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**ON A CHARACTERIZATION USING RANDOM SUMS**

J. R. CHOIKE, IGNACY I. KOTLARSKI AND V. M. SMITH

## ON A CHARACTERIZATION USING RANDOM SUMS

J. R. CHOIKE, I. I. KOTLARSKI AND V. M. SMITH

Let  $X_1, X_2,$  and  $X_3$  be independent random variables and let  $Z_1 = X_1 + X_3$  and  $Z_2 = X_2 + X_3$ . It is known that if the characteristic functions of  $X_k, k = 1, 2, 3,$  do not vanish then the distribution of  $(Z_1, Z_2)$  determines the distributions of  $X_1, X_2,$  and  $X_3$  up to a shift. The aim of this paper is to prove a result of a similar nature using sums of a random number of random variables. We shall use  $\sim$  for "has the same distribution as," r. v. for "random variable," ch. f. for "characteristic function," and p. g. f. for "probability generating function."

**THEOREM 1.** *Let  $N, X_1, X_2, \dots, Y_1, Y_2, \dots$  be independent r.v.'s where  $X_n \sim X, Y_n \sim Y, n = 1, 2, \dots,$  and  $X$  and  $Y$  are nondegenerate real-valued r.v.'s having ch.f.'s  $\varphi$  and  $\psi,$  respectively, which are of bounded variation on every finite interval. Let  $N$  be a nonnegative integer-valued r.v. with p.g.f.*

$$Q(s) = p_0 + \sum_{n=1}^{\infty} p_n s^n, \quad |s| \leq 1, \quad p_n = P(N = n)$$

and  $0 < EN = m < \infty.$  Assume that there is a neighborhood of 1 relative to the unit disk such that  $Q^{-1}$  exists in this neighborhood. Denote

$$U = 0 \text{ for } N = 0, \quad U = X_1 + X_2 + \dots + X_N \text{ for } N > 0, \text{ and} \\
V = 0 \text{ for } N = 0, \quad V = Y_1 + Y_2 + \dots + Y_N \text{ for } N > 0.$$

Then the distribution of  $(U, V)$  uniquely determines the distribution of  $N.$

*Proof.* Since  $N, X_1, X_2, \dots, Y_1, Y_2, \dots$  are independent r.v.'s, the ch.f. of  $(U, V), \varphi_{(U, V)},$  satisfies the following:

$$\begin{aligned} \varphi_{(U, V)}(r, t) &= E(e^{irU + itV}) \\ &= E(e^{irU + itV} | N = 0) \cdot P(N = 0) + \sum_{n=1}^{\infty} E(e^{irU + itV} | N = n) \cdot P(N = n) \\ &= E(1) \cdot p_0 + \sum_{n=1}^{\infty} E(e^{ir(X_1 + \dots + X_n) + it(Y_1 + \dots + Y_n)}) \cdot p_n \\ &= p_0 + \sum_{n=1}^{\infty} [E(e^{irX}) \cdot E(e^{itY})]^n \cdot p_n \\ &= p_0 + \sum_{n=1}^{\infty} [\varphi(r) \cdot \psi(t)]^n \cdot p_n \\ &= Q(\varphi(r) \cdot \psi(t)), \quad r, t \in R. \end{aligned}$$

Suppose there are other r.v.'s  $N^*$ ,  $X_1^*$ ,  $X_2^*$ ,  $\dots$ ,  $Y_1^*$ ,  $Y_2^*$ ,  $\dots$ , satisfying the assumptions. By repeating the above procedure denoting  $U^*$  and  $V^*$  similarly we obtain

$$(1) \quad \varphi_{(U^*, V^*)}(r, t) = Q^*(\varphi^*(r) \cdot \psi^*(t)), \quad r, t \in R.$$

Since  $(U^*, V^*)$  has the same distribution as  $(U, V)$ , their ch.f.'s are identical; thus,

$$(2) \quad Q^*(\varphi^*(r) \cdot \psi^*(t)) = Q(\varphi(r) \cdot \psi(t)), \quad r, t \in R.$$

Relation (2) is a functional equation and from this equation it will be shown that  $Q^* = Q$ .

The function  $Q$  is analytic inside the disk, thus the image of a domain under  $Q$  is a domain. There is a neighborhood of 1 relative to the unit disk such that  $Q^{*-1}$  exists and is analytic in this neighborhood. Thus there exists a neighborhood  $A$  of 1 relative to the unit disk such that  $Q^{*-1}$  exists and is analytic in  $Q(A)$ . Define

$$(3) \quad q(s) = Q^{*-1}(Q(s)) \quad s \in A.$$

Note that  $q$  is analytic in  $A$  and maps  $A$  into the unit disk. It can be assumed without loss of generality that  $0 \notin A$ .

