BOUNDARY VALUE PROBLEMS FOR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Sufficient conditions are given to ensure the existence of solutions for the boundary value problem

\( y(t) = T(t)\phi(0) + \int_0^t T(t-s)F(y_s)ds \quad 0 \leq t \leq b \)

\( (*) \quad My_0 + Ny_b = \psi \), \( \psi \in C(\mathbb{C}([-r,0]; B)) \) by def.\).

It is assumed that \( T(t) \), \( t \geq 0 \), is a strongly continuous semigroup of bounded linear operators on the Banach space \( B \) and \( T(t), t \geq 0 \), has infinitesimal generator \( A \). The function \( F \) is continuous from \( C \) to \( B \) and \( M \) and \( N \) are bounded linear operators defined on \( C \).

Denote by \( C \) the Banach space of continuous functions from \([-r,0]\) into the Banach space \( B \), where for each \( \phi \in C \), \( \| \phi \|_C = \sup_{-r \leq \theta \leq 0} \sup \| \phi(\theta) \| \). Let \( A \) be the infinitesimal generator of a strongly continuous semigroup of linear operators \( T(t), t \geq 0 \) mapping \( B \) into \( B \) and satisfying \( |T(t)| \leq e^{\omega t} \) for some real \( \omega \). We let \( F \) be a nonlinear continuous function from \( C \) into \( B \). If \( y(t) \) is a continuous function from \([0,T]\) to \( B \) for some \( T > 0 \), define the element \( y_t \in C \) by \( y_t(\theta) = y(t+\theta) \). Throughout this paper the reference \( y(t) \) is a solution of Equation (1) \( (*) \) will mean \( y(t) \) satisfies Equation (1) and the boundary condition \( (*) \). The statement \( y(\phi)(t) \) is a solution of Equation (1) will mean \( y(t) \) satisfies Equation (1) and the initial condition \( y_0 = \phi \). The notation Equation (1) without \( (*) \) will always denote the initial value problem.

In a recent paper [8] C. Travis and G. Webb have considered initial value problems for Equation (1). With \( F \) satisfying

\( \| F(\phi) - F(\tilde{\phi}) \| \leq L \| \phi - \tilde{\phi} \|_C \)

for some \( L > 0 \) and \( \phi, \tilde{\phi} \in C \), Travis and Webb obtain the existence of unique solutions of Equation (1) for each \( \phi \in C \). In another paper W. E. Fitzgibbon [2] has shown that global solutions of Equation (1) exist if \( F \) satisfies for each \( \phi \in C \)

\( \| F(\phi) \| \leq K_1 \| \phi \|_C + K_2 \) for some \( K_1, K_2 \in R \),

and if \( T(t), t > 0 \) is compact.

When Equation (1) has unique solutions for each \( \phi \in C \), the mapping \( U(t)\phi = y_t(\phi) \) is well defined for each \( t \geq 0 \) and \( \phi \in C \). Here \( y_t(\phi) \) represents the element of \( C \) such that \( y(\phi)(t) \) is a solution of

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Equation (1). If $F$ satisfies (2) the following estimate from [8] is true:

$$(4) \quad \|U(t)\varphi - U(t)\bar{\varphi}\|_c \leq e^{(\omega + L)t} \|\varphi - \bar{\varphi}\|_c \quad \text{if} \quad \omega \geq 0$$

for all $t \geq 0$. Throughout this paper it will be assumed that $\omega \geq 0$.

If $F$ satisfies (3), then we have for each $\varphi \in C$ and $0 \leq t \leq b$

$$\|U(t)\varphi\|_c = \|y_t(\varphi)\|_c = \sup_{-\epsilon \leq \theta \leq 0} \left| T(t + \theta)\varphi(0) + \int_0^{t+\theta} T(t + \theta - s)F(y_s)ds \right|$$

$$\leq e^{\omega t} \|\varphi\|_c + e^{\omega t} \int_0^t e^{-\omega s}K_1\|y_s(\varphi)\|_c + K_2ds .$$

This implies that

$$(5) \quad \|y_t(\varphi)\|_c \leq \bar{K}_1 \|\varphi\|_c + \bar{K}_2$$

where $\bar{K}_1 = e^{(\omega + K_1)b}$ and $\bar{K}_2 = e^{(\omega + K_1)b}K_2b$.

