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ON LINEAR FORMS AND DIOPHANTINE APPROXIMATION

JEFFREY D. VAALER

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Let \vec{x} be a vector in R^K and let $A_j(\vec{x})$, $j=1, 2, \dots, J$ be J linear forms in K variables. We prove that there is a lattice point \vec{u} in Z^K , $\vec{u} \neq \vec{0}$, for which $|A_j(\vec{u})|$ are all small (or zero) and the components of \vec{u} are not too large. The bounds that we obtain improve several previous results on this problem.

1. Introduction. Let $A_1(\vec{x}), A_2(\vec{x}), \dots, A_J(\vec{x})$ be J linear forms in K real variables x_1, x_2, \dots, x_K . We assume that $B = (b_{jk})$ is a $J \times K$ matrix with complex entries such that

$$A_j(\vec{x}) = \sum_{k=1}^K b_{jk} x_k$$

for $j = 1, 2, \dots, J$ and so \vec{x} denotes the column vector $\begin{pmatrix} x_1 \\ \dots \\ x_K \end{pmatrix}$. A basic problem in Diophantine approximation is to show that there exists a vector $\vec{u} = \begin{pmatrix} u_1 \\ \dots \\ u_K \end{pmatrix}$ in the integer lattice Z^K , $\vec{u} \neq \vec{0}$, such that each $|A_j(\vec{u})|$ is small while the components $|u_k|$ are not too large. Quantitative results on this problem are known with various hypotheses on the A_j 's; the usual method of proof involves an application of the pigeon-hole principle (Baker [1], Lemma 1, p. 13, Gel'fond [3], Lemma 1, p. 11, Mordell [7], Theorem 3, p. 32, Siegel [8], Stolarsky [9], Chapter 2). In the present paper we make improvements on previous results of this kind by using a generalization of Minkowski's linear forms theorem which we established in [10].

In order to state our main theorem we make the following assumptions. We suppose that the forms A_j are real for $j = 1, 2, \dots, p$ and that the remaining forms consist of q pairs of complex conjugate forms arranged so that $A_{p+2j-1} = \bar{A}_{p+2j}$ for $j = 1, 2, \dots, q$. Thus $J = p + 2q$. We also suppose that $\alpha_k \geq 1$ for $k = 1, 2, \dots, K$, $\beta_j > 0$ for $j = 1, 2, \dots, J$, and $\beta_{p+2j-1} = \beta_{p+2j}$ for $j = 1, 2, \dots, q$.

THEOREM 1. *Let M be a positive integer and suppose that*

$$(1.1) \quad M^2 \left\{ \prod_{i=1}^K \alpha_i^{-2} \right\} \left\{ \prod_{j=1}^J \left(1 + \beta_j^{-2} \sum_{k=1}^K \alpha_k^2 |b_{jk}|^2 \right) \right\} \leq 1.$$

Then there exist M distinct pairs of nonzero lattice points $\pm \vec{v}_m =$

$\pm \begin{pmatrix} v_{1m} \\ \dots \\ v_{Km} \end{pmatrix}$, $m = 1, 2, \dots, M$, in \mathbf{Z}^K each of which satisfies the following conditions:

$$\begin{aligned} |A_j(\pm \vec{v}_m)| &\leq \beta_j, & j = 1, 2, \dots, p, \\ |A_j(\pm \vec{v}_m)| &\leq \left(\frac{2}{\pi}\right)^{1/2} \beta_j, & j = p + 1, p + 2, \dots, J, \\ |v_{km}| &\leq \alpha_k, & k = 1, 2, \dots, K. \end{aligned}$$

Next we deduce several corollaries to Theorem 1 which are easier to use in applications. For simplicity these results are stated for the case $M = 1$.

