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**THE RADIUS OF STARLIKENESS FOR A CLASS OF REGULAR
FUNCTIONS DEFINED BY AN INTEGRAL**

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Let $F(z)$, $f(z)$, and $g(z)$ be regular in the unit disc $E = \{z: |z| < 1\}$, be normalized by $F(0) = f(0) = g(0) = 0$ and $F'(0) = f'(0) = g'(0) = 1$, and satisfy the equation $z^{c-1}(c + 1)f(z) = [F(z)g(z)^c]'$, $c \geq 0$. This paper is concerned with studying relationships between the mapping properties of these functions. The principle result is the determination of the radius of β -starlikeness of $f(z)$ when $F(z)$ and $g(z)$ are restricted to certain classes of univalent starlike functions. Conversely, a lower bound for the radius of β -starlikeness of $F(z)$ is obtained when $f(z)$ and $g(z)$ satisfy similar conditions.

Problems of this nature were first studied by Libera [9], where he showed that if $f(z)$ is a convex, starlike, or close-to-convex univalent function and $F(z)$ is defined by

$$(1) \quad F(z) = \frac{2}{z} \int_0^z f(t) dt,$$

then $F(z)$ is also convex, starlike, or close-to-convex, respectively. Livingston then considered the converse of this problem and determined that if $F(z)$ satisfies one of these geometric conditions in E and $f(z) = (F(z) + zF'(z))/2$, then $f(z)$ satisfies the same condition in $\{z: |z| < 1/2\}$ [11]. Refinements of Livingston's results can be found in [1], [2], [10], [12], and [13], while results dealing with generalizations of (1) appear in [3], [4], [5], [6], [7], and [8]. Most recently, Lewandowski et al have shown that if $f(z)$ is starlike in E and $F(z)$ is the solution of

$$(2) \quad cF(z) + zF'(z) = (1 + c)f(z),$$

then $F(z)$ is starlike whenever $\operatorname{Re} c \geq 0$ [8].

Before proceeding any further, it will be convenient to introduce the following notation. Let $S^*(\alpha)$ denote the collection of functions $f(z)$ which are regular in E , are normalized by $f(0) = 0$ and $f'(0) = 1$, and satisfy $\operatorname{Re} [zf'(z)/f(z)] \geq \alpha$ for z in E . Such functions are said to be starlike of order α . Normally one only considers α in the interval $[0, 1)$, however, in order to relate the results presented here to earlier works, it is advantageous to allow $\alpha = 1$, with the understanding that $S^*(1)$ consists only of the function $f(z) = z$.

In this paper we continue the investigation of a generalization of (1) which was introduced by the first author in [7]. Let $\mathcal{F}_1(\alpha, \gamma, c)$ denote the family of functions $F(z)$ which satisfy

$$(3) \quad F(z) = \frac{c+1}{[g(z)]^c} \int_0^z t^{c-1} f(t) dt,$$

where $f(z)$ is in $S^*(\alpha)$, $g(z)$ is in $S^*(\gamma)$ and $c \geq 0$. Let $\mathcal{F}_2(\alpha, \gamma, c)$ denote the family of functions $f(z)$ which satisfy

$$(4) \quad (c+1)f(z) = c[g(z)/z]^{c-1}g'(z)F(z) + [g(z)/z]^c zF'(z)$$

for $F(z)$ in $S^*(\alpha)$, $g(z)$ in $S^*(\gamma)$ and $c \geq 0$. Theorem 1 provides a lower bound for the radius of β -starlikeness of $\mathcal{F}_1(\alpha, \gamma, c)$ and Theorem 3 gives the radius of β -starlikeness of $\mathcal{F}_2(\alpha, \gamma, c)$.

We begin by stating a slight generalization of the result obtained by Lewandowski et al mentioned above. Since our result follows directly from the techniques used in [8], the proof will be omitted.

LEMMA 1. *If $F(z)$ and $f(z)$ satisfy (2), $f(z)$ is in $S^*(\alpha)$ and $c \geq 0$, then $F(z)$ is in $S^*(\alpha)$.*

This lemma now enables us to determine a lower bound for the radius of β -starlikeness of $\mathcal{F}_1(\alpha, \gamma, c)$.

