such that

$$\|\Phi^{(n)}_{\varepsilon,n}(t)\| \leqq C^{n+1} M_n \quad (t \geqq 0,\ n = 0, 1, 2, \cdots)\ ; \tag{3.22}$$

henceforth, C will denote diverse constants independent of n. We compute next the Laplace transform of $\Phi_{\varepsilon,n}$. Repeated integration by parts yields

$$\mathscr{L}\Phi_{\varepsilon,n}(\lambda) = \lambda^{-n} \int_0^\infty e^{-\lambda t} \Phi^{(n)}_{\varepsilon,n}(t) dt$$

hence

$$\|\mathscr{L}\Phi_{\varepsilon,n}(\lambda)\| \leqq C^{n+1} M_n |\lambda|^{-n} \quad (\operatorname{Re} \lambda \leqq 0) \tag{3.23}$$

$$\|\mathscr{L}\Phi_{\varepsilon,n}(\lambda)\| \leqq C^{n+1} M_n |\lambda|^{-n} e^{-b \operatorname{Re} \lambda} \quad (\operatorname{Re} \lambda \leqq 0) \tag{3.24}$$

for $n = 0, 1, 2, \cdots$, where $b = a + \varepsilon$. Choose now δ, $0 < \delta < 1$, and assume that for $\operatorname{Re}\lambda \leqq 0$,

$$C^{n+1}M_n|\lambda|^{-n}e^{-b\operatorname{Re}\lambda} > \delta \tag{3.25}$$

for all $n = 0, 1, \cdots$. Then

$$\begin{aligned} e^{-b\operatorname{Re}\lambda} &\geqq \sup_n \frac{\delta|\lambda|^n}{C^{n+1}M_n} \\ &> \frac{\delta}{C}\sum_{n=1}^{\infty}\frac{|\lambda|^n}{(3C)^n M_n} = \frac{\delta}{C}\,\Theta\Big(\frac{|\lambda|}{3C}\Big) \end{aligned}$$

Accordingly, if we assume, as we may, that $\varepsilon<1$ (so that $b<a+1$), and we take λ in $\Theta(\alpha, \beta, \gamma)$ with $\gamma = 1/(\alpha+1)$, $\alpha = 1/3C$, $\beta = -(a+1)^{-1}\log(\delta/C)$ and $\operatorname{Re}\lambda \leqq 0$ at least one of the inequalities (3.25) will hold and

$$\|\mathscr{L}\Phi_{\varepsilon,n}(\lambda)\|_{(E;E)} \leqq \delta\,. \tag{3.26}$$

On the other hand, (3.21) (say, for the same n) implies (3.24) for $\operatorname{Re}\lambda \geqq 0$ and $|\lambda|$ sufficiently large, thus modifying β if necessary we may assume that (3.24) holds in $\Theta(\alpha, \beta, \gamma)$. Operating with $\psi_{\varepsilon,n}$ in an entirely similar way we deduce that, if necessary adjusting the parameters α, β, γ,

$$\|\mathscr{L}\psi_{\varepsilon,n}(\lambda)\|_{(X;X)} \leqq \delta \tag{3.27}$$

in $\Theta(\alpha, \beta, \gamma)$. Returning to $\Phi_{\varepsilon,n}$ we observe that (3.26) implies that $(I - \mathscr{L}\Phi_{\varepsilon,n}(\lambda))^{-1}$ (given by its Neumann series) exists in $(E; E)$ for $\lambda \in \Theta(\alpha, \beta, \gamma)$ and

$$\|(I - \mathscr{L}\Phi_{\varepsilon,n}(\lambda))^{-1}\| \leqq (1-\delta)^{-1}$$

there. Taking the Laplace transform of the first equation (3.19) and postmultiplying by $(I - \mathscr{L}\Phi_{\varepsilon,n}(\lambda))^{-1}$ we obtain

$$\mathfrak{P}(\lambda)\mathscr{L}(\varphi_{\varepsilon,n}S)(\lambda)(I - \mathscr{L}\Phi_{\varepsilon,n}(\lambda))^{-1} = I \tag{3.28}$$

for $\lambda \in \Theta(\alpha, \beta, \gamma)$; working in a similar way with the second equation

$$(J - \mathscr{L}\psi_{\varepsilon,n}(\lambda))^{-1}\mathscr{L}(\varphi_{\varepsilon,n}S)(\lambda)\mathfrak{P}(\lambda) = J \tag{3.29}$$

thus $\mathfrak{P}(\lambda)^{-1}$ indeed exists in $\Theta(\alpha, \beta, \gamma)$. By virtue of (3.24), in order to establish the estimate (3.18) we only need to bound $\mathscr{L}(\varphi_{\varepsilon,n}S)$ in the $(E; X)$-norm. This we do as follows. Let m be an integer such that $Y_m * S$ is a $(E; X)$-valued continuous function g vanishing in $t \leqq 0$. Then

$$(\varphi_{\varepsilon,n}g)^{(m)} = \varphi_{\varepsilon,n}S + \sum_{k=0}^{m-1}\binom{m}{k}\varphi_{\varepsilon,n}^{(m-k)}g^{(k)}\,.$$

Since S is infinitely differentiable in $t > 0$, so is g and we can write

$$\varphi_{\varepsilon,n}S = (\varphi_{\varepsilon,n}g)^{(m)} + h ,$$

where h is an infinitely differentiable $(E; X)$-valued function with support in $\varepsilon/2 \leqq t \leqq \varepsilon$. Since $\varphi_{\varepsilon,n}g$ itself is a continuous function with support in $\varepsilon/2 \leqq t \leqq \varepsilon$, we obtain

$$\|\mathscr{L}(\varphi_{\varepsilon,n}S)(\lambda)\|_{(E:X)} \leqq C(1 + |\lambda|)^m e^{\varepsilon|\mathrm{Re}\,\lambda|}$$

everywhere in the complex plane. This completes the proof.