COROLLARY 2. *Suppose that $1 \leq J < K$ and that the coefficients b_{jk} satisfy $|b_{jk}| \leq T$ for some positive T . Then for each β , $0 < \beta \leq T$, there exists a lattice point $\vec{u} = \begin{pmatrix} u_1 \\ \dots \\ u_K \end{pmatrix}$, $\vec{u} \neq \vec{0}$, in \mathbf{Z}^K such that*

$$\begin{aligned} |A_j(\vec{u})| &\leq \beta, & j = 1, 2, \dots, p, \\ |A_j(\vec{u})| &\leq \left(\frac{2}{\pi}\right)^{1/2} \beta, & j = p + 1, p + 2, \dots, J, \end{aligned}$$

and

$$(1.2) \quad |u_k| \leq (\beta^{-1}T\sqrt{K+1})^{J/(K-J)}, \quad k = 1, 2, \dots, K.$$

Proof. We apply Theorem 1 with $M = 1$, $\alpha_k = \alpha \geq 1$, and $\beta_j = \beta \leq T$. Then the left hand side of (1.1) is

$$\begin{aligned} (1.3) \quad \alpha^{-2K} \prod_{j=1}^J \left(1 + \beta^{-2}\alpha^2 \sum_{k=1}^K |b_{jk}|^2\right) &\leq \alpha^{2J-2K}(\alpha^{-2} + \beta^{-2}T^2K)^J \\ &\leq \alpha^{2J-2K}(\beta^{-2}T^2(K+1))^J. \end{aligned}$$

If we choose

$$\alpha = (\beta^{-1}T\sqrt{K+1})^{J/(K-J)}$$

then $\alpha \geq 1$ and the expression on the right of (1.3) is equal to 1. Hence the corollary follows from the theorem.

We note that in previous versions of Corollary 2 (see Gel'fond [3]) the bound on $|u_k|$ was

$$|u_k| \leq 2(\beta^{-1}TK)^{J/(K-J)}.$$

However, in the special case $J = 1$ a bound similar to (1.2) was

obtained by Mahler [6].

If the coefficients b_{jk} are integers we obtain an improvement in "Siegel's lemma" (Baker [1], Siegel [8], Stolarsky [9]).

COROLLARY 3. *Suppose that $1 \leq J < K$ and that the coefficients b_{jk} are integers satisfying $|b_{jk}| \leq T$ for some $T \geq 1$. Then there exists a lattice point $\vec{u} = \begin{pmatrix} u_1 \\ \dots \\ u_K \end{pmatrix}$, $\vec{u} \neq \vec{0}$, in \mathbf{Z}^K such that*

$$(1.4) \quad A_j(\vec{u}) = 0, \quad j = 1, 2, \dots, J,$$

and

$$|u_k| \leq (T\sqrt{K+1})^{J/(K-J)}, \quad k = 1, 2, \dots, K.$$

Proof. We apply Corollary 1 with $0 < \beta < 1$, $p = J$ and $q = 0$. Since $A_j(\vec{u})$ is an integer whenever $\vec{u} \in \mathbf{Z}^K$ it follows that there exists $\vec{u} \in \mathbf{Z}^K$, $\vec{u} \neq \vec{0}$, such that (1.4) holds and

$$(1.5) \quad |u_k| \leq (\beta^{-1}T\sqrt{K+1})^{J/(K-J)}, \quad k = 1, 2, \dots, K.$$

Now among the finitely many lattice points $\vec{u} \in \mathbf{Z}^K$, $\vec{u} \neq \vec{0}$, which satisfy (1.4) and (1.5) with $\beta = 1/2$ there must be at least one which satisfies (1.4) and (1.5) for values of β arbitrarily close to 1. Thus we may take $\beta = 1$ on the right of (1.5) for some $\vec{u} \in \mathbf{Z}^K$, $\vec{u} \neq \vec{0}$.

COROLLARY 4. *Suppose that $1 \leq J < K$ and that H_1, H_2, \dots, H_K are positive integers. Then there exists a lattice point $\vec{u} = \begin{pmatrix} u_1 \\ \dots \\ u_K \end{pmatrix}$, $\vec{u} \neq \vec{0}$, in such that*

$$|u_k| \leq H_k, \quad k = 1, 2, \dots, K,$$

$$|A_j(\vec{u})| \leq \frac{2\left(\sum_{k=1}^K H_k^2 |b_{jk}|^2\right)^{1/2}}{\left(\prod_{k=1}^K H_k\right)^{1/J}}, \quad j = 1, 2, \dots, p,$$

$$|A_j(\vec{u})| \leq \frac{2\left(\frac{2}{\pi}\right)^{1/2} \left(\sum_{k=1}^K H_k^2 |b_{jk}|^2\right)^{1/2}}{\left(\prod_{k=1}^K H_k\right)^{1/J}}, \quad j = p+1, p+2, \dots, J.$$

