

Pacific Journal of Mathematics

SOME NEW RESIDUACITY CRITERIA

RICHARD HOWARD HUDSON AND KENNETH S. WILLIAMS

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Let e and k be integers ≥ 2 with e odd and k even. Set $2l = \text{L. C. M.}(e, k)$ and let p be a prime with $p \equiv 1 \pmod{2l}$ having g as a primitive root. It is shown that the index of e (with respect to g) modulo k can be computed in terms of the cyclotomic numbers of order l . By applying this result with $e = 3, k = 4$; $e = 5, k = 4$; $e = 3, k = 8$; new criteria are obtained for 3 and 5 to be fourth powers \pmod{p} and for 3 to be an eighth power \pmod{p} .

1. Introduction. Let e and k be integers greater than or equal to 2 with e odd and k even. Let p be a prime congruent to 1 modulo $2l$, where $2l = \text{L.C.M.}(e, k)$. Let g be a fixed primitive root \pmod{p} . If a is an integer not divisible by p , the index of a with respect to g is denoted by $\text{ind}(a)$ and is the least nonnegative integer b such that $a \equiv g^b \pmod{p}$. For $0 \leq h, k \leq l - 1$, the cyclotomic number $(h, k)_l$ of order l is the number of integers n ($1 \leq n \leq p - 2$) such that $\text{ind}(n) \equiv h \pmod{l}$, $\text{ind}(n + 1) \equiv k \pmod{l}$.

Using an idea due to Muskat [4: 257-258], we prove the following congruence for the index of e modulo k .

THEOREM 1.

$$\begin{aligned} \text{ind}(e) \equiv & 2 \sum_{i=1}^{k/2-1} i \sum_{j=1}^{(e-1)/2} \sum_{r=0}^{2l/k-1} \sum_{s=0}^{l/e-1} \left(i + r \frac{k}{2}, j + se \right)_l \\ & + \frac{(p-1)(e-1)^2}{8e} \pmod{k}. \end{aligned}$$

Applying Theorem 1 with $e = 3, k = 4$, we obtain the following criterion for 3 to be a fourth power \pmod{p} .

THEOREM 2. Let $p \equiv 1 \pmod{12}$ be a prime, so that there are integers x and y satisfying

$$(1.1) \quad p = x^2 + 3y^2, \quad x \equiv 1 \pmod{3}.$$

Then 3 is a fourth power \pmod{p} if and only if $x \equiv 1 \pmod{4}$.

This criterion should be compared with the classical result: 3 is a fourth power \pmod{p} if and only if

$$\begin{cases} b \equiv 0 \pmod{3}, & \text{if } p \equiv 1 \pmod{24}, \\ a \equiv 0 \pmod{3}, & \text{if } p \equiv 13 \pmod{24}, \end{cases}$$

where

$$p = a^2 + b^2, \quad a \equiv 1 \pmod{4}, \quad b \equiv 0 \pmod{2},$$

see for example [2: p. 24].

Next taking $e = 5$, $k = 4$, in Theorem 1 we obtain the following new criterion for 5 to be a fourth power (mod p).

THEOREM 3. *Let $p \equiv 1 \pmod{20}$ be a prime, so that there are integers x , u , v , and w satisfying*

$$(1.2) \quad 16p = x^2 + 50u^2 + 50v^2 + 125w^2, \quad xw = v^2 - 4uv - u^2,$$

and

$$(1.3) \quad x \equiv 1 \pmod{5}.$$

Then 5 is a fourth power (mod p) if and only if

$$\begin{cases} x \equiv 4 \pmod{8}, & \text{if } x \equiv 0 \pmod{2}, \\ x \equiv \pm 3w \pmod{8}, & \text{if } x \equiv 1 \pmod{2}. \end{cases}$$

This criterion should be compared with the well-known result (see for example [2: p. 24]):

5 is a fourth power (mod p) if and only if

$$b \equiv 0 \pmod{5}, \quad \text{where } p = a^2 + b^2, \quad a \equiv 1 \pmod{4}, \quad b \equiv 0 \pmod{2}.$$

Finally, applying Theorem 1 with $e = 3$, $k = 8$, we obtain the following new criterion for 3 to be an eighth power (mod p).