The converse of Theorem 3.5 holds under slightly different assumptions on the sequence $\mathscr{M}$.

THEOREM 3.6. *Let P be as in Theorem 3.5 a distribution in $\mathscr{E}_0'$, $\mathscr{M} = \{M_n\}$ a sequence of positive mumbers such that for any two positive integers k and m there exists a constant $K = K(k, m)$ independent of n such that*

$$M_{nk+m}^{1/k} \leqq K^n M_n \quad (n = 0, 1, \cdots) . \tag{3.30}$$

Assume P satisfies the conclusions of Theorem 3.5, that is for every $\varepsilon > 0$ there exist α, β such that $\mathfrak{P}(\lambda)^{-1}$ exists in $\Theta(\alpha, \beta, \gamma)$ and satisfies there (3.18) with m and γ independent of ε. Then $P \in (\mathscr{D}_0' \cap \mathscr{E}_+(\mathscr{M}))^{-1}$.

Proof. Let $\alpha > 0$; pick ε in the range $0 < \varepsilon \leqq 1/\alpha$ and select α', β' such that $\mathfrak{P}(\lambda)^{-1}$ exists in $\Theta(\alpha', \beta', \gamma)$ and satisfies (3.18) there. Observing that $\alpha \log|\lambda| \leqq \gamma \log \Theta(\alpha'|\lambda|) + C$ for large $|\lambda|$ we can assume, after some slight adjustments of the parameters involved, that the reversed logarithmic region $\Omega(\alpha, \beta, \omega)$ in (2.1) is contained in $\Theta(\alpha', \beta', \gamma)$. It follows that $\mathfrak{P}(\lambda)^{-1}$ satisfies (2.2) with $m + 1$ instead of m in $\Omega(\alpha, \beta, \omega)$, thus it belongs to $(\mathscr{D}_0' \cap \mathscr{E}_+^\infty)^{-1}$ and the representation (2.3) holds. Our first task is to shift the domain of integration in (2.3) to Δ, the boundary of the region $\Theta(\alpha, \beta, \gamma)$; this is easily seen to be possible taking $t > \varepsilon$. We proceed now to estimate the integrals for $t \geqq 2\varepsilon$:

$$\|S^{(n)}(t)\|_{(E:X)} \leqq C\int_\Delta |\lambda|^n e^{(\mathrm{Re}\,\lambda)t} \|\mathfrak{P}(\lambda)^{-1}\| d|\lambda|$$

$$C\int_\Delta |\lambda|^{n+m} \exp\{-(t - \varepsilon)\gamma \log \Theta(\alpha|\lambda|)d|\lambda|$$

$$\leqq C\int_\Delta |\lambda|^{n+m} \Theta(\alpha|\lambda|)^{-\gamma\varepsilon} d|\lambda|$$

where here and afterwards C indicates diverse constants independent of n. We note next that $\Theta(\alpha|\lambda|)^{-\gamma\varepsilon} \leqq M_j^{\gamma\varepsilon}/(\alpha|\lambda|)^{j\gamma\varepsilon}$ for all j. Taking $\varepsilon = 1/k\gamma$ for some integer k, and selecting $j = k(n + m + 2)$ we have

$$||S^{(n)}(t)|| \leqq C^{n+1}M_{nk+k(m+2)}^{1/k} \leqq C^{n+1}M_n \quad (n = 0, 1, \cdots)$$

in $t \geqq 2\varepsilon$. Since ε is arbitrary, this ends the proof of Theorem 3.6.

The most important particular case of the previous theory is no doubt $\mathscr{M} = \{n!\}$. By Lemma 3.4 $\mathscr{M}$ is regular, and it is easy to see that (3.3) and (3.30) are satisfied (the last with the help of Stirling's formula). Accordingly, distributions $P \in (\mathscr{E}_0' \cap \mathscr{A}_+)^{-1}$ are completely characterized by Theorems 3.5 and 3.6. Since $\Theta(t) = e^t$, $|\lambda|$ and $\operatorname{Im}|\lambda|$ are comparable in the regions $\Theta(\alpha, \beta, \gamma)$ and we may assume these are defined in terms of two parameters α, β only (both dependent on ε) by inequalities of the form

$$\operatorname{Re}\lambda \geqq -\alpha|\operatorname{Im}\lambda| + \beta \,. \tag{3.31}$$

These regions are usually called *sectors*. It is no news of course that S, being real analytic, can be extended to a region D in the complex plane containing the positive half axis. However, the previous remarks imply that D will contain a "cornet shaped" region as precised below.

LEMMA 3.7. *Let $s, \phi > 0$. Denote by $\sum(\varepsilon, \phi)$ the open sector in the complex plane defined by $|\arg(\zeta - \varepsilon)| < \phi$, $\zeta \neq \varepsilon$ and let P be a distribution in $(\mathscr{E}_0' \cap \mathscr{A}_+)^{-1}$. Then $S = P^{-1}$ can be extended as an analytic $(E; X)$-valued function to a region of the form*

$$D = \bigcup_{\varepsilon > 0} \sum(\varepsilon, \phi(\varepsilon)) \tag{3.32}$$

where ϕ is a nondecreasing function in $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$; choose $\alpha > 0$ and β such that $\mathfrak{P}(\lambda)^{-1}$ exists in the region defined by (3.31). As essentially shown in the proof of Thorem 3.6, the path of integration in 2.3 can be deformed into Δ, consisting of the two half lines $\operatorname{Re}\lambda = -\alpha|\operatorname{Im}\lambda| + \beta$ if we take $t > 2\varepsilon$:

$$S(t) = \frac{1}{2\pi i}\int_{\Delta} \mathfrak{P}(\lambda)^{-1}e^{\lambda t}d\lambda \quad (t > \varepsilon) \,. \tag{3.33}$$