Proof. Let $0 < \theta < 1$. We apply Theorem 1 with $M = 1$, $\alpha_k = H_k + \theta$ and

$$\beta_j = \psi_\theta \left(\sum_{k=1}^K \alpha_k^2 |b_{jk}|^2 \right)^{1/2},$$

where

$$\psi_\theta = \left\{ \prod_{k=1}^K (H_k + \theta)^{2/J} - 1 \right\}^{-1/2}.$$

It follows that the left hand side of (1.1) is

$$\prod_{l=1}^K (H_l + \theta)^{-2} (1 + \psi_\theta^{-2})^J = 1.$$

Thus there exists $\vec{u} \in \mathbf{Z}^K$, $\vec{u} \neq \vec{0}$, such that

$$(1.6) \quad |u_k| \leq H_k, \quad k = 1, 2, \dots, K,$$

$$(1.7) \quad |A_j(\vec{u})| \leq \psi_\theta \left(\sum_{k=1}^K (H_k + \theta)^2 |b_{jk}|^2 \right)^{1/2}, \quad j = 1, 2, \dots, p,$$

and

$$(1.8) \quad |A_j(\vec{u})| \leq \left(\frac{2}{\pi} \right)^{1/2} \psi_\theta \left(\sum_{k=1}^K (H_k + \theta)^2 |b_{jk}|^2 \right)^{1/2},$$

$$j = p + 1, p + 2, \dots, J.$$

Only finitely many $\vec{u} \in \mathbf{Z}^K$, $\vec{u} \neq \vec{0}$, satisfy (1.6) and so, as in the proof of Corollary 3, at least one of these lattice points must satisfy (1.7) and (1.8) for all θ , $0 < \theta < 1$. Thus we may take $\theta = 1$ on the right hand side of (1.7) and (1.8). Finally we observe that

$$(1.9) \quad \left(\sum_{k=1}^K (H_k + 1)^2 |b_{jk}|^2 \right)^{1/2} \leq 2 \left(\sum_{k=1}^K H_k^2 |b_{jk}|^2 \right)^{1/2}$$

and

$$(1.10) \quad \psi_1 = \left(\prod_{k=1}^K H_k \right)^{-1/J} \left\{ \prod_{l=1}^K (1 + H_l^{-1})^{2/J} - \prod_{l=1}^K H_l^{-2/J} \right\}^{-1/2}.$$

Since $K > J$ we have

$$(1.11) \quad \prod_{l=1}^K (1 + H_l^{-1})^{2/J} - \prod_{l=1}^K H_l^{-2/J} \geq \prod_{l=1}^K (1 + H_l^{-2K/J})^{1/K} - \prod_{l=1}^K H_l^{-2/J}$$

$$\geq 1 + \prod_{l=1}^K H_l^{-2/J} - \prod_{l=1}^K H_l^{-2/J} = 1,$$

where we have used Theorem 27 and 10 of [5] in the first and second inequalities respectively. Putting (1.9), (1.10) and (1.11) together gives the desired result.

Our upper bound in Corollary 4 sharpens an inequality in Stolarsky [9], p. 15.

We also remark that Corollary 4 has an interesting geometrical

interpretation. Let $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_J$ denote nonzero column vectors in \mathbf{R}^K with $\vec{b}_j^T = (b_{j1} \ b_{j2} \ \dots \ b_{jK})$. We write $A_j(\vec{x}) = \langle \vec{b}_j, \vec{x} \rangle$, $\|\vec{b}_j\| = (\sum_{k=1}^K |b_{jk}|^2)^{1/2}$ and recall that $|\langle \vec{b}_j, \vec{x} \rangle| \|\vec{b}_j\|^{-1}$ is the length of the projection of \vec{x} onto the subspace spanned by the vector \vec{b}_j . Applying the corollary with $H_1 = H_2 = \dots = H_K = H$ we find that there is always a nonzero lattice point $\vec{u} \in \mathbf{Z}^K$ with components at most H in absolute value and having a projection onto the span of each \vec{b}_j of length at most $2H^{1-K/J}$.

2. Preliminary results. The remainder of our paper is devoted to a proof of Theorem 1. This is accomplished by combining the following lemmas. Here we write δ_{jk} for the Kronecker delta and B^* for the complex conjugate transpose of the matrix B .