THEOREM 1. *If $F(z)$ is in $\mathcal{F}_1(\alpha, \gamma, c)$, then $F(z)$ is β -starlike for $|z| < \sigma = \sigma(\alpha, \beta, \gamma, c)$, where σ is the least positive root of the equation*

$$(5) \quad 1 - \beta - r[2(1 - \alpha) + 2c(1 - \gamma)] - r^2[2\alpha - 1 - \beta + 2c(1 - \gamma)] = 0.$$

Proof. If $h(z) = [(c+1)/z^c] \int_0^z t^{c-1} f(t) dt$ then $F(z) = [z/g(z)]^c h(z)$ and Lemma 1 implies $h(z)$ is in $S^*(\alpha)$. Differentiating logarithmically and applying the usual inequalities we obtain

$$\operatorname{Re} \left\{ \frac{zF'(z)}{F(z)} \right\} \geq \frac{1 + (2\alpha - 1)r}{1 + r} + \frac{2c(1 - \gamma)r}{1 - r}.$$

Thus $\operatorname{Re} \{zF'(z)/F(z)\} \geq \beta$ whenever $|z| < \sigma$ where σ is the least positive root of (5).

Before turning our attention to the principal result of this paper, we state without proof two lemmas which appear in [7] and are fundamental to what follows.

LEMMA 2. *If $\omega(z)$ is analytic and satisfies $|\omega(z)| \leq |z|$ in E and*

if $p(z) = (1 + D\omega(z))/(1 + B\omega(z))$, $-1 \leq D < B \leq 1$, then for $|z| = r < 1$ we have

$$\begin{aligned} & \operatorname{Re} \left\{ \frac{z\omega'(z)}{(1 + D\omega(z))(1 + B\omega(z))} \right\} \\ & \leq \frac{-1}{(B - D)^2} \left[\operatorname{Re} \left\{ \frac{D}{p(z)} + Bp(z) \right\} \right. \\ & \quad \left. - \frac{r^2 |Bp(z) - D|^2 - |p(z) - 1|^2}{(1 - r^2)|p(z)|} \right] + \frac{B + D}{(B - D)^2}. \end{aligned}$$

LEMMA 3. If $p(z)$ and $\omega(z)$ satisfy the conditions of Lemma 2, then for any $K \geq B$ we have on $|z| = r$

$$\begin{aligned} & \operatorname{Re} \left\{ Kp(z) + \frac{D}{p(z)} \right\} - \left[\frac{r^2 |Bp(z) - D|^2 - |1 - p(z)|^2}{(1 - r^2)|p(z)|} \right] \\ & \geq \begin{cases} P_1(r) & \text{for } R_0 \leq R_1 \\ P_2(r) & \text{for } R_0 \geq R_1 \end{cases} \end{aligned}$$

where

$$P_1(r) = P_1(r, K, B, D) = K \frac{1 + Dr}{1 + Br} + D \frac{1 + Br}{1 + Dr},$$

$$\begin{aligned} P_2(r) &= P_2(r, K, B, D) \\ &= \frac{2}{(1 - r^2)} [(1 + D)(1 + K - (B^2 + K + D(1 + K)))r^2 \\ & \quad + D(B^2 + K)r^4]^{1/2} - \frac{2(1 - BD r^2)}{1 - r^2}, \end{aligned}$$

$$R_0^2 = [(1 + D)(1 - Dr^2)] / [(1 + K) - (K + B^2)r^2],$$

and

$$R_1 = (1 + Dr)/(1 + Br).$$

The above estimates are sharp.

THEOREM 2.

$$\min_{f \in \mathcal{S}_2(\alpha, \gamma, c)} \min_{|z|=r} \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} = \begin{cases} Q_1(r), & R_0 \leq R_1 \\ Q_2(r), & R_0 \geq R_1 \end{cases}$$

where

$$(6) \quad Q_1(r) = \frac{1 + r(1 + 2D - K) + r^2 D(2 - K)}{(1 + r)(1 + Dr)},$$

$$\begin{aligned} (7) \quad Q_2(r) &= \frac{2}{1 - D} \left[\left\{ \frac{(1 + D)(1 + K)(1 - Dr^2)}{1 - r^2} \right\}^{1/2} - \frac{1 - Dr^2}{1 - r^2} \right] \\ & \quad - \frac{K - 1 + 2D}{1 - D}, \end{aligned}$$

$$(8) \quad R_0^2 = [(1 + D)(1 - Dr^2)]/[(1 + K)(1 - r^2)],$$

$$(9) \quad R_1 = (1 + Dr)/(1 + r),$$

$\delta = (\alpha + c\gamma)/(1 + c)$, $D = 2\delta - 1$, and $K = 1 + (c + 1)(1 - D)$.