THEOREM 4. *Let $p \equiv 1 \pmod{24}$ be a prime so that there are integers a , b , x and y satisfying*

$$(1.4) \quad p = a^2 + b^2 = x^2 + 3y^2,$$

and

$$(1.5) \quad a \equiv 1 \pmod{4}, \quad b \equiv 0 \pmod{4}, \quad x \equiv 1 \pmod{6}, \quad y \equiv 0 \pmod{2}.$$

Assume 3 is a fourth power (mod p), so that

$$b \equiv 0 \pmod{3}, \quad x \equiv 1 \pmod{4}.$$

Then 3 is an eighth power (mod p) if and only if

$$a \equiv 1 \pmod{3}, \quad y \equiv 0 \pmod{8},$$

or

$$a \equiv -1 \pmod{3}, \quad y \equiv 4 \pmod{8}.$$

This criterion should be compared to that of von Lienen [3: p. 114], namely, if 3 is a fourth power (mod p) then 3 is an eighth power (mod p) if and only if

$$\begin{cases} a \equiv c \pmod{3}, & \text{if } p \equiv 1 \pmod{48}, \\ a \equiv -c \pmod{3}, & \text{if } p \equiv 25 \pmod{48}, \end{cases}$$

where

$$p = a^2 + b^2 = c^2 + 2d^2$$

and

$$a \equiv 1 \pmod{4}, \quad b \equiv 0 \pmod{4}, \quad c \equiv 1 \pmod{4}, \quad d \equiv 0 \pmod{2}.$$

Combining these results, we see that if $(3/p)_4 = +1$ (equivalently $b \equiv 0 \pmod{3}$ or $x \equiv 1 \pmod{4}$), we have

$$\begin{cases} y \equiv 0 \pmod{8} \iff c \equiv 1 \pmod{3}, & \text{if } p \equiv 1 \pmod{48}, \\ y \equiv 0 \pmod{8} \iff c \equiv -1 \pmod{3}, & \text{if } p \equiv 25 \pmod{48}. \end{cases}$$

2. Proof of Theorem 1. The roots of the congruence

$$(2.1) \quad \frac{x^e - 1}{x - 1} \equiv 0 \pmod{p}$$

are

$$x \equiv g^{jf} \pmod{p}, \quad j = 1, 2, \dots, e - 1,$$

where $p - 1 = ef$, so that

$$(2.2) \quad x^{e-1} + x^{e-2} + \dots + x + 1 \equiv \prod_{j=1}^{e-1} (x - g^{jf}) \pmod{p}.$$

Taking $x = 1$ in (2.2), we obtain

$$(2.3) \quad e \equiv \prod_{j=1}^{e-1} (1 - g^{jf}) \pmod{p},$$

and so

$$(2.4) \quad \text{ind}(e) \equiv \sum_{j=1}^{e-1} \text{ind}(1 - g^{jf}) \pmod{p - 1}.$$

Next

$$\begin{aligned} & \sum_{j=(e+1)/2}^{e-1} \text{ind}(1 - g^{jf}) \\ &= \sum_{j=1}^{(e-1)/2} \text{ind}(1 - g^{(e-j)f}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{(e-1)/2} \text{ind}(1 - g^{-jf}) \\
&\equiv \sum_{j=1}^{(e-1)/2} \text{ind}(1 - g^{jf}) + \sum_{j=1}^{(e-1)/2} \text{ind}(-g^{-jf}) \pmod{p-1} \\
&\equiv \sum_{j=1}^{(e-1)/2} \text{ind}(1 - g^{jf}) + \sum_{j=1}^{(e-1)/2} \left(\frac{p-1}{2} - jf \right) \pmod{p-1},
\end{aligned}$$

so

$$(2.5) \quad \text{ind}(e) \equiv 2 \sum_{j=1}^{(e-1)/2} \text{ind}(1 - g^{jf}) + \frac{(p-1)(e-1)^2}{8e} \pmod{p-1}.$$

Next the roots of

$$x^f - g^{jf} \equiv 0 \pmod{p}$$

are

$$x \equiv g^{ei+j} \pmod{p} \quad (i = 1, 2, \dots, f),$$

so

$$(2.6) \quad x^f - g^{jf} \equiv \prod_{i=1}^f (x - g^{ei+j}) \pmod{p}.$$

Taking $x = 1$ in (2.6), we obtain

$$1 - g^{jf} \equiv \prod_{i=1}^f (1 - g^{ei+j}) \pmod{p},$$

so

$$(2.7) \quad \text{ind}(1 - g^{jf}) \equiv \sum_{i=1}^f \text{ind}(1 - g^{ei+j}) \pmod{p-1}.$$

