ANALYTIC H-SPACES, CAMPBELL-HAUSDORFF FORMULA, AND ALTERNATIVE ALGEBRAS

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Analytic H-spaces are shown to be local analytic loops (satisfying the cancellation laws). Then power associative local analytic loops are investigated and these are shown to be exactly the class to which a local loop belongs if there is a choice of coordinate system, f, for which the multiplication obeys $V(sx, tx) = sx + tx$. Here x is near 0 in R^n, each of the numbers s, t and $s + t$ is in $[0, 1]$ and V is the pulldown of the local loop multiplication via f. Homomorphism of such local loops are investigated and the set of such automorphism is shown to be isomorphic to a certain group of linear maps. Also generalizing the Lie group-Lie algebra situation, certain anti-commutative algebras are introduced to study these local loops. Finally these results are applied to local loops whose multiplication is induced by a power associative algebra. A Campbell-Hausdorff formula is shown to hold when the algebra is alternative and is related to the inverse property in the local loop. A relationship between S^7 and simple Malcev algebras is given.

Introduction. As given in [15], an H-space is a set M with multiplication function $m: M \times M \to M$ having an identity element e. As a variation of this and local groups, the triple (M, E, m) is said to be a local analytic H-space provided M is an analytic manifold, E is an open set of M containing e, and m is an analytic function from $E \times E$ to M satisfying $m(e, x) = m(x, e) = x$ for each $x \in E$. We show in §1 that these local analytic H-spaces satisfy the two-sided cancellation laws locally so that they are actually local loops and inverses exist locally.

Suppose (M, E, m) is a local analytic H-space and x is in E. Let $x^0 = e$ and if x^{n-1} is in E, let $x^n = m(x, x^{n-1})$. Then (M, E, m) is power associative if and only if for positive integers m, n

$$m(x^n, x^m) = x^{n+m}$$

whenever each of x^n and x^m is in E and x^{m+n} exists. Power associative analytic loops include Lie groups as well as seven-sphere S^7 with multiplication induced from the Cayley numbers.

In describing the structure of analytic local H-spaces it is convenient to choose a coordinate system f with domain a neighborhood of e so that $f(e) = 0$. There is then a neighborhood D of 0 in R^n so that the equation

$$V(f(x), f(y)) = f(m(x, y))$$

defines analytic function on V on $D \times D$. The triple (R^n, D, V) is a local analytic H-space locally isomorphic with (M, E, m) at e. The first main result is the following.

THEOREM 0.1. *Suppose* (M, E, m) *is a local analytic H-space. There is a neighborhood C of e in M such that $(M, C, m | C \times C)$ is power associative if and only if there is a coordinate system f at e with $f(e) = 0$ and a neighborhood D of 0 in R^n such that*

$$m(f^{-1}(tx), f^{-1}(sx)) = f^{-1}((t + s)x)$$

whenever x is in D and each of s, t and $s + t$ is in $[-1, 1]$.

In § 1 we prove this theorem and use the resulting "canonical coordinate" system f for power associative local analytic H-spaces to study analogues of Lie group-Lie algebra theorems which hold in this setting. Thus in § 2 we identify homomorphisms and kernels and show that the group of local automorphisms of (M, E, m) is a matrix group.

In § 3 we apply the previous results to power associative local analytic H-space with the multiplication being induced from an algebra with identiy. The results are similar to those for the general linear group and we concentrate on the analogue of the Campbell-Hausdorff theorem for Lie groups.

Thus suppose A is a finite dimensional power associative algebra with identity 1 over the real field. From results in § 1, there is a neighborhood D of 0 in A and a coordinate function f defined on a neighborhood of 1 in A so that the multiplication $V: D \times D \to A$ defined by

$$V(f(x), f(y)) = f(xy)$$

is analytic and satisfies

$$V(sx, tx) = (s + t)x$$

whenever s, t, $s + t$ is in $[0, 1]$ and $\|x\|$ is sufficiently small. (Here xy is the product in A.)

If V is defined in this way, then the second derivative of V at $(0, 0)$ is given by

$$V^{(2)}((0, 0)(x, y), (x, y)) = xy - yx \equiv [x, y] .$$

We investigate the Taylor's series of V in which the higher order derivatives V^n for $n > 2$ can be written in terms of V^2 analogous to the Campbell-Hausdorff theorem.

We obtain that the Taylor's series for V is given by the

Campbell-Housdorff formula if and only if A is an alternative algebra. We consider the more general problem as to when V^3 can be written in terms of V^2 and obtain algebras quasi-equivalent to an alternative algebra. This is related to the inverse properties holding in the local loop. Also we discuss the general problem as to when finitely many derivatives V^1, V^2, $\cdots$, V^N actually determine the Taylor's series for V.

1. Canonical coordinates. Let (M, E, m) be a local H-space as discussed in the introduction. If h is a homeomorphism from some neighborhood of e in M onto a neighborhood of 0 in R^n with $h(e) = 0$ then, h is said to be a *coordinate system at* e provided that there is a neighborhood D of 0 in R^n so that the function W defined by $W(h(x), h(y)) = h(m(x, y))$ has domain containing $D \times D$. For such a D the triple (R^n, D, W) is a local H-space which is said to be *induced by* h. From the definition of W it is clear that h is a local isomorphism from (M, E, m) to (R^n, D, W).

Let $\| \ \|$ denote a norm on R^n and if d is a positive number let $R(d)$ denote the ball centered at 0 with radius d. If T is a linear transformation from R^n or $R^n \times R^n$ to R^n let $|T|$ denote the operator norm of T.

We now show that (M, E, m) satisfies local cancellation laws, and hence that analytic H-spaces are local loops.

THEOREM 1.1. *If (M, E, m) is a local H-space then a coordinate system h may be chosen so that for some neighboorood D' of 0 in R^n the local H-space (R^n, D', V) induced by h satisfies local cancellation laws. Thus (M, E, m) is a local loop.*

Proof. Let f be any coordinate system at e, and let (R^n, D, V) be induced by f. If y is in D there is x in E such that $y = f(x)$. Thus $V(y, 0) = f(m(x, e)) = f(x) = y$ and similarly $V(0, y) = y$. If x is in R^n and hx is in D, then $V'(0, 0)(0, x) = \lim_{h \to 0} [V(0, hx) - V(0, 0)]/h = \lim_{h \to 0} 1/h \cdot hx = x$. Hence

$$V'(0, 0)(x, y) = V'(0, 0)[(x, 0) + (0, y)] = x + y$$

using the linearity of the derivative $V'(0, 0)$ on $R^n \times R^n$.

Choose the positive number d so that $R(d)$ is contained in D and if each of x and y is in $R(d)$, then $|V'(x, y) - V'(0, 0)| < 1/4$.

Suppose each of x, y and z is in $R(d)$. Then

$$\| y - z \| - \| V'(x, y)(0, y - z) \| \leqq \| [V'(0, 0) - V'(x, y)](0, y - z) \|$$
$$\leqq (1/4) \| y - z \| .$$

Hence $(3/4)\|y - z\| \leqq \|V'(x, y)(0, y - z)\|$. Thus,

$$(3/4)\|y - z\| - \|V(x, y) - V(x, z)\|$$
$$\leqq \|V(x, y) - V(x, z) - V'(x, y)(0, y - z)\|$$
$$= \left\|\int_0^1 dt[V'(x, z + t(y - z)) - V'(x, y)](0, y - z)\right\|.$$

Each of x, y and $z + t(y - z)$ is in $R(d)$ so this last term is $\leqq (1/4)\|y - z\|$, using $|V'(x, y) - V'(0, 0)| < 1/4$. Hence we have

$$(1/2)\|y - z\| \leqq \|V(x, y) - V(x, z)\|$$

and similarly

$$(1/2)\|y - z\| \leqq \|V(y, x) - V(z, x)\|.$$

Thus, a choice of $h = (f\,|\,f^{-1}(R(d)))$ satisfies the conclusion of Theorem 1.1.

NOTATION. We shall use "loop" instead of "H-space" to emphasize the local cancellation laws.

A variation of the following result was proved in [5] for C^1 power associative loops. We extend it to the analytic case. It is used to construct the canonical coordinate system of Theorem 0.1.

THEOREM 1.2. *Suppose (M, E, m) is a power associative local loop and (R^n, D, W) is induced from it by some coordinate system. There are positive numbers r and d such that for each x in $R(r)$, there is a unique continuous map $T_x: [0, 1] \to R(d)$ satisfying $T_x(0) = 0$, $T_x(1) = x$ and $W(T_x(s), T_x(t)) = T_x(s + t)$ whenever each of s, t and $s + t$ is in $[0, 1]$. Moreover, the function T_x is analytic on $[0, 1]$.*

The proof uses a differential equations theorem, which we paraphrase below.

THEOREM 1.3. *((10.7.5) of [3]). Suppose each of A and B is an open set in R^n and $h: A \times B \to R^n$ is analytic. Then, for each (a, b) in $A \times B$ and each number t_0 there is a segment J centered at t_0 and a ball T centered at b such that there is a unique continuous function $u: J \times T \to R^n$ with $u(t_0, z) = a$ and $u_t(t, z) = h(u(t, z), z)$. Moreover, u is analytic on $J \times T$.*

Proof of 1.2. Define $h: D \times R^n \to R^n$ by $h(x, z) = W'(x, 0)(0, z)$ and note that h is linear in the z-variable. Since W is analytic on $D \times D$, h is analytic on $D \times R^n$.