Using relations (2) and (3),

$$(4) \quad q(\varphi(r) \cdot \psi(t)) = \varphi^*(r) \cdot \psi^*(t) \quad r, t \in R, \varphi(r) \cdot \psi(t) \in A.$$

By alternately allowing  $r = 0$  and  $t = 0$  it is found that  $q(\varphi(r)) = \varphi^*(r)$  and  $q(\psi(t)) = \psi^*(t)$ . Substituting these into relation (4)

$$(5) \quad q(\varphi(r) \cdot \psi(t)) = q(\varphi(r)) \cdot q(\psi(t)) \quad r, t \in R, \varphi(r) \cdot \psi(t) \in A.$$

Since  $0 \notin A$ , there exist continuous functions  $\varphi_0$  and  $\psi_0$  such that  $\varphi(r) = e^{\varphi_0(r)}$  and  $\psi(t) = e^{\psi_0(t)}$  and  $\varphi_0(0) = \psi_0(0) = 0$  where  $\varphi(r) \cdot \psi(t) \in A$ . Since  $\varphi$  and  $\psi$  are of bounded variation on finite intervals,  $\varphi_0$  and  $\psi_0$  are of bounded variation on finite intervals. Define

$$(6) \quad q_0(b) = \ln q(e^b) \quad e^b \in A,$$

where we take the branch for which  $\ln 1 = 0$ . Then from relation (6)

$$\begin{aligned} q_0(\varphi_0(r) + \psi_0(t)) &= \ln q(e^{\varphi_0(r) + \psi_0(t)}) \\ &= \ln q(\varphi(r) \cdot \psi(t)) \\ (7) \quad &= \ln [q(\varphi(r)) \cdot q(\psi(t))] \\ &= \ln q(\varphi(r)) + \ln q(\psi(t)) \\ &= \ln q(e^{\varphi_0(r)}) + \ln q(e^{\psi_0(t)}) \\ &= q_0(\varphi_0(r)) + q_0(\psi_0(t)), \quad \varphi(r) \cdot \psi(t) \in A. \end{aligned}$$

Consider the following integrals obtained by using equation (7)

$$(8) \quad \int_0^\beta q_0(\varphi_0(\alpha) + \psi_0(t)) d\psi_0(t) = \int_0^\beta [q_0(\varphi_0(\alpha)) + q_0(\psi_0(t))] d\psi_0(t) \\ = q_0(\varphi_0(\alpha)) \cdot \psi_0(\beta) + \int_0^\beta q_0(\psi_0(t)) d\psi_0(t)$$

and

$$(9) \quad \int_0^\alpha q_0(\varphi_0(r) + \psi_0(\beta)) d\varphi_0(r) = \int_0^\alpha [q_0(\varphi_0(r)) + q_0(\psi_0(\beta))] d\varphi_0(r) \\ = \int_0^\alpha q_0(\varphi_0(r)) d\varphi_0(r) + q_0(\psi_0(\beta)) \cdot \varphi_0(\alpha)$$

where  $\alpha$  and  $\beta$  are fixed real numbers such that  $\varphi(r) \cdot \psi(t) \in A$  for  $0 \leq r \leq \alpha$  and  $0 \leq t \leq \beta$ . These integrals exist because  $\varphi_0$  and  $\psi_0$  are of bounded variation on finite intervals and  $q_0$  is analytic. Using a change of variables on relations (8) and (9), the following integrals are obtained.

$$(10) \quad \int_{\psi_0(\alpha)}^{\varphi_0(\alpha) + \psi_0(\beta)} q_0(v) dv = q_0(\varphi_0(\alpha)) \cdot \psi_0(\beta) + \int_0^{\psi_0(\beta)} q_0(v) dv .$$

$$(11) \quad \int_{\psi_0(\beta)}^{\varphi_0(\alpha) + \psi_0(\beta)} q_0(v) dv = q_0(\psi_0(\beta)) \cdot \varphi_0(\alpha) + \int_0^{\varphi_0(\alpha)} q_0(v) dv .$$

By adding equations (10) and (11) right sides to left sides the following equation is obtained,

$$(12) \quad \int_0^{\varphi_0(\alpha) + \psi_0(\beta)} q_0(v) dv + q_0(\psi_0(\beta)) \cdot \varphi_0(\alpha) \\ = \int_0^{\varphi_0(\alpha) + \psi_0(\beta)} q_0(v) dv + q_0(\varphi_0(\alpha)) \cdot \psi_0(\beta) .$$