It is now a simple matter to show, in the style of the theory of analytic semigroups ([7], Chapter IV) that the upper half line can be deformed to $\operatorname{Re}\lambda = -\delta|\operatorname{Im}\lambda| + \beta$ for $\delta < 0$ thus making possible the insertion of $t = \zeta$ complex in (3.33) as long as $0 < \arg(\zeta - \varepsilon) <$

arctg $\alpha-$arctg δ and the corresponding extension of S. A symmetric argument shows then that S can be analytically extended to the sector $\Sigma(\varepsilon, \phi)$ with $\phi=$ arctg α. We note, incidentally, that if $0<\phi'<\phi<\pi/2$ an estimate of the form

$$\|S(\zeta)\| \leqq Ce^{\omega|\zeta|}(\zeta \in \Sigma(\varepsilon, \phi')) \tag{3.34}$$

holds, with ω a function of ε. To obtain a dependence of ϕ on ε having the desired properties, define $\phi(\varepsilon)$ as the largest ϕ such that S can be analytically extended to $\Sigma(\varepsilon, \phi)$. Obviously, ϕ is nondecreasing.

Of particular interest is the case where the cornet reduces to a wedge. For $0<\phi<\pi$ denote by $\mathscr{A}(\phi; E)$ or simply $\mathscr{A}(\phi)$ the space of distributions U defined as follows:

(a) U coincides in $t>0$ with a E-valued function f admitting analytic extension to $\Sigma(0, \phi)$ and such that (3.34) holds for $|\zeta| \geqq \varepsilon$ for some ω independent of ε.

(b) For each ϕ', $|\phi'| \leqq \phi$, $U_{\phi'}(t)=f(e^{i\phi'}t)$ for $t>0$, $U_{\phi'}(t)=0$ for $t \leqq 0$ defines a distribution in $\mathscr{D}_0'(E)$.

(c) The set $\{U_{\phi'}, |\phi'| \leqq \phi\}$ is bounded in $\mathscr{D}_0'(E)$.

we have the following result, where it is interesting to note that, unlike for the others in this section, we do not require that P have compact support.

THEOREM 3.8. *Let* $P \in \mathscr{S}_0'((X; E))$. *Assume* $P \in (\mathscr{D}_0'((E; X)) \cap \mathscr{A}(\phi; (E; X)))^{-1}$ *with* $0<\phi<\pi/2$. *Then* $\rho(P)$ *contains the half-plane* $\operatorname{Re}\lambda>\omega$, $\mathfrak{P}(\lambda)^{-1}$ *can be extended from there to a sector* (3.31) *with* $\alpha=tg\phi$ *and*

$$\|\mathfrak{P}(\lambda)^{-1}\| \leqq C(1+|\lambda|)^m \tag{3.35}$$

there. Conversely, if $\mathfrak{P}(\lambda)^{-1}$ *can be extended to a region of the form* (3.31) *and the estimate* (3.35) *holds there then* $P \in (\mathscr{D}_0' \cap \mathscr{A}(\phi))^{-1}$ *for* $0<\phi<$ arctg α.

The proof is straightforward and thus left to the reader. We note that Theorem 3.8 essentially generalizes Theorem 3.2.1 in [2] (see also [3]).

4. **Inverse theorems.** Let P be a distribution in $\mathscr{D}'^{-1}$ such that $S=P^{-1}$ is smooth off the origin. It seems natural to surmise that this should imply that P must be correspondingly smooth for $t>0$. Surprisingly enough, this is true when E and X are finite dimensional but false in general (a treatment for the case $S \in \mathscr{C}_+^\infty$ was indicated in [6], the proof little more than barely sketched). The infinite dimensional counterexample is in Remark 2.4 above;

the distributions P therein have inverses S arbitrarily smooth in $t > 0$ (in fact, analytic in the entire complex plane minus the origin) but fail to have any smoothness property themselves (we may arrange P to have compact support and to be arbitrarily rough everywhere).

Inverse theorems of the type considered in this section were given by Ehrenpreis [4] in the scalar case, in the rather more general situation where t is n-dimensional. We follow here closely the treatment of Hörmander [8] for the space $\mathscr{E}_+^\infty$, which we extend to the spaces $\mathscr{E}_+(\mathscr{M})$. Its basis is the following result.

LEMMA 4.1. *Let ε_0, δ, σ denote positive constants. Then there exists a constant ρ, $0 < \rho < 1$ depending only on ε_0, δ, σ such that if $u = u(x, y)$ is a harmonic function in $x^2 + y^2 < R^2$ and*

$$u(x, y) \leqq 0,\ u(x, y) \geqq \varepsilon|y| - \delta r \qquad (x^2 + y^2 < R^2) \tag{4.1}$$

for some ε, r such that $0 \leqq \varepsilon \leqq \varepsilon_0$, $0 \leqq r \leqq \rho R$ then

$$u(x, y) \leqq \varepsilon|y| + \sigma r \qquad (x^2 + y^2 \leqq r)\ . \tag{4.2}$$

For a proof see [8], p. 179. Throughout this section $\xi = \theta(\eta)$ will denote a positive function defined in $\eta \geqq a \geqq 1$ with $\theta(a) = 0$. For $\rho, \kappa > 0$ we denote by $\Xi(\rho, \kappa)$ the region of the (ξ, η)-plane defined by

$$|\xi| \leqq \rho\theta(\kappa\eta) \qquad (\eta \geqq a/\kappa) \tag{4.3}$$

(in particular, $\Xi(1, 1)$ is the region between the curves $\xi = \theta(\eta)$ and $\xi = -\theta(\eta)$). The symbol $\Pi(\rho, \kappa)$ denotes the union of all (closed) disks with center at $(0, \eta)(\eta \geqq a/\kappa)$ and radius $\rho\theta(\kappa\eta)$.