Using 1.3 we may choose a segment J centered at the number 0 and ball T centered at 0 in R^n with unique continuous $u: J \times T \to R^n$ satisfying $u(0, z) = 0$ and $u_t(t, z) = h(u(t, z), z)$. Moreover u is analytic on $J \times T$. Now choose c in $(0, 1)$ so that c is in J and define $\bar{u}: [0, 1] \times T \to R^n$ by $\bar{u}(t, z) = u(tc, z)$. Then we have from the definition of $\bar{u}$ and the chain rule that $\bar{u}(0, z) = 0$ and $\bar{u}_t(t, z) = cu_t(tc, z) = h(\bar{u}(t, z), cz)$, using the linearity of h noted above. Thus, from the uniqueness part of Theorem 1.3 we have $\bar{u}(t, z) = u(t, cz)$ if t is in J. Hence, we may assume in our application of Theorem 1.3 that J and T are chosen with $[0, 1]$ contained in J.

From Theorem 1 of [5] we may choose positive numbers r_1 and d_1 so that for each x in $R(r_1)$ there is a unique continuous map $T_x: [0, 1] \to R(d_1)$ satisfying $T_x(0) = 0$, $T_x(1) = x$ and $W(T_x(t), T_x(s)) = T_x(s + t)$ whenever each of s, t and $s + t$ is in $[0, 1]$. From [6] each T_x is continuously differentiable. Now using the chain rule,

$$\frac{d}{ds} T_x(s + t) = T_x'(s + t) = W'(T_x(t), T_x(s))(0, T_x'(s))$$

and hence $T_x'(t) = h(T_x(t), T_x'(0))$, using the definition of h. Thus from uniqueness, we have $u(t, T_x'(0)) = T_x(t)$ whenever t is in $[0, 1]$ and $T_x'(0)$ is in the ball T.

By Lemma 1 of [5] there is $M > 0$ such that if x is in $R(r_1)$ then $\| T_x(t) \| \leq M \| x \|$. Choose $r_2 > 0$ so that if each of x and y is in $R(r_2)$ then $| W'(0, 0) - W'(x, y) | < 1/2$. Let $d = r_2/2M$. If x is in $R(d)$ then

$$\| T_x'(0) \| - \| x \| \leq \| T_x'(0) - T_x(1) \|$$
$$= \left\| \int_0^1 dt [W'(0, 0) - W'(T_x(t), 0)](0, T_x'(0)) \right\|$$
$$\leq (1/2) \| T_x'(0) \| \, ,$$

using $\| T_x(t) \| \leq M \| x \| < r_2$. Thus $\| T_x'(0) \| < 2 \| x \|$.

Hence we may choose $r > 0$ so that if x is in $R(r)$ then $T_x'(0)$ is in the ball T. Thus for x in $R(r)$ we have $T_x(t) = u(t, T_x'(0))$ and since the right side of this equality is analytic in t, T_x is analytic on $[0, 1]$.

REMARKS. Let G be a Lie group in canonical coordinates so that the corresponding local multiplication function W is given by the Campbell-Hausdorff series:

$$W(x, y) = x + y + (1/2)[x, y] + \frac{1}{12}[x[x, y]] + \frac{1}{12}[y[y, x]] + \cdots$$

for x, y near 0 in the Lie algebra g of G. Then $T_x(t) = tx$ so that

the function g given by $g(x) = T'_x(0) \ (= x)$ is an analytic diffeomorphism at 0. Now in general, we can use the notation of Theorem 1.2 to define a function on the ball $R(d)$ by $g(x) = T'_x(0)$. Then g can be used to actually obtain a canonical coordinate system on a power associative local loop. First we show the following result.

THEOREM 1.4. *There are neighborhoods* D_1 *and* D_2 *of* 0 *in* R^n *so that* g *is an analytic homeomorphism from* D_1 *onto* D_2 *and* $(g\,|\,D_1)^{-1}$ *is analytic on* D_2.

Proof. Choose u and T as in the Proof of Theorem 1.2. Define $K\colon T \to R^n$ by $K(z) = u(1, z)$. Note $K(0) = 0$ since $h(x, 0) = 0$. Next

$$K(z) = u(1, z) = u(0, z) + \int_0^1 dt\, u_t(t, z) = \int_0^1 dt\, W'(u(t, z), 0)(0, z) \ .$$

Thus

$$K'(z)(x) = \int_0^1 dt\, W''(u(t, z), 0)((0, z), (u_z(t, z)(x), 0))$$

$$+ \int_0^1 dt\, W'(u(t, z), 0)(0, x) \ .$$

In particular $K'(0)(x) = \int_0^1 dt\, W'(0, 0)(0, x) = x$.

Since $K'(0) = I$, the identity function on R^n, we may apply the Inverse Function Theorem [3, Theorem (10.2.5)] and choose neighborhoods U_1 and U_2 of 0 so that $(K\,|\,U_1)$ is an analytic homeomorphism onto U_2 and $f = (K\,|\,U_1)^{-1}$ is analytic on U_2.

From the proof of Theorem 1.2 we may choose $d' > 0$ so that if x is in $R(d')$ then $T'_x(0)$ is in U_1 and $T_x(t) = u(t, T'_x(0))$. Using this, we have $K(g(x)) = K(T'_x(0)) = u(1, T'_x(0)) = T_x(1) = x$. Thus, since f is the inverse of $K\,|\,U_1$ we see $f(x) = T'_x(0) = g(x)$, $R(d')$ is contained in U_2, and the choice $D_1 = R(d')$ and $D_2 = f(R(d'))$ satisfies the conclusion of Theorem 1.4.

We now combine Theorems 1.2 and 1.4 to obtain a proof of Theorem 0.1. Thus we show the existence of a canonical coordinate system which characterizes power associativity of the corresponding local multiplication function V by $V(sx, tx) = sx + tx$.

Proof. Suppose (M, E, m) is a power associative local loop, h is a coordinate system, and (R^n, D, W) is a local loop induced by h.

Let r, and T_x be as in the conclusion of Theorem 1.2 with r chosen in addition, using Theorem 1.4, so that the function

$g: R(r) \to R^n$ defined by $g(x) = T_x'(0)$ is an analytic homeomorphism onto a neighborhood D' of 0 and so that g^{-1} is analytic on D'.

Note that if x is in $R(r)$, s is in $[0, 1]$ and $p(t) = T_x(st)$ for each t in $[0, 1]$, we have $p(t + t') = W(p(t), p(t'))$ whenever each of t, t', and $t + t'$ is in $[0, 1]$. For this case we also have $p(0) = 0$ and $p(1) = T_x(t)$. Moreover, $p'(0) = sT_x'(0)$. Hence, from the uniqueness part of Theorem 1.2, if $T_x(s)$ is in $R(r)$ then we have

$$(1.2) \qquad\qquad g(T_x(s)) = sg(x) \ .$$

Choose $d' > 0$ so that if x is in $R(d')$ then $T_x(s)$ is in $R(r)$ for each x in $[0, 1]$. (Lemma 2 [5] again.) Let $D' = h^{-1}(R(d'))$ and define $f: D' \to R^n$ by $f(x) = g(h(x))$. Choose D'' so that the equation $V(f(x), f(y)) = f(m(x, y))$ defines V with domain $D'' \times D''$. Then using this and the definition that h is a coordinate system, we have

$$f(m(x, y)) = g(W(h(x), h(y))) = V(g(h(x)), g(h(y))) \ .$$

Thus $V(x, y) = g(W(g^{-1}(x), g^{-1}(y)))$. Hence V is analytic on $D'' \times D''$ and f is a coordinate system at e. Suppose x is in D''. Then $y = g^{-1}(x)$ is in $R(d')$ and using (1.2) we obtain $g(T_y(s)) = sg(y) = sx$ for each s in $[0, 1]$. Suppose each of s, t and $s + t$ is in $[0, 1]$. Then $V(sx, tx) = V(g(T_y(s)), g(T_y(t))) = g(W(T_y(s), T_y(t))) = g(T_y(s + t)) = (s + t)x$ which also uses the above equation involving $p(t + t') = W(p(t), p(t'))$.

If $d > 0$ is chosen so that $R(d)$ is contained in D'' then d will satisfy the conclusion of Theorem 0.1 except that we have $V(sx, tx) = (s + t)x$ only for each of s, t and $s + t$ in $[0, 1]$. The rest of the conclusion will follow from (1.5).

DEFINITION A.   A coordinate system f with induced loop $(R^n, R(d), V)$ satisfying the conclusion of Theorem 0.1 is called a *canonical coordinate system* for (M, E, m).

In terms of higher derivatives of V we have the following characterization of a power associative local loop.

THEOREM 1.5.   *Suppose (M, E, m) is a local loop. There is a neighborhood E' of e in M so that $(M, E', (m | E' \times E'))$ is power associative if and only if there is a coordinate function f such that in the induced loop (R^n, D, V), we have the derivatives $V^{(k)}(0, 0)(sx, tx)^k = 0$ for each x in R^n and each s, t in R and $k \leqq 2$.*

Proof.   Suppose (M, E, m) is a local loop and there is a coordinate system f at e such that for some open D containing 0 the induced multiplication V satisfies $V^k(sx, tx)^k = 0$ for all x in R^n, numbers s

and t, and $k \geqq 2$. (The notation V^k is used for $V^{(k)}(0, 0)$.) Choose $d > 0$ so that the Taylor's series for V converges to V on $R(d) \times R(d)$ and $R(d)$ is contained in D. Then

$$V(sx, tx) = (s + t)x + \sum_{2}^{\infty} \frac{1}{k!} V^k(sx, tx)^k = (s + t)x$$

whenever s, t are in $[0, 1]$ and x is in $R(d)$. If $E' = f^{-1}(R(d))$ then $(M, E', (m | E' \times E'))$ is power associative.