Proof. Let $s(z) = z[F(z)/z]^{1/(c+1)}[g(z)/z]^{c/(c+1)}$ where in each multi-valued expression we choose the branch which has value 1 at $z = 0$. Combining this with (4) yields

$$(10) \quad f(z) = [s(z)/z]^c z s'(z).$$

Since

$$\frac{zs'(z)}{s(z)} = \frac{1}{(1 + c)} \left[\frac{zF'(z)}{F(z)} + c \frac{zg'(z)}{g(z)} \right],$$

$s(z)$ is in $S^*(\delta)$ for $\delta = (\alpha + c\gamma)/(1 + c)$, $p(z) = zs'(z)/s(z)$ is analytic in E , $p(0) = 1$ and $\text{Re}[p(z)] \geq \delta$, z in E . Consequently, there exists a function $\omega(z)$ analytic in E and satisfying $|\omega(z)| \leq |z|$, $z \in E$, such that

$$(11) \quad p(z) = \frac{1 + D\omega(z)}{1 + \omega(z)}, \quad D = 2\delta - 1.$$

Now differentiating (10) and making use of (11), we have

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= (c + 1)p(z) + \frac{zp'(z)}{p(z)} - c \\ &= (c + 1)p(z) + \frac{z\omega'(z)(D - 1)}{(1 + D\omega(z))(1 + \omega(z))} - c \end{aligned}$$

and Lemma 2 now yields

$$\begin{aligned} \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} &\geq \frac{1}{(1 - D)} \left[\text{Re} \left\{ Kp(z) + \frac{D}{p(z)} \right\} \right. \\ &\quad \left. - \frac{r^2|p(z) - D|^2 - |p(z) - 1|^2}{(1 - r^2)|p(z)|} \right] - c - \frac{1 + D}{1 - D}, \end{aligned}$$

where $B = 1$ and $K = 1 + (c + 1)(1 - D)$. An application of Lemma 3 now completes the proof. Sharpness follows directly from the sharpness of Lemma 3.

In [7] the radius of β -starlikeness of $\mathcal{F}_2(\alpha, \gamma, c)$ is determined in the case $c = 1$ and $\alpha + \gamma \leq 1$. The following result extends this to include all permissible values of α, γ and c .

THEOREM 3. *Let $r_* = r_*(\alpha, \gamma, c, \beta)$ be the radius of β -starlikeness of $\mathcal{F}_2(\alpha, \gamma, c)$. Let $D = 2\delta - 1$, $\delta = (\alpha + c\gamma)/(1 + c)$, $c \geq 0$, $0 \leq \alpha < 1$, and $0 \leq \gamma \leq 1$. For each fixed c in $[0, \infty)$, let $r(D)$ be the*

unique solution in $(0, 1]$ of the equation

$$(12) \quad (2 + c) - (4 - 2D - 2Dc + c)r - D(5 - D + 2c - Dc)r^2 + D(1 - D - Dc)r^3 = 0.$$

If $Q_1(r)$ and $Q_2(r)$ are defined by (6) and (7) and $\mu(D) = Q_1(r(D))$, then the equation $\mu(D) = 0$ has a unique solution D_0 in $(-1, 1)$. Furthermore, if D satisfies $D_0 < D < 1$ and $0 \leq \beta \leq \mu(D)$, then r_* is the unique root in $(0, 1)$ of the equation $Q_2(r) = \beta$. For all other values of D , r_* is the unique root in $(0, 1)$ of the equation $Q_1(r) = \beta$.

Proof. Let $I(r) = \min_{f \in \mathcal{F}_2(\alpha, \gamma, c)} \min_{|z|=r} \operatorname{Re} \{zf'(z)/f(z)\}$ and let R_0 and R_1 be defined by (8) and (9). A differentiation shows R_0 is a decreasing function of r and R_1 is an increasing function of r , hence the equation $R_0 = R_1$ has a unique solution $r(D, c)$ which is the unique root in $(0, 1]$ of (12). Thus

$$I(r) = \begin{cases} Q_1(r) & 0 \leq r < r(D, c) \\ Q_2(r) & r(D, c) \leq r < 1 \end{cases},$$

with the understanding that the second inequality holds vacuously when $r(D, c) = 1$. An examination of (12) shows this happens only when $D = -1$, in which case r_* is the solution of $Q_1(r) = \beta$. Since $\alpha < 1$ implies $D < 1$, we can now restrict our attention to $D \in (-1, 1)$.

It follows from the minimum principle and the compactness of $\mathcal{F}_2(\alpha, \gamma, c)$ that $I(r)$ is a continuous, decreasing function of r . [In fact one can show $Q_1'(r(D, c)) = Q_2'(r(D, c))$ so that $I(r)$ is differentiable and $I'(r) < 0$ on $(0, 1)$.] Since $r(D, c) < 1$ for $D > -1$, $\lim I(r) (r \rightarrow 1^-) = \lim Q_2(r) (r \rightarrow 1^-) = -\infty$, and, since $I(0) = 1$, the equation $I(r) = \beta$ will always have a unique solution r_* in $(0, 1)$. Clearly r_* is always the solution of either $Q_1(r) = \beta$ or $Q_2(r) = \beta$, depending on the relationship between the roots of these equations and $r(D, c)$, or equivalently, on the relationship between $I(r(D, c))$ and β . The remainder of this argument is concerned with determining this relationship.