It is shown in [8] that if the semigroup $T(t)$, $t \geq 0$ is compact for $t > 0$, then the solution mapping $U(t)\varphi = y_t(\varphi)$ is compact in $\varphi$ for each fixed $t > r$.

Equation (1) is the integrated form of the functional differential equation

$$(6) \quad y'(t) = Ay(t) + F(y_t) \quad 0 \leq t \leq b$$

$$y_0 = \varphi .$$

Our results then can be applied to partial functional differential equations of the form

$$v_t(x, t) = v_{xx}(x, t) + f(v(x, t - r)) \quad 0 \leq t \leq b, \quad 0 \leq x \leq l$$

$$v(0, t) = v(l, t) = 0 \quad t \geq 0$$

$$\alpha(x, t)v(x, t) + \beta(x, t)v(x, b + t) = \psi(x, t) \quad -r \leq t \leq 0, \quad 0 \leq x \leq l .$$

Boundary value problems of the type Equation (6) (*) have been studied recently by R. Fennell and P. Waltman [1], G. Reddien and G. Webb [7] and P. Waltman and J. S. W. Wong [9] when $B = \mathbb{R}^n$. The work here extends results found in [7] and [9] to Equation (1) (*) when $B$ is infinite dimensional. Certain technical difficulties arise when $B$ is infinite dimensional. For example, the solution mapping $U(t)\varphi$ for Equation (1) is not compact as is the case when $B = \mathbb{R}^n$, see J. Hale [4]; this is a problem when trying to apply standard fixed point theorems. This difficulty is overcome by assuming the semigroup $T(t)$, $t \geq 0$ is compact for $t > 0$. It will become clear that our results depend on the operators $M$ and $N$, the Lipschitz constant $L$, and the length of the interval $b$. 
Define $S(b)\phi = x_b(\phi)$; $x_b(\phi)$ is the element of $C$ such that $x(\phi)(t)$ is the unique solution of the system
\[ x(t) = T(t)\phi(0) \quad t \geq 0 \]
\[ x_0 = \phi \quad \phi \in C. \]

Notice that $S(b)$ is a special case of $U(b)\phi = y_b(\phi)$ where $y(\phi)(t)$ is the solution of Equation (1) for the initial function $\phi \in C$. That is, the mapping $S(b)$ is $U(b)$ when $F \equiv 0$. Also, if the semigroup $T(t)$, $t \geq 0$ is compact for $t > 0$, we have that $U(b)$ is compact and therefore $S(b)$ is compact.

We also have need to consider the system
\[ z(t) = \int_0^t T(t - s)F(y_s(\phi))ds \quad 0 \leq t \leq b \]
\[ z = 0 \quad \text{on } [-r, 0] \]
where $y(\phi)(t)$ is the solution of Equation (1) for the initial function $\phi \in C$.

**Proposition 1.** Let $F$ satisfy condition (2).

(a) Suppose $(M + N)^{-1}$ exists with the range $R((U(b) - I))$ of $U(b) - I$ contained in $D((M + N)^{-1})$, that $\| (M + N)^{-1}N(U(b) - I) \|_{Lip} < 1$ $(b > r)$ and $\phi \in D((M + N)^{-1})$, then solutions of Equation (1) (*) exist and are unique.

(b) Suppose $(M + NS(b))^{-1}$ exists with $R(N(U(b) - S(b))) \subset D((M + NS(b))^{-1})$ and $\| (M + NS(b))^{-1}N(U(b) - S(b)) \|_{Lip} < 1$ $(b > r)$, then solutions of Equation (1) (*) exist and are unique.