Conversely if $(M, E', (m | E' \times E'))$ is power associative for some neighborhood E' of e we may use Theorem 0.1 to choose f so that the induced local loop $(R^n, R(d), V)$ satisfies $V(sx, tx) = (s + t)x$ for appropriate s, t and x. If d is chosen in addition so that the Taylor's series for V converges to V on $R(d) \times R(d)$ we have $0 = \sum_{2}^{\infty} V^k(sx, tx)^k$. This implies $V^k(sx, tx)^k = 0$ for all numbers s, t, and all x in R^n. We thus have a proof of (1.5) and the sufficiency of power associativity for the conclusion of 0.1. The necessity follows easily.

There are C^∞ power associative multiplications on R^1 which are not associative (simply disturb the graph of $+$ so that it cuts the XY plane in a nonsymmetric curve without disturbing it on $R^+ \times R^+ \cup R^- \times R^-$). Thus analyticity is a necessary hypothesis for 0.1.

Next we consider anticommutative algebras associated with a local loop which are analogous to the Lie algebra of a Lie group. In § 3 we shall investigate how these algebras determine the Taylor's series for V analogous to the Campbell-Hausdorff Theorem.

COROLLARY 1.6. *Suppose (M, E, m) is a power associative loop with canonical coordinate representation $(R^n, R(d), V)$.*

(a) *If $a(x, y) = V^2((x, 0), (0, y))$ then $a(x, y) = V^2(x, y)^2/2$, and a is bilinear and anticommutative. Thus a induces the structure of an algebra on R^n denoted by $(R^n, +, a)$.*

(b) *If (R^n, D, W) is another loop induced by some coordinate system on (M, E, m) such that $b(x, y) = W^2(x, y)^2/2$ is bilinear anticommutative, then the algebras $(R^n, +, b)$ and $(R^n, +, a)$ are isomorphic.*

(c) *If (M, E, m) is a Lie group then $(R^n, +, a)$ is its Lie algebra.*

Proof. Using the bilinearity of V^2 on $(R^n \times R^n)^2$ we expand

$$
\begin{aligned}
V^2(sx, tx)^2 &= V^2[(sx, tx), (sx, tx)] \\
&= V^2[(sx, 0) + (0, tx), (sx, 0) + (0, tx)] \\
&= s^2 V^2(x, 0)^2 + 2st V^2[(x, 0), (0, x)] + t V(0, x)^2 .
\end{aligned}
$$

Since $V^2(sx, tx)^2 = 0$ we obtain

$$V^2(x, 0)^2 = V^2[(x, 0), (0, x)] = V^2(0, x)^2 = 0 .$$

Using these we have

$$\frac{1}{2}V^2(x, y)^2 = \frac{1}{2}V^2[(x, y), (x, y)]$$

$$= \frac{1}{2}V^2[(x, 0) + (0, y), (x, 0) + (0, y)]$$

$$= \frac{1}{2}V^2(x, 0)^2 + V^2[(x, 0), (0, y)] + \frac{1}{2}V^2(0, y)^2$$

$$= V^2[(x, 0), (0, y)]$$

$$= a(x, y) .$$

Clearly a is bilinear and $a(x, x) = V^2[(x, 0), (0, x)] = 0$ so that a is anticommutative, i.e., $a(x, y) = -a(y, x)$.

Suppose (R^n, D, W) is induced from (M, E, m) by coordinate system h and f is the canonical coordinate system which induces $(R^n, R(d), V)$. Suppose furthermore that $b(x, y) = W^2(x, y)^2/2$ is bilinear anticommutative.

Let $g = f \circ h^{-1}$ on some neighborhood of 0 in R^n. Then g is analytic and $V(g(x), g(y)) = g(W(x, y))$ for all (x, y) sufficiently near $(0, 0)$.

If f, W, and V are expanded in Taylor's series about 0 and $(0, 0)$, then the above equation yields the following identity

$$a(g'(0)(x), g'(0)(y)) - g'(0)(b(x, y)) = 2g''(0)(x, y) .$$

Since the left side of this is anticommutative and the right side is symmetric we have

$$a(g'(0)(x), g'(0)(y)) = g'(0)(b(x, y)) .$$

Since h and f are coordinate systems $g'(0)$ is an isomorphism.
Part (c) is standard Lie theory [13].

Theorem 1.5 suggests the following method of constructing examples. Suppose for $k = 2, 3, \cdots$, that a_k is symmetric k-linear on $(R^n \times R^n)^k$ and satisfies $a_k(tx, sx)^k = 0$ for all t and s in R and x in R^n. Suppose furthermore that there is a positive number c so that the series $V(x, y) = x + y + \sum_2 a_k(x, y)^k$ converges on $R(c) \times R(c)$. Then for some $d \leqq c$, $(R^n, R(d), V)$ is a power associative local loop for which the identity function is a canonical coordinate system.

2. **Homomorphisms and automorphisms.** We now examine homomorphisms and kernels in the loop setting. In the case of Lie groups, differentiable group homomorphisms are analytic and are in one-to-one correspondence with Lie algebra homomorphisms. Similar

results hold for power associative loops, and we enumerate some of
these below.

DEFINITION. Suppose each of (M, E, m) and (P, U, w) is a local
H-space and h is an analytic function from some neighborhood of e
in M into a neighborhood of the identity element e' in P. h is said
to be a *homomorphism* if $h(e) = e'$ and there is a neighborhood D of
e such that $h(m(x, y)) = w(h(x), h(y))$ whenever each of x and y is
in D.

If $(P, U, w) = (M, E, m)$ and each of h and h^{-1} is a homomorphism
then we say h is an *automorphism*. Two automorphisms are said
to be equivalent if they agree on some neighborhood of e. This
defines an equivalence relation on the collection of all automorphisms
of (M, E, m). Let $[h]$ denote the equivalence class containing the
automorphism h and denote by G the set of all such equivalence
classes. Define $*: G \times G \to G$ by $[h_1] * [h_2] = [h_1 \circ h_2]$. An easy argu-
ment shows that $(G, *)$ is a group. Our next theorems give analogues
to the Lie group theorems mentioned above and show that $(G, *)$ is
isomorphic with a matrix group.

THEOREM 2.1. *Suppose each of (M, E, m) and (P, U, w) is a power
associative local loop and h is a function from a neighborhood of e
in M to a neighborhood of e' in P which is continuous and satisfies
$h(e) = e'$. If h is differentiable at e, and there is a neighborhood E
of e so that $h(m(x, y)) = w(h(x), h(y))$ whenever each of x and y is in
E, then h is a homomorphism. Moreover $im(h) \cap U$ is a local sub-
manifold of P at e'.*

Proof. From the definition of homomorphism, it suffices to show
h is analytic. Thus choose canonical coordinate functions f and g
for (M, E, m) and (P, U, w) respectively which induce local loops
(R^n, D, V) and (R^m, F, W). Let N be a neighborhood of e in M such
that if x is in $f(N)$ then $j(x) = g(h(f^{-1}(x)))$ exists. It is easy to see
that j satisfies $j(V(x, y)) = W(j(x), j(y))$ whenever each of x and y
is sufficiently close to 0 in R^n. Since h is differentiable at e we have
that j is differentiable at 0.

Since each of f and g is a canonical coordinate function we have
from the uniqueness part of Theorem 1.2 that $j(tx) = tj(x)$ for all x
sufficiently close to 0 and all t in $[0, 1]$. Differentiating both sides of
this equation with respect to t and setting $t = 0$, we obtain $j(x) =
j'(0)(x)$ for all x in some neighborhood of 0; thus j is analytic on N.
It then follows from the definition of j that h is analytic on N.
Moreover, since, near e', the image of $h = g^{-1}(im(j))$ we have $im(h)$
is a local submanifold of P at e'.

COROLLARY 2.2. *If h is a homomorphism from (M, E, m) to (P, U, w) then (with j defined as above) j is linear and $j(V^k(x, y)^k) = W^k(j(x), j(y))^k$ whenever each of x and y is in R^n and k is a positive integer.*

Proof. It is immediate from the Proof of Theorem 2.1 that j is the restriction of the linear map $j'(0)$. Since j is a homomorphism from (R^n, D, V) to (R^m, F, W) we have

$$\sum_1^\infty (1/k!) j(V^k(x, y)^k) = j\Big(\sum_1^\infty (1/k!) V^k(x, y)^k\Big)$$
$$= j(V(x, y)) = W(j(x), j(y))$$
$$= \sum_1^\infty (1/k!) W^k(j(x), j(y))^k .$$

The rest of the conclusion of the corollary follows from this equality.

Thus analogous to the Lie group case where a homomorphism of Lie groups induces a homomorphism of Lie algebras, this corollary shows that a homomorphism of local loops induces a homomorphism of the multi-linear systems (R^n, V^k) to (R^m, W^k). The next result shows the converse.

THEOREM 2.3. *If $j: R^k \to R^m$ is continuous and $j(V^k(x, y)^k) = W^k(j(x), j(y))^k$ for each positive integer k, then there is a neighborhood Q of 0 in R^n so that if $h(x) = g^{-1}(j(f(x)))$ for all x in $f^{-1}(Q)$ then h is a homomorphism of (M, E, m) to (P, U, w).*

Proof. $j(x + y) = j(V^1(x, y)) = W^1(j(x), j(y)) = j(x) + j(y)$. Since j is continuous, it follows from this that j is linear. Using j is linear and the hypothesis, we see

$$j(V(x, y)) = j\Big(\sum_1^\infty (1/k!) V^k(x, y)^k\Big)$$
$$= \Big(\sum_1^\infty (1/k!) j(V^k(x, y)^k)\Big)$$
$$= \sum_1^\infty (1/k!) W^k(j(x), j(y))^k$$
$$= W(j(x), j(y))$$

if each of x and y is sufficiently near 0. It is easy then to see from the definition of h that h is a homomorphism.

