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Let 1, β_1, β_2 be a basis of a real cubic number field K. Let $c_0 = c_0(\beta_1 \beta_2)$ be the infimum over all constants $c > 0$ such that

$$|q\beta_1 - p_1| < (c/q)^{1/2}\ , \qquad |q\beta_2 - p_2| < (c/q)^{1/2}$$

has an infinite number of solutions in integers $q > 0$, p_1, p_2. Set

$$C_0 = \sup_{\beta_1 \cdot \beta_2} c_0(\beta_1, \beta_2)\ .$$

The purpose of this note is to observe that combining a recent beautiful result in the geometry of numbers of A. C. Woods with the earlier work of the author, we obtain

THEOREM. $C_0 = 2/7$.

It is generally conjectured that the best 2-dimensional diophantine approximation constant is also 2/7 but the result here can only be taken as further evidence for the conjecture.

The statement that $C_0 \geqq 2/7$ is due to Cassels [2]. Moreover, it is shown in [1] that if 1, β_1, β_2 is the basis of a nontotally real cubic field K, then

$$c_0(\beta_1, \beta_2) \leqq 1/23^{1/2} < 2/7\ .$$

Thus we may restrict our attention to totally real fields K. The following was also proved in [1]: for a full submodule $M \subseteqq K$ (a rank 3 free $\boldsymbol{Z}$-module) set

$$m_+(M) = \inf_{\substack{\xi \in M \\ \xi > 0 \\ N\xi > 0}} N\xi\ , \qquad m_-(M) = \inf_{\substack{\xi \in M \\ \xi > 0 \\ N\xi < 0}} |N\xi|\ ,$$

then

$$C_0^2 = \operatorname*{Sup}_{K, M} \frac{4m_+(M)m_-(M)}{D_M}$$

where D_M is the discriminant of M and $N = N_{\boldsymbol{Q}}^K$ is the norm from K to $\boldsymbol{Q}$. Thus it suffices to show that for all full modules M contained in a totally real cubic number field K, we have

$$m_+(M)m_-(M) \leqq \frac{D_M}{49}\ .$$

The recent result of A. C. Woods states: if Λ is any lattice in 3-space of determinant d then for all real numbers $u > 0$ there is a point (x_1, x_2, x_3) in Λ, not the zero point, such that

$$-\frac{1}{u} \leqq \frac{7}{d} x_1 x_2 x_3 \leqq u \quad \text{and} \quad x_3 \geqq 0 .$$

Embed M into 3 space as usual: for $\xi \in M$, $\xi \to (\xi_1, \xi_2, \xi)$ where ξ_1, ξ_2 are the conjugates of ξ. The image of M is a lattice Λ_M of determinant $d = D_M^{1/2}$. Set, for any ε, $0 < \varepsilon < m_+(M)$, $u = (7/D_M^{1/2})(m_+(M) - \varepsilon)$ in Woods theorem, and we obtain a point $\xi \in M$ so that

$$-\left(\frac{7}{D_M^{1/2}}(m_+(M) - \varepsilon)\right)^{-1} \leqq \frac{7}{D_M^{1/2}} N\xi \leqq \frac{7}{D_M^{1/2}}(m_+(M) - \varepsilon)$$

and $\xi > 0$. By definition of $m_+(M)$, we have $N\xi < 0$ and so

$$m_-(M) \leqq |N\xi| \leqq \left(\frac{1}{m_+(M) - \varepsilon}\right)\frac{D_M}{49} .$$

Letting $\varepsilon \to 0$ we see that

$$m_+(M) m_-(M) \leqq \frac{D_M}{49} ,$$

thereby proving the theorem.

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