**Proof.** For an initial function $\phi \in C$ and its corresponding unique solution of Equation (1) we have
\[ My_0 + My_b = M\phi + NU(b)\phi = (M + NU(b))\phi. \]

Therefore, in order to solve the boundary value problem Equation (1) (*) we must solve the operator equation
\[ (M + NU(b))\phi = \psi. \]

In case (a) we can write Equation (6) in the form
\[ (M + N + N(U(b) - I))\phi = \psi \]
and in case (b) in the form
\[ (M + NS(b) + N(U(b) - S(b)))\phi = \psi. \]

Since $(M + N)^{-1}$ exists in (a) and $(M + NS(b))^{-1}$ exists in (b) the above equations become
and

\[ (I + (M + \tau N)^{-1}N(U(b) - I))\varphi = (M + N)^{-1}\psi, \]

\( (9') \)

\[ (I + (M + NS(b))^{-1}N(U(b) - S(b)))\varphi = (M + NS(b))^{-1}\psi \]

when \( \varphi \in D((M + N)^{-1}) \) or \( \varphi \in D((M + NS(b))^{-1}) \). The equations (9) and (9') are in the form \( x + Sx = y \) with \( \|S\|_{\text{Lip}} < 1 \) and so are uniquely solvable.

Given an initial function \( \varphi \in C \) and the solution \( y(\varphi)(t) \) of Equation (1) we can write

\[ y(\varphi)(t) = x(\varphi)(t) + z(0)(t) \quad 0 \leq t \leq b \]

(10)

where \( x(\varphi)(t) \) and \( z(0)(t) \) are solutions of Equations (7) and (8), respectively. Using the identity (10) we have the following corollary to Proposition 1(b).

**Corollary to Proposition 1(b).** If operator \( (M + NS(b))^{-1} \)
exists on \( C \) and \( \|(M + NS(b))^{-1}N\|e^{(L+\omega)\tau} < 1 \,(b > r) \), then the boundary value problem Equation (1) (*) has a unique solution.

**Proof.** We show that the mapping \( (M + NS(b))^{-1}N(U(b) - S(b)) \)
is a strict contraction:

\[
\| (M + NS(b))^{-1}N(U(b) - S(b))\varphi - (M + NS(b))^{-1}(U(b) - S(b))\varphi \|_{C} \\
\leq \| (M + NS(b))^{-1}N \| \sup_{\tau \leq \theta \leq 0} \left\| T(b + \theta - s)(F(y_{s}(\varphi)) - F(y_{s}(\varphi)))ds \right\| \\
\leq \| (M + NS(b))^{-1}N \| Le^{\omega b} \| \varphi - \varphi \|_{C} \int_{0}^{b} e^{Ls}ds \\
< \| (M + NS(b))^{-1}N \| e^{(\omega + L)b} \| \varphi - \varphi \|_{C} < \| \varphi - \varphi \|_{C},
\]

for all \( \varphi, \varphi \in C \).

The result now follows by Proposition 1(b).

**Proposition 2.** Let \( F \) satisfy condition (2). If the mapping \( M^{-1} \)
exists on \( C \) with \( \| M^{-1}N\| e^{(L+\omega)\tau} < 1 \,(b > r) \), then Equation (1) (*) has a unique solution.

**Proof.** For an initial function \( \varphi \in C \) and its corresponding solution \( y(\varphi)(t) \) of Equation (1), we have \( My_{\tau} + Ny_{\tau} = (M + NU(b))\varphi \).

Thus, for the equation \( (M + NU(b))\varphi = \psi, \ \psi \in C \), we can write

\( (I + M^{-1}NU(b))\varphi = M^{-1}\psi \). From (4) we have that

\[
\| M^{-1}NU(b)\varphi - M^{-1}NU(b)\varphi \|_{C} \leq \| M^{-1}N\| \| U(b)\varphi - U(b)\varphi \|_{C} \\
\leq \| M^{-1}N\| e^{(L+\omega)\tau} \| \varphi - \varphi \|_{C} < \| \varphi - \varphi \|_{C}
\]
for all $\varphi, \bar{\varphi} \in C$. The mapping $M^{-1}NU(b)$ is a strict contraction and so the equation $(I + M^{-1}NU(b))\varphi = M^{-1}\psi$ has a unique solution for each $\psi \in C$. The result easily follows.

Using the identity (10) we are able to extend a result found in [9].

**PROPOSITION 3.** The two point boundary value problem Equation (1) (*) has a solution if and only if $Nz_0(0) \in \psi + R(M + NS(b)), \psi \in C, b > r$.