THEOREM 2.4. *The group $(G, \ast)$ is isomorphic with a closed subgroup of $GL(R^n)$, the group of invertible linear transformations of R^n.*

Proof. Suppose each of h_1 and h_2 is an automorphism of (M, E, m) and h_1 is equivalent to h_2. Let $j_i(x) = f(h_i(f^{-1}(x)))$ for $i = 1, 2$, and x sufficiently close to 0. By Corollary 2.2 j_i is the restriction of a linear transformation and let $\bar{j}_i$ denote this linear transformation. Since h_1 and h_2 agree on a neighborhood of e, $\bar{j}_1$ and $\bar{j}_2$ coincide on a neighborhood of 0 and hence, by linearity, $\bar{j}_1 = \bar{j}_2$.

Thus, we may define the function $F: G \to GL(R^n)$ by $F([h]) = (f \circ h \circ f^{-1})'(0)$. By the preceding argument, F is well defined. Using the notation of Corollary 2.2 for $j = f \circ h \circ f^{-1}$, we see that $j = j'(0)$. Consequently if $F([h_1]) = F([h_2])$, then $j_1 = j_2$ and therefore $[h_1] = [h_2]$. Thus F is one-to-one. Next, from Theorem 2.3, the map T is in the image of F if and only if $T(V^k(x, y)^k) = V^k(Tx, Ty)^k$ for all x and y in R^n and all positive k. It follows that the image of F is closed in $GL(R^n)$.

If each of $[h]$ and $[g]$ is in G, then $F([h] * [g]) = F([h \circ g]) = [f \circ (h \circ g) \circ f^{-1}]'(0) = [(f \circ h \circ f^{-1}) \circ (f \circ g \circ f^{-1})]'(0) = (f \circ h \circ f^{-1})'(0) \circ (f \circ g \circ f^{-1})'(0) = F([h]) \circ F([g])$. Thus F is an isomorphism onto $\text{Im}(F)$.

The next theorems identify all possible homomorphisms defined on a power associative local loop.

THEOREM 2.5. *Suppose each of (R^n, D, V) and (R^m, E, W) is a power associative local loop in canonical coordinates, and $j: D \to E$ is a homomorphism. Then there is a linear transformation T from R^n to R^m such that $(T | D) = j$ and for each $k = 1, 2, \cdots$ we have $T(V^k[(x_1, y_1), \cdots, (x_k, y_k)]) = W^k[(Tx_1, Ty_1), \cdots, (Tx_k, Ty_k)]$ whenever each of $x_1, \cdots, x_k, y_1, \cdots, y_k$ is in R^n.*

Proof. Since (R^n, D, V) and (R^m, E, W) are in canonical coordinates, we may choose the maps f and g in the Proof of Theorem 2.1 to be the identity maps. Consequently, from that proof, T is $j'(0)$. From Corollary 1.3 we have $T(V^k(x, y)^k) = W^k(Tx, Ty)^k$ whenever $k \geq 1$ and each of x and y is in R^n. Let $g(x, y) = T(V^k(x, y)^k)$. From the chain rule, g is analytic. If each of x, y, x_1 and y_1 is in R^n then

$$g^1(x, y)(x_1, y_1) = \lim_{h \to 0} (1/h)[g(x + hx_1, y + hy_1) - g(x, y)]$$

$$= \lim_{h \to 0} (1/h)\left[\sum_0^k \binom{k}{s} h^s T(V^k(x, y)^{k-s}(x_1, y_1)^s) - T(V^k(x, y)^k)\right]$$

$$= k \cdot T(V^k(x, y)^{k-1}(x_1, y_1))$$

$$+ \lim_{h \to 0} h \sum_0^{k-2} \binom{k}{s} h^{k-s-1} T(V^k(x, y)^s(x_1, y_1)^{k-s})$$

$$= kT(V^k[(x, y)^{k-1}, (x_1, y_1)]) .$$

A continuation of these calculations yields

$$g^k(x, y)[(x_1, y_1), \cdots, (x_k, y_k)] = k! \, TV^k[(x_1, y_1), \cdots, (x_k, y_k)] \, .$$

A similar calculation shows

$$g^k(x, y)[(x_1, y_1), \cdots, (x_k, y_k)] = k! \, W^k[(Tx_1, Ty_1), \cdots, (Tx_k, Ty_k)] \, .$$

COROLLARY 2.6. *Let* $L = T^{-1}(\{0\})$ *be the kernel of* T. *If each of* $x_i - x_i'$ *and* $y_i - y_i'$ *is in* L *for* $i = 1, 2, \cdots, k$ *then* $V^k[(x_1, y_1), \cdots, (x_k, y_k)] - V^k[(x_1', y_1'), \cdots, (x_k', y_k')]$ *is in* L.

Proof. $T(V^k[(x_1, y_1), \cdots, (x_k, y_k)]) = W^k[(Tx_1, Ty_1), \cdots, (Tx_k, Ty_k)] = W^k[(Tx_1', Ty_1'), \cdots, (Tx_k', Ty_k')] = TV^k[(x_1', y_1'), \cdots, (x_k', y_k')]$.

A converse to this is the following theorem.

THEOREM 2.7. *Suppose* (R^n, D, V) *is a power associative local loop in canonical coordinates. Suppose* L *is a linear subspace of* R^n *so that if each of* $x_i - x_i'$ *and* $y_i - y_i'$ *is in* L *for* $i = 1, 2, \cdots, k$ *then* $V^k[(x_1, y_1), \cdots, (x_k, y_k)] - V^k[(x_1', y_1'), \cdots, (x_k', y_k')]$ *is in* L. *Then there is a power associative local loop* (R^m, E, W) *in canonical coordinates and a linear transformation* $T: R^n \to R^m$ *so that for some neighborhood* D' *of* 0 *in* R^n *we have* $(T | D')$ *is a local loop homomorphism from* (R^k, D, V) *to* (R^m, E, W). *Moreover* $L = T^{-1}(\{0\})$.

Proof. Let $\overline{R^n} = (R^n/L)$ be the linear coset space corresponding to the subspace L of R^n and $\pi: R^n \to \overline{R^n}$ be the natural projection map defined by $\pi(x) = x + L$.

If $k = 1, 2, \cdots$ and each of x_i and y_i is in R^n for $i = 1, 2, \cdots, k$ let $\overline{V^k}$ be defined on $(\overline{R^n} \times \overline{R^n})^k$ by

$$\overline{V^k}[(x_1 + L, y_1 + L), \cdots, (x_k + L, y_k + L)]$$
$$= \pi V^k[(x_1, y_1), \cdots, (x_k, y_k)] \, .$$

It is clear from our hypothesis on L that $\overline{V^k}$ is well defined and clear from $\overline{V^k}$'s definition that it is k-linear and symmetric.

If x is in R^n let $\|x + L\| = \inf \{\|z\| \mid x - z$ is in $L\}$. This defines a norm on $\overline{R^n}$ and for that norm we have $\|\pi x\| \leqq \|x\|$.

Choose D', a neighborhood of 0 in R^n so that the Taylor's series for V converges absolutely to V on $D' \times D'$. Since π is linear and onto, π is an open map. Thus $\bar{D} = \pi(D')$ is open in $\overline{R^n}$ and contains $\pi(0)$. Suppose each of $x + L$ and $y + L$ is in $\bar{D}$. Choose x' and y' so that each of x' and y' is in D' and $\pi x' = x + L$ and $\pi y' = y + L$.

Then

$$\sum_{1}^{\infty} (1/k!)\|\overline{V}^k(x + L, y + L)^k\| \leqq \sum_{1}^{\infty} (1/k!)\|\pi V^k(x', y')^k\|$$

$$\leqq \sum_{1}^{\infty} (1/k!)\|V^k(x', y')\| < \infty .$$

Thus the series $\overline{V}(\pi x, \pi y) = \sum_{1}^{\infty} (1/k!)\overline{V}^k(\pi x, \pi y)^k$ converges absolutely to an analytic function $\overline{V}$ on $\overline{D} \times \overline{D}$. Also $\overline{V}^k$ is the kth derivative of $\overline{V}$ at (L, L) and $\overline{V^k}(s\pi x, t\pi x)^k = \pi V^k(sx, tx)^k = \pi(0) = \overline{0}$ in $\overline{R}^n$ whenever πx is in $\overline{R}^n$ and $k \geqq 2$. Thus by Theorem 1.5, $(\overline{R}^n, \overline{D}, \overline{V})$ is a power associative local loop and $\overline{V}(\pi x, \pi y) = \pi V(x, y)$.

From elementary linear algebra, the dimension of $\overline{R}^n$ is $n - \dim (L) = m$. Choose S to be a linear isomorphism from $\overline{R}^n$ onto R^m and let $T = S \circ \pi$. Let $E = S(\overline{D})$ and define W on $E \times E$ by $W(S\pi x, S\pi y) = S\overline{V}(\pi x, \pi y)$. Clearly (R^m, E, W) is the local loop sought in the conclusion of Theorem 2.7 and T is the correct linear transformation, since $TV(x, y) = S\pi V(x, y) = S\overline{V}(x, y) = W(S\pi x, S\pi y) = W(Tx, Ty)$ for $x, y \in D'$.

REMARKS. Let G be a Lie group so that its multiplication, in terms of canonical coordinates, is given by $V(x, y)$. From the Campbell-Hausdorff Theorem [13] we have

$$V(x, y) = x + y + \frac{1}{2}[x, y] + \sum_{k=3}^{\infty} \frac{V^k(x, y)^k}{k!}$$

where $V^2(x, y)^2 = [x, y]$ is the multiplication on the Lie algebra g of G. Also each $V^k(x, y)^k$ is a homogeneous polynomial in the Lie subalgebra of a g generated by x and y. Thus if j is an automorphism of the Lie algebra g, we automatically have $V^k(jx, jy)^k = jV^k(x, y)^k$. Thus by Theorem 2.3 (or directly), j induces a local automorphism of G; see [13].