LEMMA 4.2. *Let θ be differentiable and nondecreasing in $\eta \geqq a$. Assume, moreover, that*

$$\theta'(\eta) \leqq c < 1 \qquad (\eta \geqq a) \tag{4.4}$$

and let $0 \leqq \rho \leqq 1$. Then there exists κ, $0 < \kappa < 1$ (depending only on c) such that

$$\Xi(\rho, \kappa) \subseteqq \Pi(\rho, \kappa) \subseteqq \Pi(1, \kappa) \subseteqq \Xi(1, 1)\ . \tag{4.5}$$

Proof. Let $\eta \geqq a$. Some elementary analysis shows that the distance $d(\eta)$ from $(0, \eta)$ to the region to the left of the positive. η-axis and the curve $\xi = \theta(\eta)$ is attained for the point $(\theta(\eta'), \eta')$ satisfying

$$\eta - \eta' = \theta(\eta')\theta'(\eta') .$$

We take now κ satisfying $0 < \kappa \leqq 1 - c^2$ but otherwise arbitrary. Observing that (4.4) implies that $\theta(\eta) \leqq c(\eta - a) \leqq c\eta$ we obtain

$$\begin{aligned} \kappa\eta \leqq \eta - c^2\eta &\leqq \eta - \theta(\eta')\theta(\eta) \\ &\leqq \eta - \theta(\eta')\theta(\eta') = \eta' . \end{aligned}$$

Accordingly $\theta(\kappa\eta) \leqq \theta(\eta') \leqq d(\eta)$ and the disk with radius $\theta(\kappa\eta)$ centered at $(0, \eta)$ is entirely contained in $\Xi(1, 1)$, proving the rightmost inclusion in (4.5). The center inclusion is evident, and the remaining one follows from the obvious fact that the vertical segment defined by $|\xi| \leqq \rho\theta(\kappa\eta)$ is contained in (is a diameter of) the disk with center at $(0, \eta)$ and radius $\rho\theta(\kappa, \eta)$. This ends the proof.

REMARK 4.3. Lemma 4.2 remains valid even if the constant c in (4.4) surpasses 1; we only have to replace $\theta(\eta)$ by $\theta(\varepsilon\eta)$ with $\varepsilon < 1/c$ and notice that $\Xi(1, \varepsilon) \subseteqq \Xi(1, 1)$.

LEMMA 4.4. *Let θ be a function satisfying the assumptions of Lemma 4.1 and such that for every κ, $0 < \kappa < 1$*

$$\log \eta = 0(\theta(\kappa\eta)) \quad (\eta \longrightarrow \infty) . \tag{4.6}$$

Let $\mathfrak{f}(\lambda) = \mathfrak{f}(\xi + i\eta)$ be a complex valued function defined and analytic in the region $\Xi(1, 1)$ of Lemma 4.2. Assume, moreover, that $\mathfrak{f}$ has no zeros there and that

$$|\mathfrak{f}(\lambda)| \leqq C_1|\lambda|^m e^{\varepsilon|\mathrm{Re}\,\lambda|} \quad (\lambda \in \Xi(1, 1)) \tag{4.7}$$

and

$$|\mathfrak{f}(i\eta)| \geqq C_2|\eta|^{-p} \quad (\eta \geqq a) \tag{4.8}$$

where m, p are nonnegative integers and $0 \leqq \varepsilon \leqq \varepsilon_0$, $\varepsilon_0, C_1, C_2 > 0$. Finally, let κ be the constant in Lemma 4.2. Then there exists ρ, $0 < \rho < 1$ independent of ε such that

$$|1/\mathfrak{f}(\lambda)| \leqq C|\lambda|^p e^{2\varepsilon|\mathrm{Re}\,\lambda|} \quad (\lambda \in \Xi(\rho, \kappa)) . \tag{4.9}$$

Proof. Let λ belong to the disk of center $\eta \geqq a/\kappa$ and radius $\rho\theta(\kappa\eta)$ for some $\rho \leqq 1$. Then

$$\begin{aligned} 1 - \rho c \leqq \frac{\eta - \rho\theta(\kappa\eta)}{\eta} &\leqq \frac{|\lambda|}{\eta} \\ &\leqq \frac{\eta + \rho\theta(\kappa\eta)}{\eta} \leqq 1 + \rho c . \end{aligned} \tag{4.10}$$

We define a harmonic function $u = u(x, y)$ in the circle $|z| \leqq \theta(\kappa\eta)$ $(z = x + iy)$ by the formula

$$u(z) = \log \frac{C_2}{(1 + c)^p \eta^p |f(i\eta + iz)|} \,. \tag{4.11}$$

It follows from (4.8) and (4.10) that $u(x, 0) \leqq 0(|x| < \theta(\kappa\eta))$. On the other hand, making use of (4.7) we obtain

$$u(x, y) \geqq -\varepsilon|y| - (p + m)\log \eta - C_3 \tag{4.12}$$

where $C_3 = \log (C_1 C_2^{-1}(1 + c)^{p+m})$. We apply now Lemma 4.1 for the constants $\varepsilon_0 = \varepsilon$, $\delta = 1$, $\sigma = \varepsilon$ in the particular case $r = \rho R$. By virtue of (4.6), if κ is the constant in Lemma 4.2 there exists $\eta_0 \geqq a/\kappa$ such that

$$\begin{gathered} u(x, y) \geqq -\varepsilon|y| - \delta\rho\theta(\kappa\eta) \\ (|z| < \theta(\kappa\eta),\ \eta \geqq \eta_0) \,. \end{gathered}$$