**Proof.** Given an initial function $\varphi \in C$, and its corresponding solution $y(\varphi)(t)$ of Equation (1) we have by (10) that

$$M y_0(\varphi) + N y_0(\varphi) = M \varphi + N(x_0(\varphi) + z_0(0)) = (M + NS(b))\varphi + Nz_0(0).$$

If $\psi \in C$ and $M y_0(\varphi) + N y_0(\varphi) = \psi$, we obtain $\psi = (M + NS(b))\varphi + Nz_0(0)$; this gives $Nz_0(0) = \psi - (M + NS(b))\varphi$ and so $Nz_0(0) \in \psi + R(M + NS(b))$.

If there exists a solution $\phi$ of $Nz_0(0) = \varphi + R(M + NS(b))\varphi$, define $v = -\varphi$. Then for the solution $y(v)(t)$ of Equation (1) we have

$$M y_0(v) + N y_0(v) = M v + N x_0(v) + Nz_0(0)$$

$$= (M + NS(b))v + Nz_0(0)$$

$$= -(M + NS(b))\varphi + Nz_0(0) = \psi.$$

Therefore the boundary value problem is solved.

The following result is due to A. Granas [3].

**PROPOSITION 4.** If $T$ is a compact operator mapping the Banach space $X$ into $X$ and satisfying $\lim_{||x|| \to \infty} ||Tx||/||x|| < 1$, then $R(I - T) = X$.

**PROPOSITION 4.** (i) Suppose the semigroup $T(t), t \geq 0$ is compact, (ii) $F$ takes closed bounded sets of $C$ into bounded sets in $B$, and $\lim_{||\varphi|| \to \infty} ||F(\varphi)||/||\varphi||_C = 0$, (iii) there exist unique solutions to the initial value problem Equation (1), $(M + NS(b))^{-1}$ exists on $C$ as a bounded operator. Then the boundary value problem Equation (1) (*) has a solution.

**Proof.** Condition (ii) implies that there exists $K_1$ and $K_2$ such that $||F(\varphi)|| \leq K_1||\varphi||_C + K_2$ for all $\varphi \in C$, so that global solutions for Equation (1) exist [2]. Furthermore, we can find constants $\bar{K}_1$ and $\bar{K}_2$ such that condition (5) is true. Let $\varphi_n$ be a sequence of functions in $C$ such that $||\varphi_n||_C \to \infty$ as $n \to \infty$ and define $\beta_n = \sup_{0 \leq t \leq b} ||y_t(\varphi_n)||_C$. Note that $\beta_n \leq \bar{K}_1||\varphi_n||_C + \bar{K}_2$ for each $n$. Let $\varepsilon$
be such that $0 < \varepsilon < 1/bK_1e^{b\|M + NS(b)^{-1}N\|}$, then by (ii) there exists $h > 0$ such that if $\|\varphi\|_c > h$, $\|F\varphi\| \leq \varepsilon \|\varphi\|_c$. We define $R = \max \{\|F(\varphi)\|: \|\varphi\|_c \leq h\}$ then

$$|| (M + NS(b))^{-1}N(U(b) - S(b))\varphi_n || \leq \sup_{-r \leq \theta \leq 0} || (M + NS(b))^{-1}N \| T(b + \theta - s) \| || F(y_s(\varphi_n)) || ds$$

$$\leq || (M + NS(b))^{-1}N \| e^{\theta b} \| \| F(y_s(\varphi_n)) \| || ds$$

$$\leq || (M + NS(b))^{-1}N \| e^{\theta b} \max \{|R, \varepsilon (K_1 + K_2)\}|.$$

If $\beta_n \to \infty$ as $n \to \infty$, we have $\lim_{n \to \infty} || (M + NS(b))^{-1}N(U(b) - S(b))\varphi_n ||_c < 1$ and if $\beta_n$ bounded as $n \to \infty$ then $\lim_{n \to \infty} || (M + NS(b))^{-1}N(U(b) - S(b))\varphi_n ||_c = 0$. Notice that $U(b)$ exists by (iii) and that by (i) $(M + NS(b))N(U(b) - S(b))$ is compact. Thus by Proposition A there is a solution to $(I + (M + NS(b))^{-1}N(U(b) - S(b)))\varphi = (M + NS(b))^{-1}\psi$ and the proposition is proved.