LEMMA 5. Let $B = (b_{jk})$ be a $J \times K$ matrix with complex entries and let $D = (d_k \delta_{jk})$ be a diagonal matrix with $d_k > 0$ for $k=1, 2, \dots, K$. Then

$$(2.1) \quad \det(D + B^*B) \leq \left(\prod_{l=1}^K d_l \right) \prod_{j=1}^J \left(1 + \sum_{k=1}^K d_k^{-1} |b_{jk}|^2 \right).$$

It is possible to bound $\det(D + B^*B)$ by using Hadamard's inequality (Bellman [2], Gantmacher [4], p. 252). But the result we obtain is

$$\det(D + B^*B) \leq \prod_{k=1}^K \left(d_k + \sum_{j=1}^J |b_{jk}|^2 \right),$$

and this is generally weaker than (2.1) if $1 \leq J < K$.

Proof of Lemma 5. Let I_K denote the $K \times K$ identity matrix. We will begin by proving that

$$(2.2) \quad \det(I_K + B^*B) \leq \prod_{j=1}^J \left(1 + \sum_{k=1}^K |b_{jk}|^2 \right).$$

If Q is a $K \times K$ unitary matrix, that is if $Q^*Q = QQ^* = I_K$, then the left and right hand sides of (2.2) are unchanged when B is replaced by BQ . Since B^*B is a positive semi-definite Hermitian matrix we may choose the unitary matrix Q so that Q^*B^*BQ is a diagonal matrix. In particular we may choose Q (see Gantmacher [4], p. 274) so that

$$Q^*B^*BQ = (BQ)^*(BQ) = (\lambda_k \delta_{jk})$$

where

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M > 0 = \lambda_{M+1} = \lambda_{M+2} = \dots = \lambda_K .$$

Thus $\text{rank}(B) = \text{rank}(B^*B) = M \leq K$. (Of course if $\text{rank}(B) = 0$ then (2.2) is trivial so we may suppose that $1 \leq M$.) By replacing B by BQ it follows that we may assume without loss of generality that $B^*B = (\lambda_k \delta_{jk})$, or equivalently that

$$(2.3) \quad \sum_{l=1}^J \bar{b}_{lj} b_{lk} = \lambda_k \delta_{jk} .$$

Taking $j = k \geq M + 1$ in (2.3) we find that $b_{jk} = 0$ if $k = M + 1, M + 2, \dots, K$.

Next we define $w_{jk} = \lambda_k^{-1/2} b_{jk}$ so that by (2.3) the $J \times M$ matrix $W = (w_{jk})$ has M orthonormal columns (and so $M \leq J$). It follows from Bessel's inequality that

$$(2.4) \quad \sum_{k=1}^M |w_{jk}|^2 \leq 1 ,$$

for $j = 1, 2, \dots, J$. Since $I_K + B^*B = (\{1 + \lambda_k\} \delta_{jk})$ we have

$$\begin{aligned} \det(I_K + B^*B) &= \prod_{k=1}^M (1 + \lambda_k) = \prod_{k=1}^M (1 + \lambda_k)^{\sum_{j=1}^J |w_{jk}|^2} \\ &= \prod_{j=1}^J \left\{ \prod_{k=1}^M (1 + \lambda_k)^{|w_{jk}|^2} \right\} . \end{aligned}$$

Thus to establish (2.2) it suffices to show that

$$(2.5) \quad \prod_{k=1}^M (1 + \lambda_k)^{|w_{jk}|^2} \leq 1 + \sum_{k=1}^K |b_{jk}|^2$$

for each $j = 1, 2, \dots, J$. If $\sum_{k=1}^M |w_{jk}|^2 = 0$ then (2.5) is trivial since the left hand side is one. If $\sum_{k=1}^M |w_{jk}|^2 > 0$ then by the arithmetic-geometric mean inequality (see [5], Theorem 9) we have

$$\begin{aligned} \prod_{k=1}^M (1 + \lambda_k)^{|w_{jk}|^2} &\leq \left(\frac{\sum_{k=1}^M |w_{jk}|^2 (1 + \lambda_k)}{\sum_{k=1}^M |w_{jk}|^2} \right)^{\sum_{k=1}^M |w_{jk}|^2} = \left(1 + \frac{\sum_{k=1}^M |b_{jk}|^2}{\sum_{k=1}^M |w_{jk}|^2} \right)^{\sum_{k=1}^M |w_{jk}|^2} \\ &\leq \left(1 + \sum_{k=1}^M |b_{jk}|^2 \right) = \left(1 + \sum_{k=1}^K |b_{jk}|^2 \right) . \end{aligned}$$