Therefore

$$\begin{gathered} u(x, y) \leqq \varepsilon|y| + \varepsilon\rho\theta(\kappa\eta) \\ (|z| \leqq \rho\theta(\kappa, \eta),\ \eta \geqq \eta_0) \,. \end{gathered} \tag{4.13}$$

Taking exponents and keeping (4.10) in mind, we obtain

$$|\mathfrak{g}(\lambda)| \leqq C|\lambda|^p e^{\varepsilon|\mathrm{Re}\,\lambda| + \varepsilon\rho\theta(\kappa\,\mathrm{Im}\,\lambda)}$$

in $\lambda \in \Xi(\rho, \kappa)$, $\mathrm{Im}\,\lambda \geqq \eta_0$ for $\mathfrak{g} = 1/\mathfrak{f}$; clearly the inequality will hold as well in $\Xi(\rho, \kappa)$, if necessary with a different constant C. We consider now the function

$$\mathfrak{h}(\lambda) = |\lambda|^{-p} e^{2\varepsilon\lambda} \mathfrak{g}(\lambda)$$

in the intersection of $\Xi(\rho, \kappa)$ with the left half-plane. By virtue of (4.8), $\mathfrak{h}$ is bounded on the imaginary axis: on the other hand, in the curve $\mathrm{Re}\,\lambda = \rho\theta(\kappa\,\mathrm{Im}\,\lambda)$,

$$|\mathfrak{h}(\lambda)| \leqq C|\lambda|^{-p} e^{2\varepsilon\rho(\kappa\,\mathrm{Im}\,\lambda) - 2\varepsilon\rho(\kappa\,\mathrm{Im}\,\lambda)}$$

hence $\mathfrak{h}$ is also bounded there. By one of the Phragmén-Lindelöf theorems, $\mathfrak{h}$ must be bounded in $\lambda \in \Xi(\rho, \kappa)$, $\mathrm{Re}\,\lambda \leqq 0$ whence (4.9) results there. A symmetric argument takes care of the intersection of $\Xi(\rho, \kappa)$ with the right half-plane.

Lemma 4.4 does not apply to the functions $\theta(\eta) = \alpha \log \eta - \beta$, essential in the abstract parabolic case. (See Theorem 2.1.) Since we only need the case $\varepsilon = 0$ here, the argument is considerably simpler.

LEMMA 4.5. *Let $\theta = \alpha \log \eta - \beta$, where β is so large that the assumptions of Lemma 4.2 are satisfied in $\eta \geqq a = e^{\beta/\alpha}$, $\mathfrak{f}$ a complex valued function defined and analytic in $\Xi(1, 1)$. Assume that* (4.7) *(with $\varepsilon = 0$) and* (4.8) *hold. Then, if κ is the constant in Lemma* 4.2

$$|1/\mathfrak{f}(\lambda)| \leqq C|\lambda|^{p'} \quad (\lambda \in \Xi(1/2, \kappa)) \tag{4.14}$$

where $p' = 4p + 3m$.

The proof is elementary: by (4.12) $v = u + (p + m)\log|\eta| + C_3 \geqq 0$ in $|z| \leqq \theta(\kappa\eta)$ whereas $v(0, 0) \leqq (p + m)\log \eta + C_3$. By Harnack's inequality,

$$u(x, y) \leqq 3(p + m)\log \eta + 3C_3 \quad (|z| \leqq \theta(\kappa\eta)/2) .$$

We obtain (4.14) from (4.11) and (4.10).

THEOREM 4.6. *Let E, X be finite dimensional, P a (X, E)-valued distribution with compact support contained in $t \geqq 0$ such that $P \in (\mathscr{D}' \cap \mathscr{E}_+^\infty)^{-1}$ (resp. $P \in (\mathscr{D}' \cap \mathscr{E}_+(\mathscr{M}))^{-1}$ with $\mathscr{M} = \{M_n\}$ a sequence satisfying the assumptions in both Theorem* 3.5 *and Theorem* 3.6, *plus*

$$(n + 1)M_n \leqq CM_{n+1} \quad (n = 0, 1, \cdots) . \tag{4.15}$$

Then P coincides for $t > 0$ with an infinitely differentiable $(X; E)$-valued function (resp. with a function in $\mathscr{E}_+(\mathscr{M})$). In particular, if $\mathscr{M}$ is quasi-analytic, P is zero in $t > 0$, hence it reduces to a differential polynomial

$$P = \sum_{j=1}^{m} \delta^{(j)} \otimes A_j,\ A_j \in (E; X) . \tag{4.16}$$

Proof. Assume that $P \in (\mathscr{D}' \cap \mathscr{E}_+^\infty)^{-1}$. Since $\mathfrak{P}(\lambda)$ must be an invertible operator for some λ it follows that $\dim X = \dim E$; we may then assume that $X = E$ and, introducing coordinates, work with matrices instead of linear operators. We use the characterization of the class $(\mathscr{D}' \cap \mathscr{E}_+^\infty)^{-1}$ in Theorem 1.3; according to it, given $\alpha > 0$ arbitrary, $\mathfrak{P}(\lambda)^{-1}$ exists in a reversed logarithmic region $\Omega = \Omega(\alpha, \beta, \omega)$ as defined in (2.1) and satisfies there an estimate of the form (2.2). Since $\log \eta \leqq \eta$ for $\eta \geqq 1$, $|\lambda|$ and $|\mathrm{Im}\, \lambda|$ are comparable in the intersection of Ω with the left half-plane and we may assume our reversed logarithmic regions are defined by inequalities of the form

$$\mathrm{Re}\, \lambda \geqq \min(\beta - \alpha \log|\mathrm{Im}\, \lambda|,\ \omega) . \tag{4.17}$$