From this it is seen that

$$(13) \quad q_0(\psi_0(\beta)) \cdot \varphi_0(\alpha) = q_0(\varphi_0(\alpha)) \cdot \psi_0(\beta) .$$

Since  $X$  and  $Y$  are nondegenerate,  $|\varphi(r)| < 1$  and  $|\psi(t)| < 1$  almost everywhere. Thus  $\varphi_0(\alpha)$  and  $\psi_0(\beta)$  are different from zero almost everywhere and

$$(14) \quad \frac{q_0(\psi_0(\beta))}{\psi_0(\beta)} = \frac{q_0(\varphi_0(\alpha))}{\varphi_0(\alpha)} .$$

Since the choice of  $\alpha$  is independent of  $\beta$

$$(15) \quad q_0(\varphi_0(\alpha)) = c\varphi_0(\alpha) \quad \text{where } c \text{ is a complex number .}$$

Since  $q_0(b) = \ln q(e^b)$ ,  $q(s) = s^c$  for  $s \in A$ .

Since  $c$  is complex,  $c = a + ib$  where  $a, b \in R$ . Thus  $Q^{*-1}(Q(s)) =$

$s^{a+ib}$  for  $s \in A$  since  $q(s) = Q^{*-1}(Q(s))$ . Since  $A$  is a relative neighborhood of 1, there is a segment of the real line  $[\delta, 1] \subset A$  where  $0 < \delta < 1$ . The function  $Q$  maps the unit disk into the unit disk, and  $Q^{*-1}$  maps  $Q(A)$  into the unit disk. For  $s \in [\delta, 1]$ ,  $s^c = e^{c \ln s} = e^{a \ln s + ib \ln s} = e^{a \ln s} \cdot e^{ib \ln s}$ . Since  $|s^c| \leq 1$ ,  $a \ln s \leq 0$  for  $s \in [\delta, 1]$ . Thus  $a \geq 0$  since  $\ln s \leq 0$ . The function  $Q(s)$  is real for  $s$  a real number and  $Q^{*-1}(Q(s))$  is real for  $Q(s)$  a real number. Thus for  $s \in [\delta, 1]$ ,  $s^c$  is a real number and  $b \ln s = 0 \pmod{2\pi}$ . Thus  $b = 0$  and  $c = a \geq 0$ .

Since  $Q^{*-1}(Q(s)) = s^c$  for  $s \in A$ , then  $Q(s) = Q^*(s^c)$  for  $s \in A$ . The functions  $Q, Q^*, s^c$  are analytic for  $0 < |s| < 1$ , thus  $Q(s) = Q^*(s^c)$  for  $0 < |s| < 1$ . Suppose  $c = 0$ . Then  $Q(s) = Q^*(1) = 1$  for  $0 < |s| < 1$ . This implies that  $EN = 0$  which is a contradiction. Thus  $c \neq 0$ . Since the expectation of  $N$  and  $N^*$  exist  $\lim_{s \rightarrow 1} Q'(s) = \lim_{s \rightarrow 1} cs^{c-1}Q^{*'}(s^c)$  or  $m = cm$ . Thus  $c = 1$  and  $Q(s) = Q^*(s)$  for all  $|s| \leq 1$ . □

**REMARK.** A characterization for the distribution of  $N$  has been found using the assumptions of Theorem 1. The following shows that the assumption “that there is a neighborhood of 1 relative to the unit disk such that  $Q^{-1}$  exists in this neighborhood” is redundant.

**THEOREM 2.** *Let  $N$  be a nonnegative integer-valued r.v. with p.g.f.*

$$Q(s) = p_0 + \sum_{n=1}^{\infty} p_n s^n, \quad |s| \leq 1, \quad p_n = P(N = n).$$

*If  $0 < EN < +\infty$ , then  $Q$  is one-to-one in a relative neighborhood of 1.*

*Proof.* Let  $D = \{s : |s| < 1, s \in C\}$  and  $Q(s) = u(s) + iv(s)$  where  $u$  and  $v$  are real-valued functions.

Let

$$G(x_1, y_1, x_2, y_2) = \begin{pmatrix} u_x(x_1, y_1), u_y(x_1, y_1) \\ v_x(x_2, y_2), v_y(x_2, y_2) \end{pmatrix}$$

where  $x_1 + iy_1, x_2 + iy_2$  are in  $\bar{D}$ . The function  $Q$  is analytic in  $D$  if and only if  $u$  and  $v$  are differentiable in  $D$  and satisfy the Cauchy-Riemann equations.