In §3 we shall investigate local loops so that the derivatives $V^k(x, y)^k$ for $k \geqq 3$ are determined by $a(x, y) = V^2(x, y)^2$ and the corresponding anti-commutative algebra $(R^n, +, a)$. The corresponding nonassociative local loops are closely related to the Cayley numbers and the sphere S^7.

The results of Theorems 2.5, 2.6 and 2.7 are also related to "ideals" as in the case of Lie algebras and Lie groups. Thus using the notation of Corollary 2.6, let L be the kernel of the homomorphism T (where $(T|D) = j$). Then for the anti-commutative multiplication a given above, we have for $x \in L$ and $y \in R^n$

$$Ta(x, y) = TV^2[(x, 0), (0, y)]$$
$$= W^2[(Tx, 0), (0, Ty)]$$
$$= W^2[(0, 0), (0, Ty)]$$
$$= 0 .$$

Consequently $a(x, y) \in L$ and since $a(x, y) = -a(y, x)$, we see L is an ideal in the algebra $(R^n, +, a)$. Thus, as expected, these calculations show we have generalizations of normal subgroups and ideals as kernels of homomorphisms.

These results, those of § 3 and the Campbell-Hausdorff Theorem lead to the following general problem: What conditions on the local loop (M, E, m) and the coordinate function h imply there exists an integer N so that the terms $V^k(x, y)^k$ for $k \leq N$ determine the terms $V^n(x, y)^n$ for $n > N$ and consequently determine the corresponding multiplication function V. By "determine" we mean that for every $x, y \in R^n$, $V^n(x, y)^n$ for $n > N$ is the subsystem of the algebraic structure $(R^n; V^1, V^2, \cdots, V^N)$ generated by x and y.

3. Alternative algebras and the Campbell-Hausdorff Theorem.

In this section we discuss H-spaces induced by nonassociative algebras and prove results analogous to those for the general linear group, $GL(n)$, its Lie algebra, $gl(n)$, and the Campbell-Hausdorff formula.

Suppose $(A, +, \cdot)$ is a finite dimensional power associative algebra over the real field with identity element 1. Let $\| \ \|$ be a norm on A. Since $\cdot$ is bilinear it is analytic and $(A, A, \cdot)$ is a power associative local loop. Since A is power associative, we can define the exponential function E on A by $E(x) = \sum_0^\infty (1/k!)x^k$. This series converges absolutely on A and we will show in this section that the function V defined on a neghborhood of $(0, 0)$ in $A \times A$ by $E(V(x, y)) = E(x) \cdot E(y)$ is induced by a canonical coordinate representation for $(A, A, \cdot)$.

Let $V^k = V^{(k)}(0, 0)$ and choose D a neighborhood of 0 in A so that $V(x, y) = \sum_0^\infty (1/k!)V^k(x, y)^k$ converges on $D \times D$.

As in the case for $GL(n)$ we can consider the power series expansion for $E(V(x, y))$ and multiply the series $E(x) \cdot E(y)$ in the algebra A. Since $E(V(x, y)) = E(x) \cdot E(y)$ in A, we then equate terms of the same degree to obtain various formulas for the terms $V^k(x, y)^k$. In particular,

$$V^2(x, y)^2 = x \cdot y - y \cdot x \equiv [x, y]$$

for each (x, y) in $A \times A$ and later in this section we use differentiation to compute more of these terms. Now if A is associative then the Campbell-Hausdorff theorem says for $k \geq 2$ that $V^k(x, y)^k$

is a specific homogeneous polynomial of degree k in the $[\ ,\ ]$ multiplication. It is easy to see that this is also the case if $(A, +, \cdot)$ is alternative which we do in Theorem 3.1. In this section we explore the consequences of assuming that V^3 is a homogeneous polynomial of degree 3 in the $[\ ,\ ]$ product and obtain some sufficient conditions for $(A, +, \cdot)$ or its complexification to be quasi-equivalent with an alernative algebra.

Suppose $(A, +, \cdot)$ is a finite dimensional power associative algebra over the real field with identity element 1 and $||\ ||$ is a norm on A. Since $\cdot$ is bilinear there is a number m so that $||x \cdot y|| < m||x|| \cdot ||y||$ for all x, y in A. It follows that for each x in A that $||x^n|| \leqq m^{n-1}||x||^n$ for $n = 1, 2, \cdots$. Hence the power series $E(x) = \sum_0^\infty (1/k!)x^k$ converges absolutely on A.

For each positive integer k and each x in A let $P_k(x)$ be the linear transformation on A defined by

$$P_k(x)(y) = x(x \cdots (x \cdot y) \cdots) + x(x \cdots (y \cdot x) \cdots) + \cdots \\ + y(x \cdots (x \cdot x) \cdots)$$

where each summand has one y and $k - 1$ x's. In particular,

$$P_1(x)(y) = y \quad\text{and}\quad P_2(x)(y) = xy + yx\ .$$

Note $||P_k(x)(y)|| \leqq km^{k-1}||x||^{k-1}||y||$ and hence the series $\sum_1^\infty (1/k!)P_k(x)$ converges absolutely in the space of linear transformations on A with operator norm.

Since the multiplication function on A is bilinear it is analytic. Hence the kth power function $f_k : x \to x^k$, which is well defined by power associativity, is analytic on A. Using some arithmetic and the choice of m we obtain for x and y in A and k a positive integer,

$$||x^k - y^k - P_k(x)(x - y)|| \\ \leqq (2^k - (k + 1))m^{k-1}||y - x||^2 \cdot \max\{||x||, ||y - x||\}^{k-2}\ .$$

Thus, if $f_k(x) = x^k$ then $f_k^1(x) = P_k(x)$. It follows from the standard theorem for term by term differentiation of convergent sequences of functions that $E^1(x) = \sum_1^\infty (1/k!)P_k(x)$.

Let $m(x, y) = x \cdot y$. An easy calculation shows $m^1(x, y)(a, b) = a \cdot y + x \cdot b$ and hence $m^1(x, 1)(0, z) = x \cdot z$. If $u(t, z) = E(tz)$ then $u(0, z) = 1$ and

$$u_t(t, z) = E'(tz)(z)$$
$$= \sum_1^\infty (1/k!)P_k(tz)(z)$$
$$= \sum_1^\infty (k/k!)t^{k-1}z^k$$
$$= E(tz) \cdot z\ .$$

Thus, making allowances for having identity element 1 instead of 0, we have from the Proof of Theorem 1.2

$$E(tx) \cdot E(sx) = E((t + s)x)$$

for all x sufficiently near 0 and all numbers s, t in some segment centered at 0.

Since $u(1, z) = E(z)$ it follows from Theorem 1.4 that there are neighborhoods D_1 and D_2 of 0 and 1 respectively so that $(E | D_1)$ is a homeomorphism onto D_2 and the logarithm function $L = (E | D_1)^{-1}$ is analytic on D_2. Choose a positive number d so that the function V defined by $E(V(x, y)) = E(x)E(y)$ is defined on $R(d) \times R(d)$ and has absolutely convergent Taylor's series there. L is canonical coordinate function constructed in Theorem 1.5 and V is the canonical coordinate representation for $(A, A, \cdot)$ near the identity element 1 which is given by $V(x, y) = L(E(x) \cdot E(y))$ as in the case of $GL(n)$.

An algebra $(A, +, \cdot)$ is said to be *alternative* provided that $(x, x, y) = (y, x, x) = 0$ whenever each of x and y is in A. Here $(x, y, z) = (xy)z - x(yz)$ is the "associator function." From [14], any power associative algebra satisfies

$$(x, x, y) + (x, y, x) + (y, x, x) = 0 \ ,$$

so if $(A, +, \cdot)$ is alternative we have also that $(x, y, x) = 0$. Again from [14] if $(A, +, \cdot)$ is alternative then $A(x, y)$, the subalgera of A generated by x, y and 1, is associative.

THEOREM 3.1. *If $(A, +, \cdot)$ is alternative and V is the canonical coordinate representation for $\cdot$ constructed in the preceding paragraphs, then V^2 determines V in the sense that V^k, for $k > 2$, is the specific homogeneous polynomial in the V^2 multiplication on A given by the conclusion of the Campbell-Hausdorff theorem.*

Proof. Suppose each of x and y is in A. Since $A(x, y)$ is associative, the Champbell-Hausdorff theorem holds for $A(x, y)$.

As before, define the function E on $A(x, y)$ by $E(z) = \sum_0^\infty (1/k!)z^k$, and let W be defined on an appropriate neighborhood of $(0, 0)$ in $A(x, y) \times A(x, y)$ by $E(W(z_1, z_2)) = E(z_1) \cdot E(z_2)$. From our previous argument, W is a canonical coordinate representation of the multiplication on $A(x, y)$. From the definition of W we have $W = (V | \mathrm{dom}\,(W))$.

From the definition of the derivative we have $V^k(x, y)^k = W^k(x, y)^k$ for each positive integer k. Thus, from the fact that the Campbell-Hausdorff formula holds for W we have that it holds for V.

Conversely, we shall see below that if the explicit formula for

V^k given in the Campbell-Hausdorff theorem holds for $(A, +, \cdot)$ in the power associative case then $(A, +, \cdot)$ must be alternative.

Toward this result we compute several terms of the Taylor's series for V. This can be done by the multiplication of power series as indicated before, or by computing derivatives as follows. Thus, let $g(x, y) = E(V(x, y)) = E(x) \cdot E(y)$. We compute the derivatives $g^i(x, y)$ for $i = 1, 2,$ and 3.