We consider now the region $\Xi(1.1)$ of Lemma 4.2 relative to the function $\theta(\eta) = \alpha \log \eta - \beta$ in $\eta = a = e^{\beta/\alpha}$; if necessary inflating β we may assume that (4.4) holds. We apply Lemma 4.4 to the function

$$\mathfrak{f}(\lambda) = \det \mathfrak{P}(\lambda)^{-1} = 1/\det \mathfrak{P}(\lambda) \tag{4.18}$$

in $\Xi(1, 1)$. Plainly $\mathfrak{f}(\lambda)$ has no zeros in $\Xi(1, 1)$. Since $\mathfrak{P}(\lambda)^{-1}$ satisfies (2.2) in, say, the matrix norm corresponding to the ordinary Euclidean norm in E, each of its entries $a_{jk}(\lambda)$ satisfy an inequality of the same type:

$$\begin{gathered} |a_{jk}(\lambda)| \leqq C_1|\lambda|^m, \ |\mathfrak{f}(\lambda)| \leqq C_1|\lambda|^{mr} \\ (\lambda \in \Xi(1, 1)) \end{gathered} \tag{4.19}$$

with $r = \dim E$. We observe next that, since $\mathfrak{P}(\lambda)$ is the Laplace transform of the tempered distribution P it must grow polynomially in the right half-plane; hence the same is true of its determinant $1/\mathfrak{f}(\lambda)$:

$$|\mathfrak{f}(i\eta)| \geqq C_2|\eta|^p \quad (\eta \geqq a) . \tag{4.20}$$

We are then in condition to apply Lemma 4.5 and deduce the existence of a parameter, κ, $0 < \kappa < 1$ such that

$$|1/\mathfrak{f}(\lambda)| \leqq C|\lambda|^{p'} \quad (\lambda \in \Xi(1/2, \kappa)) \tag{4.21}$$

where κ does not depend on α. We point out finally that each entry of $\mathfrak{P}(\lambda) = (\mathfrak{P}(\lambda)^{-1})^{-1}$ is a sum of products of $r-1$ entries of $\mathfrak{P}(\lambda)^{-1}$ divided by $\mathfrak{f}(\lambda)$, the determinant of $\mathfrak{P}(\lambda)^{-1}$. Hence we obtain from the first inequality (4.17) and from (4.19) an estimate of the form

$$||\mathfrak{P}(\lambda)|| \leqq C|\lambda|^{m'}$$

in $\Xi(1/2, \kappa)$, thus in reversed logarithmic regions $\Omega(\alpha/2, \beta, \omega)$ where α can be taken arbitrarily large and m' does not depend on α. It follows then that a representation of the type of (2.3) ($n = 0$) holds for $\mathfrak{P}$ itself and an argument very similar to that in [6], Theorem 6.1 yields the desired conclusion. Details are omitted.

We consider next the case where $P \in (\mathscr{D}' \cap \mathscr{C}_+(\mathscr{M}))^{-1}$, making use of Theorem 3.5. Let $\alpha, \gamma > 0$, Θ the function in (3.17),

$$\theta(\eta) = \gamma \log \Theta(\alpha\eta) - \beta .$$

Under assumption (4.15), $\Theta'(\eta) \leqq C\Theta(\eta)$ so that θ satisfies (4.4) for $\alpha \leqq \alpha_0 < 1/c$; if necessary increasing β (or decreasing γ) we may assume that $\beta = \gamma \log \Theta(\alpha a)$ for some $a \geqq 1$ and consider θ in $\eta \geqq a$.

It is obvious that (4.6) holds. Since $\theta(\eta) \leqq c\eta$, again $|\lambda|$ and $|\operatorname{Im} \lambda|$ are comparable in the intersection of any region $\Theta(\alpha, \beta, \gamma)$ with the left half-plane and we may assume that these regions are defined by inequalities of the type

$$\begin{aligned} \operatorname{Re} \lambda &\geqq -\gamma \log \Theta(\alpha |\operatorname{Im} \lambda|) + \beta \\ &= -\theta(|\operatorname{Im} \lambda|) + 2\beta \,. \end{aligned} \tag{4.22}$$

Given $\varepsilon > 0$, $\mathfrak{P}(\lambda)^{-1}$ exists in one of them, thus in $\Xi(1, 1)$, its entries and determinant satisfying there

$$\begin{aligned} |a_{jk}(\lambda)| &\leqq C_1 |\lambda|^m e^{\varepsilon |\operatorname{Re} \lambda|} \,, \\ |\mathfrak{f}(\lambda)| &\leqq C_1 |\lambda|^{mr} e^{r\varepsilon |\operatorname{Re} \lambda|} \,; \end{aligned} \tag{4.23}$$

obviously, $\mathfrak{f}$ has no zeros in $\Xi(1.1)$ and (4.20) holds. We deduce from Lemma 4.3 that there exist two constants ρ, κ independent of $\alpha \leqq \alpha_0$ and of ε if $\varepsilon \leqq \varepsilon_0$ such that

$$\begin{aligned} |1/\mathfrak{f}(\lambda)| &\leqq C |\lambda|^p e^{r\varepsilon |\operatorname{Re} \lambda|} \\ &(\lambda \in \Xi(\rho, \kappa)) \,. \end{aligned} \tag{4.24}$$