Further, working modulo $k/2$, we have

$$\begin{aligned}
&\sum_{i=1}^f \text{ind}(1 - g^{ei+j}) \\
&= \sum_{\substack{n=2 \\ \text{ind}(n) \equiv j \pmod{e}}}^{p-1} \text{ind}(1 - n) \\
&\equiv \sum_{\substack{n=2 \\ \text{ind}(n) \equiv j \pmod{e}}}^{p-1} \text{ind}(n-1) + \sum_{\substack{n=2 \\ \text{ind}(n) \equiv j \pmod{e}}}^{p-1} \text{ind}(-1) \\
&\equiv \sum_{\substack{n=1 \\ \text{ind}(n+1) \equiv j \pmod{e}}}^{p-2} \text{ind}(n) + \frac{p-1}{2} \sum_{\substack{n=1 \\ \text{ind}(n) \equiv j \pmod{e}}}^{p-1} 1 \\
&\equiv \sum_{\substack{i=0 \\ \text{ind}(n) \equiv i \pmod{k/2}}}^{k/2-1} \sum_{\substack{n=1 \\ \text{ind}(n+1) \equiv j \pmod{e}}}^{p-2} i,
\end{aligned}$$

that is

$$(2.8) \quad \sum_{i=1}^f \text{ind}(1 - g^{e_i+j}) \equiv \sum_{i=1}^{k/2-1} i \sum_{r=0}^{2l/k-1} \sum_{s=0}^{l/e-1} (i + rk/2, j + se)_i .$$

The result now follows from (2.5), (2.7) and (2.8).

3. Proof of Theorem 2. Taking $e = 3, k = 4$, so that $l = 6$, in Theorem 1, we obtain, for $p \equiv 1 \pmod{12}$,

$$(3.1) \quad \text{ind}(3) \equiv 2 \sum_{r=0}^2 \sum_{s=0}^1 (1 + 2r, 1 + 3s)_6 + \frac{p-1}{6} \pmod{4} .$$

Defining x and y , as in [6: p. 68], by

$$x = 6(0, 3)_6 - 6(1, 2)_6 + 1$$

and

$$y = (0, 1)_6 - (0, 5)_6 - (1, 3)_6 + (1, 4)_6 ,$$

so that x and y satisfy (1.1), from the tables for the cyclotomic numbers of order 6, we obtain

$$\sum_{r=0}^2 \sum_{s=0}^1 (1 + 2r, 1 + 3s)_6 = \frac{1}{6}(p - x - 3y) .$$

Hence, from (3.1), we obtain

$$\text{ind}(3) \equiv \frac{1}{3}(p - x) - y + \frac{p-1}{6} \pmod{4} .$$

Now

$$y \equiv \begin{cases} 0 \pmod{4}, & \text{if } p \equiv 1 \pmod{24}, \\ 2 \pmod{4}, & \text{if } p \equiv 13 \pmod{24}, \end{cases}$$

that is

$$y \equiv \frac{1}{6}(p - 1) \pmod{4} ,$$

giving

$$\text{ind}(3) \equiv \frac{1}{3}(p - x) \equiv \frac{1}{3}(1 - x) \pmod{4} ,$$

which completes the proof of Theorem 2.

4. Proof of Theorem 3. Taking $e = 5, k = 4$, so that $l = 10$, in Theorem 1, we obtain for $p \equiv 1 \pmod{20}$,

$$(4.1) \quad \text{ind } (5) \equiv 2 \sum_{j=1}^2 \sum_{r=0}^4 \sum_{s=0}^1 (1 + 2r, j + 5s)_{10} + \frac{2}{5}(p-1) \pmod{4}.$$

Define m by $2 \equiv g^m \pmod{p}$. Replacing g by an appropriate power of g , we may suppose that $m \equiv 0$ or $1 \pmod{5}$. Next we define x, u, v, w by

$$\begin{aligned} 3x &= -p + 14 + 25(0, 0)_5, \\ u &= (0, 2)_5 - (0, 3)_5, \\ v &= (0, 1)_5 - (0, 4)_5, \\ w &= (1, 3)_5 - (1, 2)_5, \end{aligned}$$

so that x, u, v, w is a solution of (1.2) satisfying (1.3) (see for example [5: p. 100]). From the tables of Whiteman [5: pp. 107-109] for the cyclotomic numbers of order 10, we obtain in the case $m \equiv 0 \pmod{5}$, that is, 2 is a fifth power \pmod{p} or equivalently, $x \equiv 0 \pmod{2}$ [1: p. 13]:

$$\begin{aligned} \sum_{j=1}^2 \sum_{r=0}^4 \sum_{s=0}^1 (1 + 2r, j + 5s)_{10} \\ = \frac{1}{20} \{4p + x - 15u + 15v - 30w\}, \end{aligned}$$

so

$$\begin{aligned} \text{ind } (5) &\equiv \frac{1}{10} \{4p + x - 15u + 15v - 30w\} \pmod{4} \\ &\equiv \frac{1}{10}(x + 4) - \frac{3}{2}(u - v) + w \pmod{4}. \end{aligned}$$