From the chain rule we have

$$g^1(x, y)(a, b) = E^1(V(x, y))(V^1(x, y)(a, b))$$

and from the product rule

$$g^1(x, y)(a, b) = E^1(x)(a) \cdot E(y) + E(x) \cdot E^1(y)(b) \, .$$

Recall, if $h(x, y) = g^1(x, y)(a, b)$ then $g^2(x, y)(a, b)^2 = h^1(x, y)(a, b)$. Thus, from the product rule and the chain rule

$$g^2(x, y)(a, b)^2 = E^2(V(x, y))(V^1(x, y)(a, b))^2$$
$$+ E^1(V(x, y))(V^2(x, y)(a, b)^2)$$

and recalling the notation $(z)^k = (z, \cdots, z)$ with z occurring k times,

$$g^2(x, y)(a, b)^2 = E^2(x)(a)^2 \cdot E(y) + 2E^1(x)(a) \cdot E^1(y)(b)$$
$$+ E(x) \cdot E^2(y)(b)^2 \, .$$

Similarly,

$$g^3(x, y)(a, b)^3 = E^3(V(x, y))(V^1(x, y)(a, b))^3$$
$$+ 3E^2(V(x, y))(V^1(x, y)(a, b), V^2(x, y)(a, b)^2)$$
$$+ E^1(V(x, y))(V^3(x, y)(a, b)^3)$$

and

$$g^3(x, y)(a, b)^3 = E^3(x)(a)^3 \cdot E(y) + 3E^2(x)(a)^2 \cdot E^1(y)(b)$$
$$+ 3E^1(x)(a) \cdot E^2(y)(b)^2 + E(x) \cdot E^3(y)(b)^3 \, .$$

We wish to evaluate the preceding derivatives at $(0, 0)$ so to do this we compute $E^2(0)(x, y)$.

$$E^2(0)(x, y) = \lim_{h \to 0} 1/h(E^1(hy)(x) - E^1(0)(x))$$

$$= \lim_{h \to 0} 1/h\Big(\sum_1^\infty (h^{k-1}/k!)P_k(y)(x) - x\Big)$$

$$= \lim_{h \to 0} \Big(\frac{1}{2}P_2(y)(x) + h \sum_3^\infty (h^{k-3}/k!)P_k(y)(x)\Big)$$

$$= \frac{1}{2}(xy + yx) \, .$$

Putting $(x, y) = (0, 0)$ in the expression for $g^i(x, y)(a, b)^i$ and using $E^k(0)(x)^k = x^k$ we have

(3.2)
$$V^1(0, 0)(a, b) = a + b$$

(3.3)
$$(a + b)^2 + V^2(0, 0)(a, b)^2 = a^2 + 2ab + b^2$$

(3.4) $(a + b)^3 + (3/2)[(a + b) \cdot V^2(0, 0)(a, b)^2 + V^2(0, 0)(a, b)^2 \cdot (a + b)]$
$$+ V^3(0, 0)(a, b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 .$$

From (3.3), $V^2(0, 0)(a, b)^2 = ab - ba$. Substituting this in (3.4) and computing yields

$$2V^3(0, 0)(a, b)^3 = 6a^2b - 5a(ab) + a(ba) - 3(ab)a - 2ba^2 + 3(ba)a$$
$$+ 4ab^2 - 5b(ab) + b(ba) - 3(ab)b + 3(ba)b .$$

To arrange this in more comprehensible form let $[a, b] = V^2(a, b)^2 = ab - ba$ and recall the power associative identity [14]; $(a, a, b) + (a, b, a) + (b, a, a) = 0$.

Using this notation the above expression becomes

(3.5) $V^3(0, 0)(a, b)^3 = V^3(a, b)^3$

$$= 4(a, a, b) + 2(b, a, a) + \frac{1}{2}[a, [a, b]]$$

$$- 2(b, b, a) - 4(a, b, b) + \frac{1}{2}[b, [b, a]] .$$

We summarize these calculations as follows.

THEOREM 3.6.   *Suppose $(A, +, \cdot)$ is a power associative algebra with identity 1 and V is the canonical coordinate representation of $(A, A, \cdot)$ induced by the exponential function E. Then with the notation $(x, 0)^r(0, y)^s = ((x, 0), \cdots, (x, 0), (0, y), \cdots, (0, y))$ with $(x, 0)$ occurring r times and $(0, y)$ occurring s times, we have*

(3.7)
$$V^1(x, y) = x + y$$

(3.8)
$$V^2(x, y)^2 = [x, y] = xy - yx$$

(3.9) $V^3(x, 0)^2(0, y) = 4/3(x, x, y) + 2/3(y, x, x) + 1/6[x, [x, y]]$

(3.10) $V^3(y, 0)(0, x)^2 = -\dfrac{2}{3}(x, x, y) - \dfrac{4}{3}(y, x, x) + \dfrac{1}{6}[x[x, y]] .$

Proof.   The first two of these are immediate from our calculations. To see (3.9) and (3.10) we observe from (3.5) that

$$1/stV^3(tx, sy)^3 = t\left\{4(x, x, y) + 2(y, x, x) + \frac{1}{2}[x, [x, y]]\right\}$$

$$+ s\left\{-2(y, y, x) - 4(x, y, y) + \frac{1}{2}[y, [y, x]]\right\},$$

if each of s and t is a number. But from Theorem 1.5 and the fact that V^3 is symmetric trilinear we have also that

$$1/stV^3(tx, sy)^3 = 3tV^3(x, 0)^2(0, y) + 3sV^3(x, 0)(0, y)^2.$$

The remainder of Theorem 3.6 is immediate from these two formulas.

COROLLARY 3.11. $(A, +, \cdot)$ *is alternative if and only if* $V^3(x, 0)^2(0, y) = V^3(y, 0)(0, x)^2 = (1/6)[x, [x, y]]$.

Proof. If $(A, +, \cdot)$ is alternative, the expressions V^3 are immediate from Theorem 3.6 and $(x, x, y) = (y, x, x) = 0$.

If we have the indicated expressions for V^3 then from Theorem 3.6 we have

$$2(x, x, y) + (y, x, x) = 0 \quad \text{and} \quad -(x, x, y) - 2(y, x, x) = 0.$$

The only solution to this system is $(x, x, y) = (y, x, x) = 0$ so that $(A, +, \cdot)$ is alternative.

We now investigate some of the consequences of V^2 determining V^3. Thus assume $V^3(x, y)^3$ may be written as a polynomial expression in terms of V^2. Since $V^2(x, y)^2 = [x, y]$ we shall investigate the *hypothesis H*: There are real numbers a and b such that

$$V^3(x, y)^3 = 3a[x, [x, y]] + 3b[y, [y, x]].$$

Conditions which imply hypothesis H are discussed in Theorem 3.31. Since $V^3(x, y)^3 = 3V^3[(x, 0)^2, (0, y)] + 3V^3[(x, 0), (0, y)^2]$ we see that H implies

$$V^3[(x, 0)^3, (0, y)] = a[x, [x, y]] \quad \text{and}$$
$$V^3[(y, 0), (0, x)^2] = b[x, [x, y]].$$

In view of Theorem 3.6 this is the same as saying the power associative algebra $(A, +, \cdot)$ satisfies the identities

(3.12) $(x, x, y) = \left(a + \dfrac{b}{2} - \dfrac{1}{4}\right)[x, [x, y]]$

(3.13) $(x, y, x) = \dfrac{1}{2}(b - a)[x, [x, y]]$

(3.14) $$(y, x, x) = \left(-b - \frac{a}{2} + \frac{1}{4}\right)[x, [x, y]] .$$

Since undetermined a, b enter into the identities for A, we take up the slack by considering quasi-equivalent algebras.

DEFINITION. Let A be an algebra and let u, v be numbers so that $u + v = 1$ and $u - v \neq 0$. Let A^0 denote the algebra with vector space A and multiplication $x \circ y = uxy + vyx$. A is said to be *quasi-equivalent* to an algebra B in case there are numbers u, v so that $A^0 = B$ as algebras.

We shall now consider the possibility that if A satisfies hypothesis H, then it is quasi-equivalent to an alternative algebra. Denoting $(x \circ y) \circ z - x \circ (y \circ z)$ by $(x, y, x)^0$ an easy calculation shows

$$(x, y, x)^0 = (u - v)(x, y, x)$$
$$= (v - u)q[x, [x, y]]$$
$$(x, x, y)^0 = u(x, x, y) - v(y, x, x) + uv[x, [x, y]]$$
$$= (pu - rv + uv)[x, [x, y]]$$

where $p = a + b/2 - 1/4$, $q = (1/2)(b - a)$ and $r = -b - a/2 + 1/4$. Also note that if A is power associative, so is A^0.

If A^0 is to be alternative, then noting $u \neq v$ we must have $a = b$ or $[x, [x, y]] = 0$ for all x, y in A. In the second case A is alternative if A satisfies H. Thus we now consider a case when $a = b$.

LEMMA 3.15. *If A is a power associative algebra which satisfies condition H and if A contains an idempotent e not in the center of A, then $a = b$. In this case A^0 satisfies $(x, y, x)^0 = 0$ for all x, y in A^0.*

Proof. Suppose the idempotent e of A is not in the center of A. Define the linear transformation T on A by $T(x) = (1/2)(ex + xe)$. Then from [14, p. 131] T satisfies $2T^3 - 3T^2 + T = 0$ and consequently T has the three simple characteristic roots $0, 1, 1/2$. Thus A has the direct sum Peirce decomposition

$$A = A_0 + A_1 + A_{1/2}$$

where $A_i = \{x: T(x) = ix\}$ for $i = 0, 1, 1/2$. From [14, p. 131] it follows that $x \in A_i$ for $i = 0, 1$ if and only if $ex = xe = ix$. Also $x \in A_{1/2}$ if and only if $xe + ex = x$.