Arguing as before on the basis of this estimate and of the first inequality (4.20) we obtain

$$\|\mathfrak{P}(\lambda)\| \leqq C |\lambda|^{m'} e^{\varepsilon' |\operatorname{Re} \lambda|} \quad (\lambda \in \Xi(\rho, \kappa)) \tag{4.25}$$

where $m' = m(r - 1) + p$ and $\varepsilon' = 2(2r + 1)\varepsilon$. Since $\Theta(\eta) \geqq \eta^n / M_n$ for all η the present hypotheses imply that the $P \in (\mathscr{D}' \cap \mathscr{E}_+^{\infty})^{-1}$ and we know that P admits a representation of the form (2.3); we next show that the contour of integration there can be deformed to Δ, the curve consisting of the two infinite arcs $\xi = \rho\theta(-\kappa\eta)(\eta \leqq -a/\kappa)$ joined by an arc lying in the region $\Theta(\alpha, \beta, \gamma)$. This follows easily from (4.25). We can then estimate P in the same way S was estimated at the end of Theorem 3.6. The representation (4.16) in the quasi-analytic case follows from the fact that a distribution with support $\{0\}$ must be a polynomial in δ and its derivatives. This ends the proof.

We note that the sequence $\mathscr{M} = \{n!\}$ satisfies all the hypotheses adopted in this section, thus the results apply to the class $\mathscr{A}$ of real analytic functions. The following result handles the distributions considered in Theorem 4.8: since these do not necessarily have compact support, Theorem 4.7 is not a particular case of Theorem 4.6, although the proof is totally similar.

THEOREM 4.7. *Let E, X be finite dimensional, $P \in \mathscr{S}_0'((E; X))$ such that $P \in (\mathscr{D}_0'((E; X)) \cap \mathscr{A}(\phi; (E; X)))^{-1}$ for some $\phi > 0$. Then P is of the form* (4.16).

Proof. An examination of the proof of Theorem 4.6 reveals that only the properties of $\mathfrak{P}(\lambda)^{-1}$ in the regions $\Theta(\alpha, \beta, \gamma)$ were essential; since these properties are the same as those used in Theorem 4.6 (with the added advantage that $\varepsilon = 0$) the result follows.

We close this section with a comment on the infinite dimensional case. Although the example in Remark 2.4 sweeps away all hopes of deducing smoothness of P from smoothness of P^{-1} it is interesting to note that distributions in $(\mathscr{D}_0' \cap \mathscr{E}_+(\mathscr{M}))^{-1}$ behave in some respects as smooth functions off the origin, at least when $\mathscr{M}$ is quasi-analytic. One example is the following unique continuation property, where E and X are general Banach spaces.

LEMMA 4.8. *Let $\mathscr{M}$ be a quasi-analytic sequence and let P_1, P_2 be distributions in $(\mathscr{D}_0'(E; X) \cap \mathscr{E}_+(\mathscr{M}; (E; X)))^{-1}$ such that $P_1 = P_2$ in $t < a$ $(a > 0)$. Then $P_1 = P_2$ in $-\infty < t < \infty$.*

Proof. Since $P_1 = P_2$ in $t < a$ we have $S_1 = P_1^{-1} = P_2^{-1} = S_2$ in $t < a$ so that $S_1 - S_2$ vanishes in $t < a$. Since S_1, S_2 belong to $\mathscr{M}$, $S_1 = S_2$ and $P_1 = S_1^{-1} = S_2^{-1} = P_2$.

On the other hand, it is easy to see that this continuation property does not extend to intervals other than $(0, a)$.

5. **Continuation properties.** We call a subspace $\mathscr{F} \subseteqq \mathscr{D}'$ *local* if it is defined by local conditions, i.e., by conditions that can be verified in arbitrarily small neighborhoods. For instance $\mathscr{D}'$ itself, $\mathscr{E}_+^\infty$, $\mathscr{E}_+(\mathscr{M})$ (for any sequence $\mathscr{M}$, not necessarily quasi-analytic), in particular $\mathscr{A}$, are local, while $\mathscr{S}$ and $\mathscr{E}$ are not.

The following natural question arises. Let $P \in \mathscr{D}'$ and assume P has a convolution inverse S which belongs to $\mathscr{F}$ in $t < a$. Can we conclude (a) that P has an inverse S in $-\infty < t < \infty$ and that (b) S belongs to $\mathscr{F}$ in $-\infty < t < \infty$? As we shall see in the next Lemma 5.1 the answer to (a) is always affirmative. Since the existence of inverse of P in $\mathscr{F}$ for $t < a$ does not bring into play the restriction of P to $t > a$ it is natural to surmise that the answer to (b) must be in the negative if the support of P is not contained in $t < a$, except possibly in the case $\mathscr{F} = \mathscr{D}'$. That the answer is actually affirmative here is a consequence of the following result.

LEMMA 5.1. *Let $P \in \mathscr{D}_0'((X; E))$. Assume there exists $S_a \in \mathscr{D}_0'((-\infty; a); (E; X))$ such that* (1) *holds in $t < a$. Then there exists $S \in \mathscr{D}_0'((E; X))$ such that* (1) *holds in $-\infty < t < \infty$.*

Proof. Let $\varphi \in \mathscr{H}_b$ with $0 < b < a/2$. Then it is plain that

φS can be thought of as a distribution in $\mathscr{D}_0'((E; X))$ and that (2.5) holds with $\Phi = P*(1-\varphi)S \in \mathscr{D}'((E; E))$ and $\psi = (1-\varphi)S*P \in \mathscr{D}'((X; X))$. Since Φ has support in $t \geqq a$, Φ^{*n} has support in $t \geqq na$ and this assures convergence in $\mathscr{D}'$ of both series (2.7) and thus existence of S (details may be found in § 2).