In the last inequality we have used (2.4) together with the observation that $(1 + (c/x))^x$ is an increasing function of x for $x > 0$ and any fixed $c \geq 0$. This proves (2.2).

To complete the proof of the lemma we note that

$$\begin{aligned} \det(D + B^*B) &= \det(D^{1/2}) \det(I_K + D^{-1/2} B^* B D^{-1/2}) \det(D^{1/2}) \\ &= \left(\prod_{k=1}^K d_k \right) \det(I_K + (B D^{-1/2})^* (B D^{-1/2})) \end{aligned}$$

$$\leq \left(\prod_{k=1}^K d_k \right) \prod_{j=1}^J \left(1 + \sum_{k=1}^K d_k^{-1} |b_{jk}|^2 \right).$$

Next we suppose that $L_j(\vec{x})$, $j = 1, 2, \dots, N$ are N linear forms in K variables,

$$L_j(\vec{x}) = \sum_{k=1}^K a_{jk} x_k,$$

so that $A = (a_{jk})$ is an $N \times K$ matrix. We assume that the forms L_j are real for $j = 1, 2, \dots, r$ and that the remaining forms consist of s pairs of complex conjugate forms arranged so that $L_{r+2j-1} = \bar{L}_{r+2j}$ for $j = 1, 2, \dots, s$. Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N$ be positive with $\varepsilon_{r+2j-1} = \varepsilon_{r+2j}$ for $j = 1, 2, \dots, s$. We define the $N \times N$ diagonal matrix E by $E = (c_j \delta_{jk})$ where $c_j = \varepsilon_j^{-1}$ if $j = 1, 2, \dots, r$ and $c_j = (2/\pi)^{1/2} \varepsilon_j^{-1}$ if $j = r + 1, r + 2, \dots, N$.

LEMMA 6. *Let M be a positive integer and suppose that*

$$M |\det A^* E^2 A|^{1/2} \leq 1.$$

Then there exist at least M distinct pairs of nonzero lattice points $\pm \vec{v}_m$, $m = 1, 2, \dots, M$, in \mathbf{Z}^K such that

$$|L_j(\pm \vec{v}_m)| \leq \varepsilon_j$$

for each $j = 1, 2, \dots, N$ and each $m = 1, 2, \dots, M$.

For a proof of Lemma 6 we refer to [10].

3. **Proof of Theorem 1.** Let $N = J + K$. We apply Lemma 6 with

$$\begin{aligned} L_j(\vec{x}) &= x_j, & j &= 1, 2, \dots, K, \\ L_{K+j}(\vec{x}) &= A_j(\vec{x}), & j &= 1, 2, \dots, J. \end{aligned}$$

Thus $r = K + p$ and $s = q$. The matrix A can then be partitioned as

$$(3.1) \quad A = \begin{pmatrix} I_K \\ B \end{pmatrix}.$$

We also let

$$\begin{aligned} \varepsilon_j &= \alpha_j, & j &= 1, 2, \dots, K, \\ \varepsilon_{K+j} &= \beta_j, & j &= 1, 2, \dots, p, \\ \varepsilon_{K+j} &= \left(\frac{2}{\pi}\right)^{1/2} \beta_j, & j &= p + 1, p + 2, \dots, J. \end{aligned}$$

Using (3.1) it follows that

$$(3.2) \quad A^*E^2A = D + (GB)^*(GB)$$

where $D = (\alpha_k^{-2}\delta_{jk})$ is a $K \times K$ diagonal matrix and $G = (\beta_j^{-1}\delta_{jk})$ is a $J \times J$ diagonal matrix. Combining (1.1), (3.2) and Lemma 5 we find that

$$M^2 \det(AE^2A^*) \leq 1.$$

Thus the conclusion of Theorem 1 follows as an application of Lemma 6.

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