As in Lie algebras, let ade be defined on A by

$$ade(x) = [e, x] = ex - xe .$$

If x is in A_0 or A_1, then $ade(x) = 0$. Since e is not in the center of A we must have some $x_{1/2}$ in $A_{1/2}$ so that $ade(x_{1/2}) \neq 0$ and consequently $A_{1/2}$ is not the zero subspace.

Choose y in $A_{1/2}$ with $y \neq 0$. Then using identity (3.12) with $x = e$ and $ey + ye = y$ we obtain

$$
\begin{aligned}
p(ade)^2(y) &= (e, e, y) \\
&= ey - e(ey) \\
&= ey - e(y - ye) \\
&= e(ye) \ .
\end{aligned}
$$

Similarly using identity (3.14) with $x = e$ we obtain

$$
\begin{aligned}
r(ade)^2(y) &= (y, e, e) \\
&= (ye)e - ye \\
&= (y - ey)e - ye \\
&= -(ey)e \ .
\end{aligned}
$$

But

$$
\begin{aligned}
(ade)^2(y) &= ade(ey - ye) \\
&= e(ey) - e(ye) - (ey)e + (ye)e \\
&= e(y - ye) - e(ye) - (ey)e + (y - ey)e \\
&= ey - 2e(ye) - 2(ey)e + ye \\
&= y - 2p(ade)^2(y) + 2r(ade)^2(y) \ .
\end{aligned}
$$

Consequently $(1 + 2p - 2r)(ade)^2(y) = y \neq 0$ since we have assumed $y \neq 0$ in $A_{1/2}$. Thus we may conclude that $1 + 2p - 2r = 3a + 3b \neq 0$ and using the direct sum $A = A_0 + A_1 + A_{1/2}$ we see $(ade)^2$ has a matrix of the form

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \lambda I
\end{pmatrix}
$$

where $\lambda = 1/(1 + 2p - 2r)$. Thus trace $(ade)^2 \neq 0$.

Now defining endomorphisms L and R on A by $L(x) = ex$ and $R(x) = xe$ we see from identity (3.13),

$$
(RL - LR)(x) = (e, x, e) = \frac{1}{2}(b - a)(ade)^2(x)
$$

so that $RL - LR = (1/2)(b - a)(ade)^2$. Since trace $(RL - LR) = 0$ and trace $(ade)^2 \neq 0$ we obtain $b - a = 0$ which proves the lemma.

THEOREM 3.16. *Let A be a power associative algebra which*

satisfies condition H. If A contains an idempotent e which is not in the center of A, then A or the complexification of A is quasi-equivalent to an alternative algebra.

Proof. From Lemma 3.15 we have $(x, y, x) = 0$ and consequently $(x, y, x)^0 = 0$ for any u, v used to define A^0. Thus to make A^0 alternative it suffices to choose u and v to make $(x, x, y)^0 = 0$. This uses again the identity $(x, x, y)^0 + (x, y, x)^0 + (y, x, x)^0 = 0$, since A^0 is also power associative.

We do this as follows. Since $(x, x, y)^0 = (p + uv)[x, [x, y]]$ with $u + v = 1$ it suffices to choose u so that

$$0 = p + u(1 - u) = p + u - u^2 .$$

From the lemma, $a = b \neq 0$ and therefore $p = (3/2)a - 1/4$. Thus the discriminant of the quadratic equation is $6a$. So if a is positive we may choose $u = 1/2 \pm (1/2)\sqrt{6a}$ to obtain the conclusion.

If a is negative we use the complexification $A_C = C \times A$ of A. Thus A_C can be regarded as formal linear combinations $x + iy$ for $x, y \in A$ and $i = \sqrt{-1}$. Next since A is power associative, so is A_C. From identities (3.12), (3.13), (3.14) we replace x by $x + z$ and simplify to obtain the identities

(3.17) $(x, z, y) + (z, x, y) = p[x, [z, y]] + p[z, [x, y]]$

(3.18) $(x, y, z) + (z, y, x) = q[x, [z, y]] + q[z, [x, y]]$

(3.19) $(y, x, z) + (y, z, x) = r[x, [z, y]] + r[z, [x, y]]$

where p, q, r are as before. Using these, a straightforwad calculation shows A_C also satisfies the identities (3.12), (3.13), (3.14) which define A. Thus for A_C we also have $(x, y, x)^0 = 0$ in A_C^0. By choosing $u = 1/2 \pm (1/2)\sqrt{6a}$ in C we obtain $(x, x, y)^0 = 0$ in A_C^0 so that $(A_C^0, +, \circ)$ is alternative.

As previously noted when A is alternative, a local loop induced by the multiplication in A satisfies the Campbell-Hausdorff Theorem. The above theorem shows that if a Campbell-Hausdorff Theorem holds —or actually just the dependency of $V^{(3)}$ on $V^{(2)}$, then the idempotent condition implies A is essentially alternative. We now replace the idempotent condition with that of semi-simplicity to obtain additional results.

LEMMA 3.20. *Let I be an ideal of $(A, +, \cdot)$. Then $I^0 = I$ (as sets) is an ideal of $(A, +, \circ)$. Conversely, if I^0 is an ideal of $(A, +, \circ)$ then I is an ideal of $(A, +, \cdot)$.*

Proof. Let $x \in I$ and $y \in A$, then since $xy, yx \in I$ we see that from the definition of $x \circ y$ that $x \circ y$ and $y \circ x$ are in I. Thus I is also an ideal of A^0.

Conversely, if I^0 is an ideal of A^0, then since quasi-equivalence is symmetric, I is an ideal of A.

Using the previous notation $(x, y, z)^0$ and $[x, y]^0 = x \circ y - y \circ x$ in A^0 we note

$$[x, y]^0 = (u - v)[x, y] \qquad \text{and} \qquad [x, [x, y]^0]^0 = (u - v)^2[x, [x, y]] .$$

Thus let A be a power associative algebra satisfying condition H. Then A satisfies equations (3.12), (3.13), (3.14) and, with p, q, r as before, we see A^0 satisfies similar identities

$$(x, x, y)^0 = \frac{pu - rv + uv}{(u - v)^2}[x, [x, y]^0]^0$$

$$(x, y, x)^0 = \frac{q}{u - v}[x, [x, y]^0]^0$$

$$(y, x, x)^0 = \frac{(ru - pv - uv)}{(u - v)^2}[x, [x, y]^0]^0$$

where $u + v = 1$, $u - v \neq 0$. We now choose u, v such that

$$0 = ru - pv - uv$$
$$= u^2 + (r + p - 1)u - p .$$

Thus if $2p - 2r + 1 \neq 0$, then $u \neq 1/2$ and for $v = 1 - u \neq u$ we see A^0 satisfies

(3.21) $(y, x, x)^0 = 0$

(3.22) $(x, x, y)^0 = \lambda[x, [x, y]^0]^0$

where $\lambda = (pu - rv + uv)/(u - v)^2 = (p + r)/(2u - 1)$ using $u + v = 1$, $u \neq 1/2$ and $ru - pu - uv = 0$.

Algebras satisfying these identities were considered in [4] from which we use the following results. Let M be the subspace of A spanned by the associators of the form (x, x, y) and (x, y, x) for all $x, y \in A$. Then M is an ideal of A such that $M^2 = 0$. Consequently if A is simple or, more generally, semi-simple (i.e., a direct sum of simple ideals), then $M = 0$ and therefore A is alternative. Using the fact that our algebra A^0 and its complexification $(A_c)^0$ satisfies these identities (note the proof of Theorem 3.16) we use Lemma 3.20 to obtain the following result.

THEOREM 3.23. *Let A be a power associative over the real numbers such that A or A_c is semi-simple. If the algebra A^0 satisfies*

the identities (3.21) *and* (3.22) *above, then* A^0 *is a semi-simple alternative algebra.*

REMARKS.   Using Theorems 3.16 and 3.20 we can obtain further information about the original algebra A depending on the two cases $\lambda \neq 0$ or $\lambda = 0$ for λ given in equation (3.22).

Case $\lambda \neq 0$ implies from [4] that A^0 is a direct sum of associative, commutative integral domains.   Since A^0 is finite dimensional, this implies A^0 is a direct sum of fields isomorphic to the reals or complex numbers [14].   Since A^0 is commutative with $x \circ y = uxy + vyx$, we see $xy = x \circ y$ and $(x, y, z) = (x, y, z)^0 = 0$.   Consequently A is commutative and associative.   Thus A is a direct sum of fields isomorphic to the real or complex numbers and $V(x, y) = x + y$.

Case $\lambda = 0$ implies

$$0 = \lambda = (p + r)/2u - 1$$

using the formula following equation (3.22).   Thus $p = -r$ and from the definition of p and r (preceding Lemma 3.15) we see $a = b$.   Thus using equations (3.12), (3.13), (3.14) the original power associative algebra A satisfies

$$(x, y, x) = 0 \quad \text{and} \quad (x, x, y) = \frac{1}{2}[x, [x, y]]$$

where $p = (3/2)a - 1/4$ and $u^2 - u - p = 0$.   (We can assume $p \neq 0$ using Corollary 3.11.)   Algebras satisfying these identities are studied in [7] and they need not be alternative.

From Theorem 3.23 the algebra A^0 is alternative.   Consequently for the induced local loop (R^n, D, V^0) we have V^0 given by $E^0(x) \circ E^0(y) = E^0(V^0(x, y))$ where E^0 is the exponential map in A^0. But since the powers in A and A^0 are equal, $E(x) = E^0(x)$ and therefore

$$E(V^0(x, y)) = uE(x)E(y) + vE(y(Ex)$$
$$= uE(V(x, y)) + vE(V(y, x)) \ .$$

From this we obtain a formula for the original $V(x, y)$ in terms of $V^0(x, y)$ and consequently in terms of the Campbell-Hausdorff formula by $E(V(x, y)) = (u/(u - v))E(V^0(x, y)) + (v/(v - u))E(V^0(y, x))$.