Let

$$f(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \quad x + iy \in C.$$

Thus  $f(x, y)$  is differentiable in  $D$ ,  $f'$  may be represented by the Jacobian matrix of  $f$ ,

$$f'(x, y) = \begin{pmatrix} u_x(x, y), u_y(x, y) \\ v_x(x, y), v_y(x, y) \end{pmatrix} \quad x + iy \in C,$$

and  $f'(x, y)$  has continuous existension to  $\bar{D}$ .

The mapping  $\det [G(x_1, y_1, x_2, y_2)]: R^4 \rightarrow R$  is continuous on  $\bar{D} \times \bar{D} \subset R^4$ . But

$$\det [G(x_1, y_1, x_2, y_2)] = u_x(x_1, y_1) \cdot v_y(x_2, y_2) - u_y(x_1, y_1) \cdot v_x(x_2, y_2).$$

Since  $\det [G(1, 0, 1, 0)] = |Q'(1)|^2 \neq 0$ , there exists a convex neighborhood of 1 such that  $\det [G(x_1, y_1, x_2, y_2)] \neq 0$  is this convex (closed) neighborhood. Without loss of generality, we assume  $\det [G(x_1, y_1, x_2, y_2)] \neq 0$  for all  $x_1 + iy_1, x_2 + iy_2 \in \bar{D}$ .

Let  $\vec{c}, \vec{d} \in \bar{D}$ . By the Mean Value Theorem for vector-valued functions

$$f(\vec{c}) - f(\vec{d}) = G(\vec{c}_1, \vec{c}_2)(\vec{c} - \vec{d})$$

where  $\vec{c}_j = (1 - t_j)\vec{c} + t_j\vec{d}$ ,  $j = 1, 2$ , for some  $t_j \in (0, 1)$ . Note that  $\vec{c}_j \in D$ ,  $j = 1, 2$ .

Since  $\det G[(x_1, y_1, x_2, y_2)] \neq 0$ , the matrix  $G(x_1, y_1, x_2, y_2)$  represents a one-to-one linear map. Thus, if  $\vec{c} \neq \vec{d}$ , then  $f(\vec{c}) \neq f(\vec{d})$ . Thus,  $Q$  is one-to-one in a relative neighborhood of 1.  $\square$

Note that in Theorem 1 nothing is said about the distributions of  $X$  and  $Y$ . The following example will show that more assumptions are needed in order to determine the distributions of  $X$  and  $Y$ .

**EXAMPLE 1.** Let  $N$  and  $N^*$  be distributed according to the p.g.f.  $Q(s) = s^2$ ,  $|s| \leq 1$ . Let  $X$  be distributed according to the characteristic function  $\varphi(r) = 1 - 2|r|/\pi$  for  $|r| \leq \pi$  and  $\varphi(r)$  is periodic with period  $2\pi$ , and let  $X^* \sim |\varphi(r)|$ . Let  $Y \sim Y^* \sim \psi(t)$  where  $\psi(t)$  is any nonvanishing real-valued ch.f.  $(U, V) \sim (U^*, V^*)$  since

$$Q^*(\varphi^*(r) \cdot \psi^*(t)) = Q(\varphi(r) \cdot \psi(t)), \quad r, t \in R \quad \text{although} \quad \varphi^*(r) \neq \varphi(r).$$

Thus more conditions must be imposed in order to prove Theorem 3.

**THEOREM 3.** *Let  $N, X_1, X_2, \dots, Y_1, Y_2, \dots$  be r.v.'s satisfying the assumptions of Theorem 1, and  $U$  and  $V$  be defined as in Theorem 1. Then the distribution of  $(U, V)$  uniquely determines the distributions of  $X$  and  $Y$  if one of the following conditions holds:*

- (i) The characteristic functions  $\varphi$  and  $\psi$  are analytic at zero.  
(ii) There is a relative neighborhood  $B$  of 1 such that  $\varphi(r) \cdot \psi(t) \in B$ ,  $r, t \in R$ , and  $Q$  is one-to-one on  $B$ .

*Proof.* From the proof of Theorem 1  $Q^* = Q$  and

$$(1) \quad Q(\varphi^*(r) \cdot \psi^*(t)) = Q(\varphi(r) \cdot \psi(t)) \quad r, t \in R.$$

Thus by alternately letting  $r = 0$  and  $t = 0$

$$(2) \quad Q(\varphi^*(r)) = Q(\varphi(r)) \quad \text{and} \quad Q(\psi^*(t)) = Q(\psi(t)) \quad r, t \in R.$$

If condition (ii) is assumed, then it is clear that  $\varphi^* = \varphi$  and  $\psi^* = \psi$ .