Let $c \in [0, \infty)$ be fixed, let $r(D) = r(D, c)$ and let $\mu(D) = Q_1(r(D)) = Q_2(r(D))$. We will show $\mu(D)$ is a strictly increasing function of D mapping $(-1, 1)$ onto $(-\infty, 1)$. Now

$$\mu'(D) = \frac{d}{dD} Q_1(r(D)) > 0$$

if and only if

$$(13) \quad r'(D) < \frac{r(D)(1 + r(D))}{1 - D} \cdot \frac{(1 + c)(1 + Dr(D))^2 + 1 + r(D)}{(1 + c)(1 + Dr(D))^2 + 1 - Dr(D)^2}.$$

Since the second factor in the right hand side of (13) is clearly greater than 1, it is sufficient to show

$$(14) \quad r'(D) < r(D)(1 + r(D))/(1 - D) .$$

Differentiating (12) implicitly yields

$$(15) \quad r'(D) = [2(1 + c)r(D) + (2D - 5 - 2c + 2Dc)r(D)^2 \\ + (1 - 2D - 2Dc)r(D)^3]/[(4 - 2D - 2Dc - c) \\ + 2D(5 - D + 2c - Dc)r(D) - 3D(1 - D - Dc)r(D)^2] ,$$

and, before substituting (15) in (14), we must determine the sign of the denominator in (15). Let

$$p(r) = (1 + K)(1 - r)(1 + r)^2(R_0^2 - R_1^2) \\ = (2 + c) - (4 - 2D - 2Dc + c)r \\ - D(5 - D - 2c - Dc)r^2 + D(1 - D - Dc)r^3$$

so that $p(r(D)) = 0$ and the denominator in (15) is $-p'(r(D))$. Since R_0 is decreasing and R_1 is increasing, $p(r)$ changes sign at $r(D)$ and must have a zero of order 1 or 3 at $r(D)$. If $r(D)$ is a root of order 3 then $p''(r(D)) = 0$ which implies

$$r(D) = (5 + 2c - D - Dc)/(3(1 - D - Dc)) .$$

However this last expression is not in $(0, 1)$ for $D \in (-1, 1)$ and $c \in [0, \infty)$, hence $r(D)$ is a root of order 1 and, since $P(r)$ is decreasing at $r(D)$, $p'(r(D)) < 0$. Thus the denominator in (15) is positive and substituting (15) in (14) then shows that (14) is equivalent to

$$(16) \quad (2 - c) + (9 + D + 3c - 2Dc)r(D) \\ + (6Dc - D^2c + 10D - 1 - D^2)r(D)^2 \\ - 3D(1 - D - Dc)r(D)^3 > 0 .$$

Using the fact that $r(D)$ satisfies (12) to eliminate $r(D)^3$ in (16), we find that (16) is equivalent to

$$r(D)(7 - 5r(D))(D + 1) + (8 - 10r(D) + 4r(D)^2) \\ + 2D^2r(D)^2 + 2c(1 + Dr(D))^2 > 0 ,$$

which is obviously valid for $r(D)$ in $(0, 1)$, D in $(-1, 1)$ and $c \geq 0$. Thus $\mu(D)$ is increasing on $(-1, 1)$.

An examination of (12) shows $r(D) \rightarrow 1$ when $D \rightarrow 1$ or $D \rightarrow -1$, hence $\mu(D) \rightarrow -\infty$ as $D \rightarrow -1$, $\mu(D) \rightarrow 1$ as $D \rightarrow 1$, and the equation $\mu(D) = 0$ has a unique solution D_0 in $(-1, 1)$. If $-1 < D \leq D_0$, then $\mu(D) = Q_1(r(D)) \leq 0$ and r_* is the root of $Q_1(r) = \beta$. If $D_0 < D < 1$, then r_* is the root of $Q_1(r) = \beta$ when $\mu(D) \leq \beta$ and r_* is the root of $Q_2(r) = \beta$ when $\beta \leq \mu(D)$. This completes the proof.

If we take $c = \gamma = 1$ and $\alpha = \beta = 0$, then we obtain as a special case Livingston's result [11]. If we let $\gamma = 1$ and $\alpha = \beta = 0$, then we obtain Theorem 1 in [4]. Letting $c = \gamma = 1$ yields results found in [1], [2], [10], [13] and, as we have already noted, the case $c = 1$ and $\alpha + \gamma \leq 1$ appears in [7].

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