Emma Lehmer [1: p. 13] has shown in this case that

$$x \equiv u \equiv v \equiv w \equiv 0 \pmod{4}, \quad u \equiv v \pmod{8},$$

so that

$$\text{ind } (5) \equiv \frac{1}{10}(x + 4) \equiv \frac{x}{2} + 2 \pmod{4},$$

completing the proof of Theorem 3 in this case.

When $m \equiv 1 \pmod{5}$, 2 is not a fifth power \pmod{p} and $x \equiv 1 \pmod{2}$. From the tables of Whiteman [5: pp. 107-109], in this case, we obtain

$$\begin{aligned} \sum_{j=1}^2 \sum_{r=0}^4 \sum_{s=0}^1 (1 + 2r, j + 5s)_{10} \\ = \frac{1}{40} \{8p - 3x + 10u + 20v - 25w\}, \end{aligned}$$

so that

$$4 \operatorname{ind}(5) \equiv 8p - 3x + 10u + 20v - 25 \pmod{16},$$

which shows that $w \equiv 1 \pmod{2}$.

Since

$$400(0, 2)_{10} = 4p - 36 + 17x + 50u - 25w,$$

we have (as $x \equiv w \equiv 1 \pmod{2}$)

$$10u \equiv 3x + 5w \pmod{16},$$

so that

$$\operatorname{ind}(5) \equiv v + w \pmod{4}.$$

As

$$200(0, 9)_{10} = 2p - 18 - 4x + 25u - 25v + 25w$$

and

$$200(1, 2)_{10} = 2p + 2 + x + 25u + 25v - 50w$$

we have

$$\begin{cases} u - v \equiv 4 - w \pmod{8}, \\ u + v \equiv 4 + 2w - x \pmod{8}, \end{cases}$$

so

$$u \equiv \frac{1}{2}(w - x) \pmod{4}, \quad v \equiv \frac{1}{2}(3w - x) \pmod{4}.$$

Hence we have

$$(4.2) \quad \operatorname{ind}(5) \equiv \frac{1}{2}(5w - x) \pmod{4}.$$

Since all solutions of (1.2) satisfying (1.3) are given by (see for example [1: p. 13])

$$(x, u, v, w), \quad (x, v, -u, -w), \quad (x, -u, -v, w), \quad (x, -v, u, -w),$$

(4.2) gives

$$\operatorname{ind}(5) \equiv 0 \pmod{4} \iff x \equiv \pm 3w \pmod{8},$$

and

$$\operatorname{ind}(5) \equiv 2 \pmod{4} \iff x \equiv \pm w \pmod{8},$$

which completes the proof of Theorem 3.

5. **Proof of Theorem 4.** Taking $e = 3$, $k = 8$ so that $l = 12$, in Theorem 1, we obtain, for $p \equiv 1 \pmod{24}$,

$$(5.1) \quad \text{ind}(3) \equiv 2 \sum_{i=1}^3 i \sum_{r=0}^2 \sum_{s=0}^3 (i + 4r, 1 + 3s)_{12} + \frac{1}{6}(p-1) \pmod{8}.$$

Following Whiteman [6: p. 64], we define m and m' by $2 \equiv g^m \pmod{p}$ and $3 \equiv g^{m'} \pmod{p}$ respectively. As $p \equiv 1 \pmod{8}$ we have $m \equiv 0 \pmod{2}$. Replacing g by an appropriate power of g we may suppose that $m \equiv 0$ or $2 \pmod{3}$, so that $m \equiv 0$ or $2 \pmod{6}$. Further, as we are assuming 3 is a fourth power \pmod{p} , we have $m' \equiv 0 \pmod{4}$. Next we define x and y (as in [6: p. 68]) by

$$\begin{aligned} x &= 6(0, 3)_6 - 6(1, 2)_6 + 1, \\ y &= (0, 1)_6 - (0, 5)_6 - (1, 3)_6 + (1, 4)_6, \end{aligned}$$