To prove Proposition 5 we need the following result of Z. Nashed and J. S. W. Wong [5].

**Proposition B.** If $A_1$ is a strict contraction on a Banach space $X$, i.e., $\|A_1x - A_2y\| \leq \gamma \|x - y\|$ ($0 < \gamma < 1$), $x, y \in X$, and $A_2$ is a compact mapping on $X$ such that $\lim_{\|x\| \to \infty} \|A_2x\|/\|x\| = \beta < 1 - \gamma$, then $R(I - (A_1 + A_2)) = X$.

**Proposition 5.** (i) If the semigroup $T(t), t \geq 0$ is compact for $t > 0$, (ii) $F$ takes closed bounded sets of $C$ into bounded sets in $B$, and $\lim_{\|\varphi\|_c \to \infty} \|F(\varphi)\|/\|\varphi\| = 0$, (iii) there exist unique solutions to the initial value problem Equation (1), (iv) $M^{-1}$ exists on $C$ as a bounded operator and $\|M^{-1}N\|e^{\theta b} < 1 (b > r)$. Then the boundary value problem Equation (1) (*) has a solution.

**Proof.** Given an initial function $\varphi \in C$, we can write

$$y_s(\varphi)(\theta) = T(b + \theta)\varphi(0) + \int_0^{b+\theta} T(b + \theta - s)F(y_s(\varphi))ds$$

where $y(\varphi)(t)$ is the solution of Equation (1) corresponding to $\varphi$. Define the operators $A_1$ and $A_2$ on $C$ as follows:

$$(A_1\varphi)(\theta) = T(b + \theta)\varphi(0) \quad \text{and} \quad (A_2\varphi)(\theta) = \int_0^{b+\theta} T(b + \theta - s)F(y_s(\varphi))ds.$$

The operator $A_2$ is compact by (i) and for $\varphi, \bar{\varphi} \in C$ we have

$$\|M^{-1}NA_1\varphi - M^{-1}NA_1\bar{\varphi}\|_c \leq \|M^{-1}N\|e^{\theta b}\|\varphi - \bar{\varphi}\|_c.$$
By (iv) the operator $M^{-1}NA_1$ is Lipschitz with Lipschitz constant $\gamma \leq ||M^{-1}N||e^{\omega b} < 1$.

Let $\varphi_n \in C$ such that $||\varphi_n||_c \to \infty$ as $n \to \infty$ and define $\beta_n = \sup_{0 \leq t \leq b} ||y_t(\varphi_n)||_c$. As in the proof of Proposition 4 we have constants $K_1$, $K_2$, $\bar{K}_1$, $\bar{K}_2$ such that $||F(\varphi)|| \leq K_1 ||\varphi||_c + K_2$ and $||y_t(\varphi)||_c \leq \bar{K}_1 ||\varphi||_c + \bar{K}_2$; therefore, we have $\beta_n \leq \bar{K}_1 ||\varphi_n||_c + \bar{K}_2$. If the sequence $\beta_n$ has limit infinity as $n$ approaches infinity, then by (ii)

$$\lim_{n \to \infty} ||M^{-1}NA_2\varphi_n||_c/||\varphi_n||_c \leq \lim_{n \to \infty} ||M^{-1}N||e^{\omega b} \epsilon \int_0^b (\bar{K}_1 ||\varphi_n||_c + \bar{K}_2)ds/||\varphi_n||_c$$

$$\leq ||M^{-1}N||e^{\omega b} \epsilon \in \bar{K}_1,$$

where $\epsilon > 0$ is arbitrary. Thus if we choose $\epsilon < 1 - \gamma/||M^{-1}N||e^{\omega b}\bar{K}_1$, then $\lim_{n \to \infty} ||M^{-1}NA_2\varphi_n||_c/||\varphi_n||_c < 1 - \gamma$. If the sequence $\beta_n$ is bounded, then $\lim_{n \to \infty} ||M^{-1}NA_2\varphi_n||_c/||\varphi_n||_c = 0 < 1 - \gamma$. Applying Proposition B, we see that for each $\psi \in C$ there exists a solution $\varphi$ of

$$(I + M^{-1}N(A_1 + A_2))\varphi = M^{-1}\psi.$$  

From the above equation we can solve the boundary value problem Equation (1) (*).

