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**THE BEST TWO-DIMENSIONAL DIOPHANTINE
APPROXIMATION CONSTANT FOR CUBIC IRRATIONALS**

WILLIAM WELLS ADAMS

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Let $1, \beta_1, \beta_2$ be a basis of a real cubic number field K . Let $c_0 = c_0(\beta_1, \beta_2)$ be the infimum over all constants $c > 0$ such that

$$|q\beta_1 - p_1| < (c/q)^{1/2}, \quad |q\beta_2 - p_2| < (c/q)^{1/2}$$

has an infinite number of solutions in integers $q > 0, p_1, p_2$. Set

$$C_0 = \sup_{\beta_1, \beta_2} c_0(\beta_1, \beta_2).$$

The purpose of this note is to observe that combining a recent beautiful result in the geometry of numbers of A. C. Woods with the earlier work of the author, we obtain

THEOREM. $C_0 = 2/7$.

It is generally conjectured that the best 2-dimensional diophantine approximation constant is also $2/7$ but the result here can only be taken as further evidence for the conjecture.

The statement that $C_0 \geq 2/7$ is due to Cassels [2]. Moreover, it is shown in [1] that if $1, \beta_1, \beta_2$ is the basis of a nontotally real cubic field K , then

$$c_0(\beta_1, \beta_2) \leq 1/23^{1/2} < 2/7.$$

Thus we may restrict our attention to totally real fields K . The following was also proved in [1]: for a full submodule $M \subseteq K$ (a rank 3 free \mathbb{Z} -module) set

$$m_+(M) = \inf_{\substack{\xi \in M \\ \xi > 0 \\ N_\xi > 0}} N_\xi, \quad m_-(M) = \inf_{\substack{\xi \in M \\ \xi > 0 \\ N_\xi < 0}} |N_\xi|,$$

then

$$C_0^2 = \sup_{K, M} \frac{4m_+(M)m_-(M)}{D_M}$$

where D_M is the discriminant of M and $N = N_Q^K$ is the norm from K to \mathbb{Q} . Thus it suffices to show that for all full modules M contained in a totally real cubic number field K , we have

$$m_+(M)m_-(M) \leq \frac{D_M}{49}.$$

The recent result of A. C. Woods states: if A is any lattice in 3-space of determinant d then for all real numbers $u > 0$ there is a point (x_1, x_2, x_3) in A , not the zero point, such that

$$-\frac{1}{u} \leq \frac{7}{d}x_1x_2x_3 \leq u \quad \text{and} \quad x_3 \geq 0.$$

Embed M into 3 space as usual: for $\xi \in M$, $\xi \rightarrow (\xi_1, \xi_2, \xi)$ where ξ_1, ξ_2 are the conjugates of ξ . The image of M is a lattice A_M of determinant $d = D_M^{1/2}$. Set, for any ε , $0 < \varepsilon < m_+(M)$, $u = (7/D_M^{1/2})(m_+(M) - \varepsilon)$ in Woods theorem, and we obtain a point $\xi \in M$ so that

$$-\left(\frac{7}{D_M^{1/2}}(m_+(M) - \varepsilon)\right)^{-1} \leq \frac{7}{D_M^{1/2}}N\xi \leq \frac{7}{D_M^{1/2}}(m_+(M) - \varepsilon)$$

and $\xi > 0$. By definition of $m_+(M)$, we have $N\xi < 0$ and so

$$m_-(M) \leq |N\xi| \leq \left(\frac{1}{m_+(M) - \varepsilon}\right)\frac{D_M}{49}.$$

Letting $\varepsilon \rightarrow 0$ we see that

$$m_+(M)m_-(M) \leq \frac{D_M}{49},$$

thereby proving the theorem.

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