If condition (i) is assumed, then as before,  $Q$  has a local inverse at one and  $\varphi^*(r) = \varphi(r)$  and  $\psi^*(t) = \psi(t)$  for  $r, t$  in some neighborhood of zero. But since the functions are analytic ch.f.'s,  $\varphi^* = \varphi$  and  $\psi^* = \psi$ .

Thus the distributions of  $X$  and  $Y$  are determined uniquely.

The following theorem has a proof very similar to that of Theorem 1.

**THEOREM 4.** Let  $N, X_1, X_2, \dots, Y_1, Y_2, \dots$  be independent r.v.'s with  $X_n \sim X, Y_n \sim Y, n = 1, 2, \dots$ , where  $X$  and  $Y$  are symmetric real-valued nondegenerate r.v.'s having ch.f.'s  $\varphi$  and  $\psi$ , respectively, with  $0 \leq \varphi(r) \leq 1$  and  $0 \leq \psi(t) \leq 1, r, t \in R$ . Let  $N$  be a nonnegative integer-valued r.v. with p.g.f.

$$Q(s) = p_0 + \sum_{n=1}^{\infty} p_n s^n, \quad |s| \leq 1, \quad p_n = P(N = n)$$

where  $0 < EN = m < \infty$ .

Denote  $U$  and  $V$  as in Theorem 1.

Then the distribution of  $(U, V)$  uniquely determines the distributions of  $X, Y$ , and  $N$ .

*Proof.* The proof of this theorem is the same as the proof of Theorem 1 up to relation (2). At this point the fact that  $\varphi$  and  $\psi$  are nonnegative real-valued functions can be used to simplify the proof. Since  $EN > 0$  and  $EN^* > 0$ ,  $Q$  and  $Q^*$  are strictly increasing on the interval  $[0, 1]$ . Thus the inverse of  $Q$  and  $Q^*$  exist as functions from  $[p_0, 1]$  and  $[p_0^*, 1]$ , respectively, onto  $[0, 1]$ . Without loss of generality  $p_0^* \leq p_0$ . By letting

$$(1) \quad q(s) = Q^{*-1}(Q(s)) \quad s \in [0, 1]$$

and using relation (2) in Theorem 1

$$(2) \quad q(\varphi(r) \cdot \psi(t)) = \varphi^*(r) \cdot \psi^*(t) \quad r, t \in R.$$

Note that  $q$  is continuous since  $Q^*$  and  $Q$  are continuous. Taking alternately  $r = 0$  and  $t = 0$  and substituting in equation (2) gives

$$(3) \quad q(\varphi(r) \cdot \psi(t)) = q(\varphi(r)) \cdot q(\psi(t)) \quad r, t \in R.$$

Denote  $A = \{a: a = \varphi(r), r \in R\}$  and  $B = \{b: b = \psi(t), t \in R\}$ . Since  $X$  and  $Y$  are nondegenerate,  $\varphi$  and  $\psi$  are not identically equal to 1. Since  $\varphi$  and  $\psi$  are real-valued, continuous, and  $\varphi(0) = \psi(0) = 1$ , there is an interval  $[c, 1]$ ,  $0 < c < 1$ , such that  $[c, 1] \subset A \cap B$ . Thus

$$(4) \quad q(ab) = q(a) \cdot q(b) \quad \text{for } a, b, ab \in [c, 1].$$

From [1],  $q(s) = s^k$  for  $s \in [c, 1]$  and  $k$  some real number. Using the same argument as in Theorem 1,  $k = 1$  and  $Q^*(s) = Q(s)$ ,  $|s| \leq 1$ . Thus the distribution of  $N$  is uniquely determined.

Using relation (1),  $q(s) = s$ , and relation (2) yields  $\varphi^*(r) = \varphi(r)$ ,  $r \in R$ , and  $\psi^*(t) = \psi(t)$ ,  $t \in R$ . Thus the distributions of  $X$  and  $Y$  are uniquely determined.

REMARKS. In each of the theorems we have assumed  $0 < EN = m < +\infty$ . This assumption can be replaced by the assumption, "There exists a fixed smallest positive index  $j_0$ , such that  $p_{j_0} > 0$ ." The theorems can be generalized if  $X$  and  $Y$  are random variables taking values in a locally compact Abelian group or taking values in a locally convex topological vector space if appropriate assumptions are made.

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