To illustrate our results we consider the partial functional differential equation

$$w_t(x, t) = w_{xx}(x, t) + f(w(x, t - r)) \quad 0 \leq t \leq b \quad 0 \leq x \leq l$$

$$w(0, t) = w(l, t) = 0 \quad t \geq 0.$$  

Here $f$ is a real-valued, Lipschitz continuous and continuously differentiable function. We let $B = L_2[0, l]$, and define $A$ and $F$ respectively as:

$A: D(A) \to B$ by $Au = \ddot{u}$, $D(A) = \{u \in B | u$ and $\ddot{u}$ are absolutely continuous, $\ddot{u} \in B$ and $u(0) = u(l) = 0\}$ and $F: C \to B$ by $F(\varphi)(x) = f(\varphi(-r)(x))\varphi(\epsilon)$ and $\epsilon \in [0, l]$. It is known that $A$ generates a strongly continuous semigroup $T(t)$, $t \geq 0$ such that $T(t)$ is compact for $t > 0$ and $w = 0$, see A. Pazy [6, pages 9 and 47]. The function $F$ is Lipschitz continuous and continuously differentiable.

If we let $M = I$, $N = 1/4 I$, then $(M + N)^{-1} = 4/5 I$ and

$$||M^{-1}N(U(b) - I)\varphi - (M + N)^{-1}N(U(b) - I)\bar{\varphi}||_c$$

$$\leq ||(M + N)^{-1}N||(||U(b)\varphi - U(b)\bar{\varphi}||_c + ||\varphi - \bar{\varphi}||_c)$$

$$\leq 1/5(||y_s(\varphi) - y_s(\bar{\varphi})||_c + ||\varphi - \bar{\varphi}||_c) \leq 1/5(e^{\epsilon b}||\varphi - \bar{\varphi}||_c + ||\varphi - \bar{\varphi}||_c)$$

$$\leq 1/5(e^{\epsilon b} + 1)||\varphi - \bar{\varphi}||_c.$$  

Part (a) of Proposition 1 is applicable if $1/5(e^{\epsilon b} + 1) < 1$. This is true if $Lb < ln4$. 
If the operators $M = I$ and $N = -1/4 \, I$ then

$$||M + NS(b)\varphi||_c = \sup_{-r \leq \theta \leq 0} ||(M + NS(b)\varphi)(\theta)||$$

$$= \sup_{-r \leq \theta \leq 0} ||\varphi(\theta) - 1/4 T(b + \theta)\varphi(0)||$$

$$\geq \sup_{-r \leq \theta \leq 0} ||\varphi(\theta)|| - 1/4 ||\varphi(0)|| \geq ||\varphi||_c - 1/4 ||\varphi||_c = 3/4 ||\varphi||_c .$$

The above estimate implies that $(M + NS(b))^{-1}$ exists on $C$ and $||(M + NS(b))^{-1}|| \leq 4/3$, furthermore

$$||(M + NS(b))^{-1}N(U(b) - S(b))\varphi - (M + NS(b))^{-1}N(U(b) - S(b))\varphi||_c$$

$$\leq ||(M + NS(b))^{-1}N|| (U(b) - S(b))\varphi - (U(b) - S(b))\varphi||_c$$

$$\leq ||(M + NS(b))^{-1}N|| e^{Lb} ||\varphi - \varphi||_c \leq 4/3 \cdot 1/4e^{Lb} ||\varphi - \varphi||_c$$

$$= 1/3e^{Lb} ||\varphi - \varphi||_c .$$

Here if $Lb < ln 3$ then $1/3e^{Lb} < 1$, and the corollary to Proposition 1(b) applies.

If $M = I$ and $N = -1/2I$ Proposition 1 is not readily applicable since we can obtain only the following estimate:

$$||(M + N)^{-1}N(U(b) - I)||_{L^p} \leq ||(M + N)^{-1}N|| (e^{Lb} + 1) \leq e^{Lb} + 1 .$$

The term $e^{Lb} + 1$ cannot be less than 1 for any positive numbers $L$ and $b$. Similarly we have

$$||(M + NS(b))^{-1}N(U(b) - S(b))||_{L^p} \leq ||(M + NS(b))^{-1}N|| e^{Lb} \leq e^{Lb}$$

and $e^{Lb}$ cannot be less than 1 and positive for any $L$ and $b$. Proposition 2, however, is easily applied since $||M^{-1}N||e^{Lb} = 1/2e^{Lb} < 1$ if $0 < Lb < ln 2$.