and a and b by equations (4.4) and (4.5) in [6] (a replaces Whiteman's x , b replaces Whiteman's $2y$). Then x, y, a, b satisfy (1.4) and (1.5). Whiteman [6: pp. 69-73] gives the cyclotomic numbers of order 12 in terms of x, y, a and b , as defined above. When $m \equiv 0 \pmod{6}$, we must use Tables 9 and 10 of [6] and, when $m \equiv 2 \pmod{6}$, we must use Tables 3 and 4. By considering the cyclotomic numbers $(3, 6)_{12}$ in Table 9; $(2, 4)_{12}$ in Table 10; $(1, 2)_{12}$ in Table 3; $(2, 8)_{12}$ in Table 4; it is easy to check that Whiteman's quantity $c = \pm 1$ (see [6: pp. 64-65]) satisfies

$$(5.2) \quad \begin{cases} c = +1 \iff a \equiv 1 \pmod{3}, \\ c = -1 \iff a \equiv 2 \pmod{3}. \end{cases}$$

We remark that $a \not\equiv 0 \pmod{3}$ as 3 is assumed to be a fourth power \pmod{p} .

Next we set

$$\sum_i = \sum_{r=0}^2 \sum_{s=0}^3 (i + 4r, 1 + 3s)_{12} \quad (i = 1, 2, 3),$$

so that

$$(5.3) \quad \text{ind}(3) \equiv 2 \left(\sum_1 + 2 \sum_2 + 3 \sum_3 \right) + \frac{1}{6}(p-1) \pmod{8}.$$

From Whiteman's tables, we obtain

$$\begin{aligned} 12 \sum_1 &= \begin{cases} p - 2b - x - 3y, & \text{if } a \equiv 1 \pmod{3}, \\ p + 2b - x - 3y, & \text{if } a \equiv -1 \pmod{3}, \end{cases} \\ 12 \sum_2 &= \begin{cases} p - 2a + x + 3y, & \text{if } a \equiv 1 \pmod{3}, \\ p + 2a + x + 3y, & \text{if } a \equiv -1 \pmod{3}, \end{cases} \end{aligned}$$

$$12 \sum_3 = \begin{cases} p + 2b - x - 3y, & \text{if } a \equiv 1 \pmod{3}, \\ p - 2b - x - 3y, & \text{if } a \equiv -1 \pmod{3}. \end{cases}$$

From (5.3) and (5.4) we obtain

$$(5.5) \quad \text{ind } (3) \equiv \begin{cases} 1 - \frac{1}{3}(2a - 2b + x) - y + \frac{1}{6}(p - 1) \pmod{8}, & \text{if } a \equiv 1 \pmod{3}, \\ 1 + \frac{1}{3}(2a - 2b - x) - y + \frac{1}{6}(p - 1) \pmod{8}, & \text{if } a \equiv -1 \pmod{3}. \end{cases}$$

Also, from Whiteman's tables, we have in every case,

$$p + 1 - 8a + 6x \equiv 0 \pmod{16},$$

so

$$\text{ind } (3) \equiv \begin{cases} 1 + 2a - 2b + \frac{p+1}{2} - 4a - y + \frac{1}{6}(p-1) \pmod{8}, & \text{if } a \equiv 1 \pmod{3}, \\ 1 - 2a + 2b + \frac{p+1}{2} - 4a - y + \frac{1}{6}(p-1) \pmod{8}, & \text{if } a \equiv -1 \pmod{3}, \\ -y \pmod{8}, & \text{if } a \equiv 1 \pmod{3}, \\ 4 - y \pmod{8}, & \text{if } a \equiv -1 \pmod{3}, \end{cases}$$

which completes the proof of Theorem 4.

REFERENCES

1. Emma Lehmer, *The quintic character of 2 and 3*, Duke Math. J., **18** (1951), 11-18.
2. ———, *Criteria for cubic and quartic residuacity*, Mathematika, **5** (1958), 20-29.
3. H. von Lienen, *Primzahlen als achte Potenzreste*, J. Reine Angew. Math., **266** (1974), 107-117.
4. J. B. Muskat, *On the solvability of $x^e \equiv e \pmod{p}$* , Pacific J. Math., **14** (1964), 257-260.
5. A. L. Whiteman, *The cyclotomic number of order ten*, Proceedings of the Symposia in Applied Mathematics **10**, pp. 95-111, Amer. Math. Soc., Providence, Rhode Island, 1960.
6. ———, *The cyclotomic numbers of order twelve*, Acta Arith., **6** (1960), 53-76.

Received October 26, 1979. Research supported by the Natural Sciences and Engineering Research Council Canada Grant No. A-7233.

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