Lemma 5.1 has the somewhat curious consequence that the restriction of P to $t > a$, where $a > 0$ is arbitrarily small is of no consequence in deciding whether $P \in \mathscr{D}_0'^{-1}$ or not.

Consider now the case of a general local space $\mathscr{F}$, and let $P \in \mathscr{D}'$ have support in an interval $0 \leqq t \leqq a$. The space $\mathscr{F}$ is said to be *continuable with respect to* P if the fact that P has an inverse in $t \leqq b$ $(b > a)$ implies that $P \in (\mathscr{D}_0' \cap \mathscr{F})^{-1}$. If $\mathscr{F}$ is continuable with respect to P for every P we simply say that $\mathscr{F}$ is *continuable.*

The next result follows from a cursory examination of the proofs of Theorems 3.5 and 3.6.

LEMMA 5.2. *The spaces* $\mathscr{F} = \mathscr{C}_+^\infty, \mathscr{C}_+(\mathscr{M})$ *(for any sequence* $\mathscr{M}$ *satisfying the assumptions of Theorems* 3.5 *and* 3.6), *in particular,* $\mathscr{F} = \mathscr{A}$ *are continuable.*

In fact, the functions Φ and ψ in Theorem 3.5 only take into account the restriction of S to $0 \leqq t \leqq a + \varepsilon$ for ε arbitrarily small.

On the other hand, the spaces $\mathscr{C}_+^{(n)}$ are not continuable (see, however, the end of the section) as the present example (used in [5] in a different way) shows:

EXAMPLE 5.3. Let $E = H$ be a separable Hilbert space, $\{e_n; n \geqq 1\}$ an orthonormal basis of E, A the normal operator defined by $Au = \Sigma \lambda_n (u \mid e_n) e_n$ with

$$\lambda_n = \frac{1}{a} \log n + \frac{i}{a}(n^2 - (\log n)^2)^{1/2} \tag{5.3}$$

where $a > 0$. Finally, let $X = D(A) = \{u \in H; \Sigma |\lambda_n|^2 |(u \mid e_n)|^2 < \infty\}$ endowed with its graph norm. It is not difficult to see that $P = \delta'' \otimes I - \delta' \otimes A \in \mathscr{D}'$, the inverse S being described as follows. Let $r \geqq 1$ be an integer. For $0 \leqq t \leqq a(r+1)$ define

$$S_r(t)u = \sum_{n=1}^{\infty} \lambda_n^{-(r+1)}(e^{\lambda_n t} - p_{n,r}(t))(u \mid e_n)e_n$$

where $p_{n,r}(t) = 1 + \lambda_n t + \cdots + (\lambda_n t)^r / r!$. Since

$$a\frac{(n^{t/a} - |p_{n,r}(t)|)}{n^{r+1}} \leqq \frac{|e^{\lambda_n t} - p_{n,r}(t)|}{|\lambda_n|^{r+1}} \leqq a\frac{n^{t/a} + |p_{n,r}(t)|}{n^{r+1}}$$

and $\|S_r(t)\|_{(E;X)} = \sup |\lambda_n|^{-r} |e^{-\lambda_n t} - p_{n,r}(t)|$ it follows that $S_r(t)$ is a continuous $(E;X)$-valued function in $-\infty < t \leqq ar$ (but not in $t > ar$) and it is not difficult to check that

$$S_0 = P^{-1} = D^r S_r \quad (-\infty < t < ar) .$$

If $t < a(r-m)$ we have $D^m S_r = S_{r-m}$ in $(E;X)$, therefore each S_r is m times continuously differentiable in $-\infty < t < a(r-m)$, but not in $t > a(r-m)$. We define then $P_m = Y_{-(m+1)} * P = P^{(m+1)}$, so that $S_m = Y_{m+1} * S_0 = S_{m+1}$ in $t < a(m+1)$, which belongs to $\mathscr{E}_+^{(m)}((E;X))$ in $-\infty < t \leqq a$ but loses differentiability step by step in each interval $(a, 2a], \cdots, ((m-1)a, ma]$ until it ceases altogether to be a $(E;X)$-valued function for $t > m$.

On the other hand, since, say, a semigroup or distribution semigroup S belongs to $\mathscr{E}_+^{(m)}$ in $t > 0$ if it belongs to $\mathscr{E}_+^{(m)}$ in $0 < t < a$, a arbitrarily small, the spaces $\mathscr{E}_+^{(m)}$ are continuable with respect to $P = \delta' \otimes I - \delta \otimes A$. Similar results hold for distributions of the form $\delta'' \otimes I - \delta \otimes A$ and others.

6. Several variables. Many of the results in the previous sections and in [6] can be extended to distributions in several variables; in these extensions, the role of the half-line $t \leqq 0$ containing the supports of distributions in $\mathscr{D}_0'$, $\mathscr{S}_0'$, $\mathscr{E}_0'$, is taken over by a sufficiently regular cone $K \subseteqq R^n$ having the property that $K \cap (\{t\} - K)$ is compact for all $t \in R^n$ (for instance, the positive quadrant R_+^n) such as to make possible the convolution of arbitrary distributions with support in K. As seen in [6], the theory of the convolution inverse in one variable is related to initial value (or Cauchy) problems in $t \geqq 0$; in the n-dimensional case the relation is to problems of Goursat type where the aim is to solve differential (or convolution) equations in a suitable cone with values prescribed outside (on the surface, when the equation is purely differential). The point of view taken in [4], [8] and other works on convolution equations in the scalar case is different: here P is assumed to have compact support, so that convolution by an arbitrary distribution is possible. Roughly speaking, this corresponds to studying differential or convolution equations in the whole space in the absence of boundary or initial conditions.

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