In the case that the original semi-simple algebra is alternative, $A = A_1 \oplus \cdots \oplus A_n$ as a direct sum of simple associative algebras or simple 8-dimensional Cayley algebras.   If some A_i is the 8-dimensional Cayley numbers, then the invertible elements of A_i form a Moufang loop and consequently induces a local loop (D_i, V_i).   The anti-commu-

tative algebra induced by $V_i^2(x, y)^2$ is the 8-dimensional Malcev algebra A_i^- (i.e., the vector space A_i with multiplication $[x_i, y_i]$). If R denotes the real numbers, then the algebra A_i^- has the one-dimensional center $R1$. The simple 7-dimensional anti-commutative algebra $A_i^-/R1$ is the "tangent algebra" to the 7-sphere S^7, in A_i consisting of all those vectors of norm one [10].

We now summarize the general results as follows.

THEOREM 3.24. *Let A be a power associative algebra with 1 so that A or A_C is semi-simple. Let (R^n, D, V) be the canonical coordinate representation of $(A, A, \cdot)$ as before. Then hypothesis H holds if and only if A is quasi-equivalent to a semi-simple alternative algebra.*

Proof. If hypothesis H holds for (R^n, D, V) then A is quasi-equivalent to an algebra A^0 satisfying equations (3.21) and (3.22). Thus by Theorem 3.23, we obtain the desired conclusion.

Conversely, if A is quasi-equivalent to an alternative algebra A^0, then using the equations preceding Lemma 3.15 we obtain

$$0 = (x, y, x)^0 = (u - v)(x, y, x)$$

so that $(x, y, x) = 0$. Using this and A is power associative we have

$$(x, x, y) + (y, x, x) = 0 .$$

Consequently,

$$
\begin{aligned}
0 &= (x, x, y)^0 \\
&= u(x, x, y) - v(y, x, x) + uv[x, [x, y]] \\
&= (u + v)(x, x, y) + uv[x, [x, y]] \\
&= (x, x, y) + uv[x, [x, y]] .
\end{aligned}
$$

Thus from equation (3.9) and the above formulas we obtain

$$
\begin{aligned}
V^3(x, 0)^2(0, y) &= \frac{4}{3}(x, x, y) + \frac{2}{3}(y, x, x) + \frac{1}{6}[x, [x, y]] \\
&= \frac{2}{3}(x, x, y) + \frac{1}{6}[x, [x, y]] \\
&= \left(\frac{1}{6} - \frac{2}{3}uv\right)[x, [x, y]] \\
&= a[x, [x, y]]
\end{aligned}
$$

where $3a = 1/6 - (2/3)uv$. Similarly

$$V^3(x, 0)(0, y)^2 = b[y, [y, x]]$$

so that hypothesis H is satisfied.

We now investigate conditions on a power associative local loop so that it satisfies "hypothesis H". Thus we have seen that if the canonical coordinate multiplication function V satisfies hypothesis H and if the local loop multiplication is induced from an algebra A, then A is essentially an alternative algebra. Next note that an alternative division algebra A also satisfies the inverse property identities

$$x^{-1}(xy) = y = (yx)x^{-1}$$

for all nonzero x, y in A; see [1, 2, 8]. We shall make a similar assumption on a local power associative loop and observe that hypothesis H is satisfied.

Thus let (R^n, D, V) be a local power associative loop in canonical coordinates. Then using $V(sx, tx) = (s + t)x$ we see that the local inverse of x is $-x$ and the inverse property identities become

(3.25) $$V(-x, V(x, y)) = y = V(V(y, x), -x) .$$

We shall assume this holds for the local loop and show this implies hypothesis H as stated in Theorem 3.31 and Corollary 3.32.

As before let

$$V(x, y) = x + y + a(x, y) + \sum_{k=3}^{\infty} \frac{V^k(x, y)^k}{k!}$$

where we have seen $a(x, y) = (1/2)V^2(x, y)^2$ satisfies $a(x, y) = -a(y, x)$ and in general for s, t in R and $k \geqq 2$,

$$V^k(sx, tx)^k = 0 .$$

This implies for $k = 3$ that

$$0 = 3V^3[(sx, 0)^2, (0, tx)] + 3V^3[(sx, 0), (0, tx)^2]$$
$$= 3s^2tV^3[(x, 0)^2, (0, x)] + 3st^2V^3[(x, 0), (0, x)^2]$$

which implies

(3.26) $$V^3[(x, 0)^2, (0, x)] = 0 \quad \text{and} \quad V^3[(x, 0), (0, x)^2] = 0 .$$

Next we expand equation (3.25) into its Taylor's series up to terms of order 3 to obtain

$$y = V(x, V(-x, y))$$
$$= x + V(-x, y) + a(x, V(-x, y)) + \frac{V^3(x, V(-x, y))^3 + \cdots}{3!}$$

$$= x + \left(-x + y + a(-x, y) + \frac{V^3(-x, y)^3 + \cdots}{3!}\right)$$

$$+ a(x, -x + y + a(-x, y) + \cdots) + \frac{V^3(x, -x + y)^3 + \cdots}{3!}$$

$$= y + \frac{V^3(-x, y)^3}{3!} + \frac{V^3(x, -x + y)^3}{3!} - a(x, a(x, y)) \cdots .$$

This implies

$$a(x, a(x, y)) = \frac{V^3(-x, y)^3}{3!} + \frac{V^3(x, -x + y)^3}{3!}$$

and using multi-linearity to expand the right side of this equation, we obtain

$$3! \, a(x, a(x, y)) = 3V^3[(-x, 0)^2, (0, y)] + 3V^3[(-x, 0), (0, y)^2]$$
$$+ 3V^3[(x, 0)^2, (0, -x + y)] + 3V^3[(x, 0), (0, -x + y)^2]$$
$$= 3V^3[(x, 0)^2, (0, y)] - 3V^3[(x, 0), (0, y)^2]$$
$$- 3V^3[(x, 0)^2, (0, x)] + 3V^3[(x, 0)^2, (0, y)]$$
$$+ 3V^3[(x, 0), (0, -x + y), (0, -x + y)]$$
$$= 6V^3[(x, 0)^2, (0, y)] - 6V^3[(x, 0), (0, x), (0, y)]$$

using equations (3.26).   Therefore

$$(3.27) \qquad a(x, a(x, y)) = V^3[(x, 0)^2, (0, y)] - V^3[(x, 0), (0, x), (0, y)] \, .$$

Similarly expanding $y = V(V(y, -x), x)$ we obtain

$$(3.28) \qquad a(a(y, x), x) = V^3[(y, 0), (0, x)^2] - V^3[(y, 0), (x, 0), (0, x)] \, .$$

Next in (3.26) we replace x by $x + y$ and use multi-linearity to obtain

$$0 = V^3[(x + y, 0)^2, (0, x + y)]$$
$$= V^3[(x, 0)^2, (0, x)] + V^3[(x, 0)^2, (0, y)]$$
$$+ 2V^3[(x, 0), (y, 0), (0, x)] + 2V^3[(x, 0), (y, 0), (0, y)]$$
$$+ V^3[(y, 0)^2, (0, x)] + V^3[(y, 0)^2, (0, y)] \, .$$

Now using (3.26) and considering terms of the same degree in x and y, we obtain

$$(3.29) \qquad V^3[(x, 0)^2, (0, y)] + 2V^3[(y, 0), (x, 0), (0, x)] = 0 \, .$$

Using the other equation in (3.26) we similarly obtain

$$(3.30) \qquad V^3[(y, 0), (0, x)^2] + 2V^3[(x, 0), (0, x), (0, y)] = 0 \, .$$

Substituting (3.29) into (3.28) and (3.30) into (3.27) and using anti-commutativity, we obtain

$$a(x, a(x, y)) = V^3[(y, 0), (0, x)^2] + \frac{1}{2}V^3[(x, 0)^2, (0, y)] \qquad \text{and}$$

$$a(x, a(x, y)) = \frac{1}{2}V^3[(y, 0), (0, x)^2] + V^3[(x, 0)^2, (0, y)] \ .$$

These can be regarded as a system of equations in the V^3 and have solution

$$V^3[(x, 0)^2, (0, y)] = \frac{2}{3}a(x, a(x, y)) \qquad \text{and}$$

$$V^3[(y, 0), (0, x)^2] = \frac{2}{3}a(x, a(x, y)) \ .$$

Denoting $a(x, y) = (1/2)V^2(x, y)^2 = (1/2)[x, y]$ this gives

$$V^3(x, 0)^2(0, y) = V^3(y, 0)(0, x)^2 = \frac{1}{6}[x, [x, y]] \ .$$

With the above equations, we obtain from Corollary 3.11 and the definition of "hypothesis H" the following result.

THEOREM 3.31.   *Let (R^n, D, V) be a local power associative loop given in canonical coordinates.   Suppose the inverse properties (3.25) are satisfied.   Then hypothesis H is satisfied.   In case the local loop (R^n, D, V) is induced by a power associative algebra A, then A is alternative.*

Combining previous results with those in [1, 2] gives the following.

COROLLARY 3.32.   *Let the local loop (R^n, D, V) given in canonical coordinates be induced by a power associative algebra A.   Then A is alternative if and only if the Campbell-Hausdorff formula holds for V if and only if (R^n, D, V) satisfies the inverse property identities.   Thus in this case the first two derivatives of V at $(0, 0)$ determine the Taylor's series for V and the corresponding local loop is a local Moufang loop.*

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