If we define $F(\varphi)(x) = f(\varphi(-r)(x)) = \varphi^{1/2}(-r)(x)$, then

$$||F(\varphi)||/||\varphi||_c = \left(\int_0^1 |\varphi^{1/2}(-r)(x)| \, dx \right)^{1/2} \sup_{-r \leq \theta \leq 0} \int_0^1 |\varphi^3(\theta)(x)| \, dx$$

$$\leq l^{1/2} \left(\int_0^1 |\varphi^3(-r)(x)| \, dx \right)^{1/2} \sup_{-r \leq \theta \leq 0} \int_0^1 |\varphi^3(\theta)(x)| \, dx$$

$$\leq l^{1/2} \left(\sup_{-r \leq \theta \leq 0} \int_0^1 |\varphi^3(\theta)(x)| \, dx \right)^{1/2} \sup_{-r \leq \theta \leq 0} \int_0^1 |\varphi^3(\theta)(x)| \, dx$$

and $\lim_{||\varphi||_c \to \infty} ||F(\varphi)||/||\varphi|| = 0$. Furthermore, $F$ takes closed bounded sets of $C$ into bounded sets of $B = L_2[0, 1]$. Letting $M = I$ and $N = -1/4 \, I$, both $(M + NS(b))^{-1}$ and $M^{-1}$ exist, and Propositions 4 and 5 can be applied to obtain solutions of

(11) $y(t) = T(t)\varphi(0) + \int_0^t T(t + \theta - s)y_s^{1/4}(-r)(\cdot) \, ds$

(* ) $My_0 + N y_b = \psi \, \, \, b > r$. 
Notice that the length of the interval $b$ does not enter into the discussion for the above example, other than $b$ is required to be greater than $r$.

The next theorem handles periodic boundary conditions, i.e., the boundary condition $y_0 = y_b$.

**Proposition 6.** Suppose $F$ satisfies condition (2). If the operator $M + NS(b)$ has a bounded inverse defined on $C$ such that $\| (M + NS(b))^{-1} \| < d$ for some $d > 0$ and for all $(r, \gamma)$ where $\gamma$ satisfies $\gamma > r$ and $d \| N \| e^{(L+\omega)r} = 1$, then the boundary value problem Equation (1) (*) has a unique solution.

**Proof.** For a function $\varphi \in C$ define the mapping $H : C \to C$ by

$$H \varphi = (M + NS(b))^{-1} \varphi - (M + NS(b))^{-1} N(U(b) - S(b)) \varphi.$$ 

We have for $\varphi, \bar{\varphi} \in C$

$$\| H \varphi - H \bar{\varphi} \|_C = \| (M + NS(b))^{-1} N(U(b) - S(b)) \varphi$$

$$- (M + NS(b))^{-1} N(U(b) - S(b)) \bar{\varphi} \|_C$$

$$\leq \| (M + NS(b))^{-1} N \| \sup_{0 \leq \theta \leq b} \| z_b(\varphi) - z_b(\bar{\varphi}) \|_C$$

$$\leq d \| N \| \int_0^b e^\theta \| F(y_s(\varphi)) - F(y_s(\bar{\varphi})) \| ds$$

$$\leq d \| N \| e^{b} \int_0^b e^{-\omega s} \| y_s(\varphi) - y_s(\bar{\varphi}) \| ds$$

$$\leq d \| N \| e^{Lb} \| \varphi - \bar{\varphi} \|_C.$$ 

The operator $H$ is a contraction if $b$ is sufficiently small and the boundary value problem is uniquely solvable.

**Remark.** Proposition 4 also handles periodic boundary conditions since again the only requirement on $M$ and $N$ is the existence of $(M + NS(b))^{-1}$. The inverse of $M + NS(b)$ exists with domain $C$ if and only if the boundary value problem Equation (7)(*) has a unique solution for each $\varphi \in C$.

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**References**


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